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On linear processes of approximation (I)

by

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1. Let $C_{2\pi}$ be the space of 2π-periodic continuous functions and let $\{T_n(f(\tau); t)\}$ be a sequence of linear operators over $C_{2\pi}$. Let us suppose the T_n's to be trigonometric polynomials. Several known theorems state that for appropriately chosen operator-sequences $\{T_n\}$ every function satisfying e. g. a Lipschitz condition

$$(1.1)\qquad |f(t+h)-f(t)| \leqslant c_1|h|^\alpha, \quad -\pi \leqslant h \leqslant \pi, \quad 0<\alpha\leqslant 1,$$

admits an approximation

$$(1.2)\qquad |T_n(f(\tau); t)-f(t)| \leqslant c_2 c_1 n^{-\alpha}.$$

The question of characterizing such sequences in general was studied only recently. An approach to the question would be what we are going to call the *method of test-functions*. This method was in fact introduced, though not stated explicitly, as early as the first investigations of D. Jackson. Jackson considers a sequence $\{T_n\}$ satisfying

$$(1.3)\qquad \|T_n(f)\| \leqslant c_3\|f\|,$$

such that: if functions satisfy (1.1) for $\alpha=1$ and $t\in[0, 2\pi]$, then these functions satisfy (1.2) for $\alpha=1$ and $t\in[0,2\pi]$. He concludes therefrom (1.2) as a consequence of (1.1) for *every* $0<\alpha\leqslant 1$. This argument may be interpreted as saying that the class of "test-functions" Lip 1 tests the degree of approximation of functions in the class Lip α, $0<\alpha<1$. The test is whether the degree of approximation of functions belonging to the "test-class" is $O(1/n)$. It is easy to see that the test-class in the above case can be replaced by the smaller class of continuously differentiable functions. The twice continuously differentiable functions form a still smaller test-class. The degree of approximation required here is $O(1/n^2)$; this follows from a more general theorem of one of us [1].

If we suppose that each term of $\{T_n\}$ is a positive operation, i. e.,

$$(1.4)\qquad T_n(f) \geqslant 0 \quad \text{if} \quad f\geqslant 0,$$

we have — as shown by Korovkin [3] — a surprisingly simple test-class; it consists of three functions

$$f_0(t) = 1, \quad f_1(t) = \sin t, \quad f_2(t) = \cos t$$

and the test conditions are

$$T_n(f_0; t) \equiv 1, \quad T_n(f_1; t) = \sin t + O(1/n^2), \quad T_n(f_2; t) = \cos t + O(1/n^2)$$

(see [3], also [2]).

The remarkable difference between these two sorts of tests is that in the latter case the test-class is finite, while in the previous case it is infinite. It seems almost sure that, if we do not suppose positiveness od $\{T_n\}$ or something else which would be stronger than (1.3), no finite set of functions serves as a test-class.

In this paper we shall construct another test-class, consisting of rather simple functions, giving in particular a sufficient condition that (1.1) implies (1.2) in case $a < 1$. In the forthcoming continuation of this paper we intend to supply similar test-class conditions covering also the case of $a = 1$.

2. Let X be one of the following normed linear function-spaces, consisting of 2π-periodic, L-integrable functions:

(a) $L^p[-\pi, \pi]$, $p \geqslant 1$, with the usual norm $\|f\| = \{\int_{-\pi}^{\pi} |f(\tau)|^p d\tau\}^{1/p}$,

(b) $C_{2\pi}$ with the norm $\|f\| = \max_{-\pi \leqslant \tau \leqslant \pi} |f(\tau)|$.

It may be noted that either of the above-introduced norms satisfies the inequality

$$(2.1) \qquad \left\| \int_{-\pi}^{\pi} |f(\tau+u) - f(\tau)| \varphi(u)\, du \right\| \leqslant \int_{-\pi}^{\pi} \|f(\tau+u) - f(\tau)\| \cdot |\varphi(u)|\, du$$

for every continuous $\varphi(u)$. In what follows, X is one of the spaces (a), (b), however, it should be remarked that we might be concerned as well with any other normed space $X[-\pi, \pi]$ for which the condition (2.1) would be satisfied.

Let $\{B_n(f; t)\}$ denote a sequence of linear operators mapping X into X and mapping $C_{2\pi}$ into $C_{2\pi}$. The norms of the operators $\{B_n\}$ are supposed to form a bounded sequence, i. e.,

$$(2.2) \qquad \|B_n(f; t)\| \leqslant c_4 \|f\|,$$

where c_4 depends neither upon f nor on n. We shall use throughout the O-notation and it will be understood that constants involved with O are numerical after having fixed X and $\{B_n\}$.

THEOREM 1. *Let $\{\lambda_n\}$ be a monotonously increasing sequence of positive integers and let us suppose, in addition to* (2.2),

$$(2.3) \qquad B_n(1; t) = 1,$$

$$(2.4) \qquad B_n(e^{i\tau}; t) = e^{it} + O\left(\frac{1}{\lambda_n}\right),$$

$$(2.5) \qquad B_n\left(e^{ik\tau} \sin^2 \frac{\tau - t}{2}; t\right) = O\left(\frac{1}{\lambda_n^2}\right), \quad k = 1, 2, \ldots, \lambda_n.$$

Then for every $f \in X$

$$(2.6) \qquad \|B_n(f(\tau); t) - f(t)\| = O(1)\left\{\int_{-\infty}^{\infty} \|f(\tau + y/\lambda_n) - f(\tau)\| \min(1, y^{-2})\, dy\right\}.$$

COROLLARY. *Let*

$$(2.7) \qquad \|f(t+h) - f(t)\| = O(|h|^a), \quad 0 < a < 1.$$

Then the statement (2.6) *is just*

$$(2.8) \qquad \|B_n(f(\tau); t) - f(t)\| \leqslant \frac{c_5(f)}{\lambda_n^a}.$$

Examples. As an example of principal interest, let us consider the Fejér means

$$\sigma_n(f; t) = \frac{S_0(f; t) + S_1(f; t) + \ldots + S_{n-1}(f; t)}{n}$$

of the Fourier series of $f(\tau)$. Inserting σ_n for B_n and $n-1$ for λ_n, a direct calculation shows that expression (2.5) vanishes identically, and also (2.3), (2.4) are satisfied. We obtain the well-known result (see [6]):

$$\|\sigma_{n+1}(f; t) - f(t)\| = O(1)\left\{\int_{-\infty}^{\infty} \|f(\tau + y/\lambda_n) - f(\tau)\| \min(1, y^{-2})\, dy\right\}.$$

This estimate is after investigations of Steckin [5] at least in the case $X = C_{2\pi}$ best-possible. We conclude from this example that (2.6) cannot be improved. Inspite of the fact that Fejér kernels are positive, Korovkin's theorem would yield only a less sharp estimate (see [3], p. 75) in this case. Nevertheless, the estimate above is anticipated in the proof of the theorem itself.

As a second example let us consider the Rogosinski-sums

$$B_n(f; t) = \frac{1}{2}\left\{S_n\left(f; t + \frac{\pi}{2n+1}\right) + S_n\left(f; t - \frac{\pi}{2n+1}\right)\right\} \stackrel{\text{def}}{=} R_n(f; t),$$

where $S_n(f; t)$ stands for the n-th partial sum of the Fourier series of $f(\tau)$ at the point $\tau = t$. As well-known these sums satisfy (2.2) (see [4]).

Putting $\lambda_n = n-1$, $n \geqslant 2$, the conditions (2.3), (2.4) and (2.5) can be easily verified to the familiar effect that

$$\|R_n(f;t)-f(t)\| \leqslant \frac{c_6(f)}{n^a}$$

provided that (2.7) is satisfied.

3. In order to prove our theorem we introduce the Fejér-sums

$$\sigma_{\lambda_n} = \sigma_{\lambda_n}(f;t) = \sum_{|k|\leqslant\lambda_n}\left(1-\frac{|k|}{\lambda_n}\right)a_k e^{ikt}, \qquad a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}dy. \tag{3.1}$$

The following estimate may be regarded as classical:

$$\|\sigma_{\lambda_n}(f;t)-f(t)\| = O(1)\int_{-\infty}^{\infty}\|f(\tau+y/\lambda_n)-f(\tau)\|\min(1,\ y^{-2})\,dy; \tag{3.2}$$

in the case of $C_{2\pi}$-norm see e. g. [5], for L^p-norm proof of (3.2) would run along the same lines owing to (2.1).

We use the formula

$$B_n(f;t)-f(t) = \{\sigma_{\lambda_n}(f;t)-f(t)\}+B_n(f-\sigma_{\lambda_n};t)+\{B_n(\sigma_{\lambda_n};t)-\sigma_{\lambda_n}(f;t)\}.$$

The first term of it can be estimated to the required order by (3.2). The second term contributes the same owing to (2.2) and (3.2). The only thing which remains to be proved is that

$$\|B_n(\sigma_{\lambda_n};t)-\sigma_{\lambda_n}(f;t)\| = O(1)\int_{-\infty}^{\infty}\|f(\tau+y/\lambda_n)-f(\tau)\|\min(1,y^{-2})\,dy. \tag{3.3}$$

By the linearity of B_n we obtain from (3.1)

$$\Delta_n(f;t) \overset{\text{def}}{=} B_n(\sigma_{\lambda_n};t)-\sigma_{\lambda_n}(f;t) = \sum_{|k|\leqslant\lambda_n}\left(1-\frac{|k|}{\lambda_n}\right)a_k e^{ikt}\eta_n(k;t),$$

where

$$\eta_n(k;t) \overset{\text{def}}{=} B_n(e^{ik(\tau-t)};t)-1. \tag{3.4}$$

Using the definition of a_k's we come to

$$\Delta_n(f;t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y+t)\sum_{|k|\leqslant\lambda_n} e^{-iky}\eta_n(k;t)\left(1-\frac{|k|}{\lambda_n}\right)dy.$$

Putting $f\equiv 1$ the above expression vanishes. Using this fact we get

$$\Delta_n(f;t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(y+t)-f(t)\}\sum_{|k|\leqslant\lambda_n} e^{-iky}\eta_n(k;t)\left(1-\frac{|k|}{\lambda_n}\right)dy. \tag{3.5}$$

We anticipate the estimate

$$\Gamma_n(t,y) \overset{\text{def}}{=} \sum_{|k|\leqslant\lambda_n} e^{-iky}\eta_n(k;t)\left(1-\frac{|k|}{\lambda_n}\right) = O\left(\min\left(\lambda_n,\ \frac{1}{\lambda_n y^2}\right)\right) \tag{3.6}$$

which will be shown in the next section to be a consequence of (2.3), (2.4) and (2.5). From (3.5) and (3.6) we conclude

$$|\Delta_n(f;t)| = O(1)\int_{-\pi}^{\pi}|f(y+t)-f(t)|\min\left(\lambda_n,\ \frac{1}{\lambda_n y^2}\right)dy;$$

using (2.1) we have

$$\|\Delta_n(f;t)\| = O(1)\int_{-\pi}^{\pi}\|f(y+t)-f(t)\|\min\left(\lambda_n,\ \frac{1}{\lambda_n y^2}\right)dy.$$

Substituting y/λ_n for y in the above integral, we obtain (3.3) as required.

4. We turn to the last step of our proof, i. e. to the estimation (3.6). Let a_k, b_k, $k = 0,1,\ldots,\nu$, be arbitrary numbers, further let $b_{\nu+1} = b_{\nu+2} = 0$. We put

$$A_k = a_0+a_1+\ldots+a_k, \qquad k = 0,1,\ldots,\nu,$$

$$A_k^{(1)} = A_0+A_1+\ldots+A_k, \qquad k = 0,1,\ldots,\nu,$$

and further

$$\Delta b_k = b_k-b_{k+1}, \qquad \Delta^2 b_k = b_k-2b_{k+1}+b_{k+2}, \qquad k = 0,1,\ldots,\nu.$$

Using this notation the following formula may be easily verified:

$$\Omega \overset{\text{def}}{=} \sum_{k=0}^{\nu}\left(1-\frac{k}{\nu}\right)a_k b_k = \sum_{k=0}^{\nu}\left(1-\frac{k}{\nu}\right)\Delta^2 b_k A_k^{(1)}+\frac{2}{\nu}\sum_{k=1}^{\nu}\Delta b_{k+1}A_k^{(1)}. \tag{4.1}$$

If we choose $a_0 = 1$, $a_1 = a_2 = \ldots = a_\nu = 0$, we obtain the identity

$$\sum_{k=0}^{\nu}\left(1-\frac{k}{\nu}\right)\Delta^2 b_k(k+1)+\frac{2}{\nu}\sum_{k=1}^{\nu}\Delta b_{k+1}(k+1) = b_0. \tag{4.2}$$

Substituting in turn $a_k = e^{-iky}$, which yields

$$A_k^{(1)} = \frac{k+1}{1-e^{-iy}}-\frac{e^{-iy}}{(1-e^{-iy})^2}(1-e^{-i(k+1)y}) = \frac{k+1}{1-e^{-iy}}+O(y^{-2}),$$

we get from (4.1), taking also account from (4.2),

$$\Omega = \frac{b_0}{1-e^{-iy}}+O(y^{-2})\left\{\sum_{k=0}^{\nu}\left(1-\frac{k}{\nu}\right)|\Delta^2 b_k|+\frac{1}{\nu}\sum_{k=0}^{\nu}|\Delta b_{k+1}|\right\}. \tag{4.3}$$

Setting $b_k = \eta_n(k; t)$, $k = 0, 1, \ldots, \nu$, $\nu = \lambda_n$, and observing that owing to (3.4) in this case, for $k = 0, 1, \ldots, \lambda_n - 2$,

$$\begin{aligned}\Delta^2 b_k &= B_n(e^{ik(\tau-t)} - 2e^{i(k+1)(\tau-t)} + e^{i(k+2)(\tau-t)}; t)\\ &= 2B_n(e^{i(k+1)(\tau-t)}\{\cos(\tau-t)-1\}; t)\\ &= -4e^{-i(k+1)t} B_n\left(e^{i(k+1)\tau}\sin^2\frac{\tau-t}{2}; t\right),\end{aligned}$$

we get by (2.5)

$$\Delta^2 b_k = O(\lambda_n^{-2}), \qquad k = 0, 1, \ldots, \lambda_n - 2. \tag{4.4}$$

Owing to (2.3), (2.4) and (3.4) we have

$$b_0 = 0, \qquad b_1 = O(\lambda_n^{-1}). \tag{4.5}$$

From

$$\Delta b_k = b_0 - b_1 - \{\Delta^2 b_0 + \Delta^2 b_1 + \ldots + \Delta^2 b_{k-1}\}, \qquad k = 1, 2, \ldots, \lambda_n - 1,$$

$$b_k = b_0 - \{\Delta b_0 + \Delta b_1 + \ldots + \Delta b_{k-1}\}, \qquad k = 1, 2, \ldots, \lambda_n,$$

and from (4.4), (4.5)

$$\Delta b_k = O(\lambda_n^{-1}), \qquad k = 1, 2, \ldots, \lambda_n - 1, \tag{4.6}$$

$$b_k = \eta_n(k; t) = O(k\lambda_n^{-1}), \qquad k = 1, 2, \ldots, \lambda_n. \tag{4.7}$$

Substituting (4.4), (4.5), (4.6) and (4.7) into (4.3) we come to

$$\Omega = O\left(\frac{1}{\lambda_n y^2}\right), \qquad |y| \leqslant \pi. \tag{4.8}$$

Noting that $\eta_n(k; t) = \overline{\eta_n(-k; t)}$, we find $\Gamma_n(t, y) = \Omega + \overline{\Omega}$, so that, by (4.8),

$$\Gamma_n(t, y) = O\left(\frac{1}{\lambda_n y^2}\right), \qquad |y| \leqslant \pi. \tag{4.9}$$

We also have as a direct corollary of the definition (3.6) of $\Gamma_n(t, y)$ and (4.7)

$$\Gamma_n(t, y) = O(\lambda_n)$$

which together with (4.9) yields the required estimate (3.6).

5. The argument applied in sections 3 and 4 yields also the following

THEOREM 2. *Suppose in the notation of Theorem 1 formulae* (2.3), (2.4) *and* (2.5) *to be satisfied only at points belonging to a set T. Replacing, further, the condition* (2.2) *by*

$$B_n(g; t) = O(1) \max_{-\pi \leqslant \tau \leqslant \pi} |g(\tau)| \quad \text{for every} \quad g \in C_{2\pi} \text{ and } t \in T, \tag{5.1}$$

we have, for $f \in C_{2\pi}$ and $t \in T$,

$$B_n(f; t) = f(t) + O(1)\int_{-\infty}^{\infty} \max_{|\tau| \leqslant \pi} |f(\tau + y/\lambda_n) - f(\tau)| \min(1, y^{-2})\, dy. \tag{5.2}$$

Using Theorem 2 we can obtain a version of Theorem 1 in which the condition of 2π-periodicity is dropped.

THEOREM 3. *Let $\{B_n(f; t)\}$ be a sequence of linear operators mapping $C[-1, 1]$ — the space of functions continuous on $[-1, 1]$ — into itself. Suppose, further*

$$B_n(g; t) = O\left(\max_{|\tau| \leqslant 1} |g(\tau)|\right) \quad \text{for all} \quad g \in C[-1, 1] \text{ and } t \in [-1, 1].$$

If (2.3), (2.4) *and* (2.5) *hold for $|t| \leqslant 1$, then, for any $f \in C[-1, 1]$, $t \in [-1, 1]$,*

$$B_n(f; t) = f(t) + O(1)\left\{\int_1^{\infty} \frac{\omega(y/\lambda_n)}{y^2}\, dy + \frac{\log \lambda_n}{\lambda_n} |f(1) - f(-1)|\right\}, \tag{5.3}$$

where $\omega(\delta)$ stands for the modulus of continuity of f.

Proof. First we extend the definition of $B_n(\varphi; t)$ for $\varphi \in C_{2\pi}$:

$$\tilde{B}_n(\varphi; t) \overset{\text{def}}{=} \begin{cases} B_n(\varphi; t) & \text{if} \quad t \in [-1, 1], \text{ where } \varphi \text{ is restricted to } [-1, 1],\\ B_n(\varphi; -\pi) = B_n(\varphi; \pi) = 0, & \\ \text{linear otherwise.} & \end{cases}$$

This $\tilde{B}_n(\varphi; t)$ maps $C_{2\pi}$ into $C_{2\pi}$ and is linear. Also

$$\tilde{B}_n(\varphi; t) = O(1) \max_{|\tau| \leqslant \pi} |\varphi(\tau)|, \qquad t \in [-\pi, \pi].$$

Then we apply Theorem 2 for the 2π-periodic function f_1 which we obtain by defining $f_1(t) = f(t)$, $|t| \leqslant 1$, $f_1(2\pi - 1) = f(-1)$ and $f_1(t)$ linear in $[1, 2\pi - 1]$.

From the definition of $f_1(t)$ it follows immediately

$$\max_{|t| \leqslant \pi} |f_1(t + y) - f_1(t)| \leqslant \omega(|y|) + \frac{|f(1) - f(-1)|}{2\pi - 2} |y|.$$

Inserting this estimate into (5.2) and putting $T = [-1, 1]$ we come straightaway to (5.3).

We conclude our paper with the remark that our theorems are applicable also in cases when condition (2.3) is violated. Let us suppose for this purpose that the sequence $\{B_n^*(f; t)\}$ of linear operators satisfies (2.4) and (2.5), but

$$B_n^*(1; t) = 1 + \varepsilon_n(t), \qquad \varepsilon_n(t) = O(1/\lambda_n^2).$$

We introduce the new sequence of operators,

$$B_n(f;t) = B_n^*(f;t) - \varepsilon_n(t)\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\,d\tau,$$

for which all the conditions of our theorems are satisfied. Thus putting B_n^* in place of B_n, an additional term

$$\frac{1}{2\pi}\|\varepsilon_n(t)\|\int_{-\pi}^{\pi}|f(\tau)|\,d\tau, \quad \text{resp.} \quad \frac{1}{2\pi}|\varepsilon_n(t)|\int_{-\pi}^{\pi}|f(\tau)|\,d\tau$$

will occur in estimates (2.6), resp. (5.2) and (5.3).

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Über unbedingte Konvergenz der Orthogonalreihen mit strukturellen Bedingungen

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In dieser Note beweisen wir solche Sätze, welche sich aus der auf die beste Annäherungsgrad von $f(x)$ bezüglichen Bedingung, bzw. aus dem Stetigkeitsverhältnis von $f(x)$ ergeben und sich auf die Konvergenz der Orthogonalreihen beziehen.

Es sei $\{\varphi_n(x)\}$ $(n = 1, 2, \ldots)$ ein bezüglich der Verteilung $d\mu(x)$(1) im Intervall (a, b) orthonormiertes Funktionensystem. Bezeichne E_n die im Raum L_μ^2 beste Annäherungsgrad von $f(x)$ mit Linearformen

$$\sum_{k=1}^{n} d_k\varphi_k(x),$$

dann ist nach einem bekannten Satz von Gram [2] (s. z. Alexits [1], S. 14)

$$E_n = \Big\{\int_a^b \Big(f(x) - \sum_{k=1}^{n} c_k\varphi_k(x)\Big)^2 d\mu(x)\Big\}^{1/2}$$

mit

$$c_k = \int_a^b f(x)\varphi_k(x)\,d\mu(x).$$

Sei $\Phi(x)$ eine für $x \geqslant 0$ definierte, positive, monoton wachsende Funktion. Bezeichne $R_\Phi^{(\mu)}(\{\varphi_n(x)\})$ die Klasse aller Funktionen $f(x) \in L_\mu^2$, für welche $E_n^2 = O\big(1/\Phi(n)\big)$.

Wir werden zuerst den folgenden Satz beweisen:

Satz I. *Ist*

$$\sum_{k=1}^{\infty}\frac{E_k}{k} < \infty, \tag{1}$$

(1) Derivierte $\mu'(x)$ verschwindet höchstens auf einer Nullmenge.