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On the differentiability of weak solutions of certain non-elliptic equations II

by

H. MARCINKOWSKA (Warszawa)

Introduction

In the first part of this paper (which will be quoted here as [I]) a theorem was given concerning the periodic weak solutions of certain partial differential equations of non-elliptic type. The differentiability properties of such solutions were described with the aid of some Hilbert spaces, which have been defined in the first chapter of [I]. In the present paper we are going to prove some further properties of these Hilbert spaces and to study the differentiability of weak solutions of the mentioned equations under some special boundary conditions.

We recall some definitions and notations of [I]. Let Ω be the product of two domains: $\overset{1}{\Omega}$ of the space E^R, and $\overset{2}{\Omega}$ of the space E^S ($R+S=N$), and let $C_2^\infty(\Omega)$ be the class of all complex-valued functions which are infinitely differentiable in Ω and whose all the derivatives are square summable in Ω. We denote by B a linear subset of the class $C_2^\infty(\Omega)$ containing the class $C_0^\infty(\Omega)$ which has the following properties:

1° for each function $\varphi\in C_0^\infty(\overset{1}{\Omega})$ or $\psi\in C_0^\infty(\overset{2}{\Omega})$, and for each $u\in B$, the functions φu and ψu are also in B,

2° for each $u\in B$ all the derivatives of u are also in B.

$B_{0,-}$ denotes the subset of the class B consisting of all functions $u(x,y)$ which vanish for $x\in\overset{1}{\Omega}-K$ and $y\in\overset{2}{\Omega}$, when K is a compact contained in $\overset{1}{\Omega}$ (depending on u). $B_{-,0}$ has the same meaning when the roles of x and y are interchanged. Let especially Ω be the N-dimensional cube; so B_p denotes the class of all functions infinitely differentiable in E^N which are periodic with Ω as period-parallelogram.

We have defined in [I] the two-indices norms for $u\in C_2^\infty(\Omega)$ as follows:

$$\|u\|_{0,k}^2 = \sum_{0\leqslant|\beta|\leqslant k} \|D_y^\beta u\|_{L^2(\Omega)}^2,$$

$$\|u\|^2_{m,q} = \sum_{0 \leqslant |\alpha| \leqslant m} \|D^\alpha_x u\|^2_{0,q},$$

$$\|u\|_{-m,q} = \sup_{v \in B} \frac{|(u,v)_{L^2(\Omega)}|}{\|v\|_{m,-q}}$$

(p, q arbitrary integers; m, k non-negative integers). The corresponding Hilbert space $H_{p,q}(\Omega, B)$ has been introduced as the completion of B in the norm $\| \; \|_{p,q}$. If $u \in H_{m,-k}(\Omega, B)$, the *strong derivative* of u in the norm $\| \; \|_{0,-k}$ with respect to x of order α ($0 \leqslant |\alpha| \leqslant m$) has been defined as the limit in the norm $\| \; \|_{0,-k}$ of the sequence $\{D^\alpha_x u_n\}$, $\{u_n\}$ being the sequence contained in B which tends to u in the norm $\| \; \|_{m,-k}$.

Our paper deals with the differential operators of the form (for sufficiently differentiable u)

$$Lu = \sum_{\substack{0 \leqslant |\alpha|, |\alpha'| \leqslant m \\ 0 \leqslant |\beta|, |\beta'| \leqslant n}} (-1)^{m+n} D^\alpha_x D^\beta_y (a_{\alpha\alpha'\beta\beta'} D^{\alpha'}_x D^{\beta'}_y u)$$

satisfying the following conditions:

1° $a_{\alpha\alpha'\beta\beta'}$ are complex-valued functions infinitely differentiable in Ω and having all the derivatives bounded in Ω,

2° $a_{\alpha\alpha'\beta\beta'}(x, y) = \overline{a_{\alpha'\alpha\beta'\beta}}(x, y)$ for $|\alpha| = |\alpha'| = m$, $|\beta| = |\beta'| = n$, $(x, y) \in \Omega$,

3° there is some positive constant d, such that

$$\sum_{\substack{|\alpha| = |\alpha'| = m \\ |\beta| = |\beta'| = n}} a_{\alpha\alpha'\beta\beta'}(x, y)\, \zeta_{\alpha,\beta} \overline{\zeta}_{\alpha',\beta'} \geqslant d \sum_{\substack{|\alpha| = m \\ |\beta| = n}} |\zeta_{\alpha,\beta}|^2$$

for all $(x, y) \in \Omega$ and all systems of complex numbers $\{\zeta_{\alpha,\beta}\}$ ($|\alpha| = m$, $|\beta| = n$). The class of all such differential operators was denoted in [I] by Λ. We have proved for such operators the inequality

$$(0) \qquad |(L_{r,s} u, u)| \geqslant c \|u\|^2_{m+r,n+s}$$

assuming that the functions $\operatorname{Re} a_{\alpha\alpha 00}$ ($|\alpha| = m$), $\operatorname{Re} a_{00\beta\beta}$ ($|\beta| = n$), $\operatorname{Re} a_{0000}$, have positive sufficiently large lower bound, and the function u is smooth in $\overline{\Omega}$ and satisfies such boundary conditions that after the integration by parts the boundary integrals vanish (here $L_{r,s}$ denotes some of the operators $\Delta^r_x \Delta^s_y L$, $\Delta^r_x L \Delta^s_y$, $\Delta^s_y L \Delta^r_x$, $L \Delta^r_x \Delta^s_y$, Δ being the identity operator minus the Laplacean).

In the sequel we shall deal with the spaces $H_{p,q}$ belonging to the different domains of the Euclidean space E^N and to the different classes B; accordingly the corresponding norms will be denoted by $\| \; \|_{H_{p,q}(\Omega,B)}$ (instead of the brief notation $\| \; \|_{p,q}$) when it will be necessary to avoid a misunderstanding. In such cases the bilinear form $(\, ,)$ considered on the product $H_{p,q}(\Omega, B) \times H_{-p,-q}(\Omega, B)$ will be denoted by $(\, ,)_{(\Omega,B)}$. Different positive constants which do not depend on the function u, will be denoted by the same letter c.

All the bibliographical references are given in [I].

1. Some properties of the spaces $H_{p,q}(\Omega, B)$

1.1. The spaces $H_{p,q}(\Omega, B)$ have been defined in such a way that theorem 1 [I] holds, but there is of course a defect in this definition: when in the definition of the norm $\| \; \|_{m,-k}$ ($m, k > 0$) the roles of x and y are interchanged, we do not get the norm $\| \; \|_{-k,m}$. Therefore the definition of the spaces $H_{p,q}(\Omega, B)$ in the case when p and q have different signs seems to be asymmetric with respect to x and y. We shall prove that in the case of some special boundary conditions the space $H_{-m,k}(\Omega, B)$ can be identified with the completion of B in the norm $||| \; |||_{-m,k(\Omega,B)}$ (when

$$|||u|||^2_{-m,k,(\Omega,B)} \overset{\mathrm{df}}{=} \sum_{0 \leqslant |\beta| \leqslant k} \|D^\beta_y u\|^2_{H_{-m,0}(\Omega,B)}$$

for $u \in B$), so that in this case there is in fact no asymmetry.

Let $\Omega = \overset{1}{\Omega} \times \overset{2}{\Omega}$ be a domain of the Euclidean space E^N with $\overset{1}{\Omega}$ being the R-dimensional cube and $\overset{2}{\Omega}$ being an arbitrary domain of the space E^S ($R + S = N$). We denote by $B_{p,0}$ the class of all functions φ satisfying the following conditions:

(C_1) $\varphi \in C^\infty(E^R \times \overset{2}{\Omega})$;

(C_2) $\varphi(-, y)$ is the periodic function of x with $\overset{1}{\Omega}$ being the period-parallelogram for arbitrary fixed $y \in \overset{2}{\Omega}$;

(C_3) there is a compact $K \subset \overset{2}{\Omega}$ (depending on φ) such that φ vanish on $E^R \times (\overset{2}{\Omega} - K)$.

The following inequality can be obtained in the same manner as lemma 4 [I], when we notice that after the integration by parts the boundary integrals vanish:

$$(1) \qquad \|u\|_{H_{p,q}(\Omega, B_{p,0})} \geqslant \|D^\alpha_x D^\beta_y u\|_{H_{p-|\alpha|,q-|\beta|}(\Omega, B_{p,0})} \qquad (u \in B_{p,0}).$$

It is easy to show that in the case of an arbitrary class B the elements of the spaces $H_{m,-k}(\Omega, B)$ can be characterized by the following conditions:

1° u is an element of the space $H_{0,k}(\Omega, B)$,

2° there is a sequence $\{u_n\} \subset B_{-,0}$ such that $\|u_n - u\|_{0,-k} \xrightarrow[n\to\infty]{} 0$, and all sequences $\{D^\alpha_x u_n\}$ for $0 \leqslant |\alpha| \leqslant m$ are fundamental in the norm $\| \; \|_{0,-k}$.

The limit of $\{D_x^\alpha u_n\}$ in the norm $\| \ \|_{0,-k}$ can be called *strong derivative* in the norm $\| \ \|_{0,-k}$ with respect to x of order α. So the space $H_{m,-k}(\Omega, B)$ can be treated as the set of all elements of the space $H_{0,-k}(\Omega, B)$ which have all the strong derivatives in the norm $\| \ \|_{0,-k}$ with respect to x of order not exceeding m. From lemma 4 [I] it follows that each element of the space $H_{-m,k}(\Omega, B)$ has all strong derivatives in the norms $\| \ \|_{-m,0}$ with respect to y of order not exceeding k, but $H_{-m,k}(\Omega, B)$ contains in general not all the elements of $H_{-m,0}(\Omega, B)$ having such property.

Let now Ω be the N-dimensional cube. From lemma 4 [I] it follows that

$$\|u\|^2_{H_{-m,k}(\Omega,B_p)} \geqslant c \sum_{0\leqslant|\beta|\leqslant k} \|D_y^\beta u\|^2_{H_{-m,0}(\Omega,B_p)} \qquad (u \in B_p), \tag{2}$$

and

$$\|D_y^\alpha u\|_{H_{-m,q}(\Omega,B_p)} \leqslant \|D_y^\gamma u\|_{H_{-m,0}(\Omega,B_p)} \qquad (u \in B_p)$$

for $q \leqslant |\gamma|-|\alpha|$. Putting in the last inequality $q = k-2n$ (when n is fuch an integer that $k-2n \leqslant 0$) and $\alpha = 2\beta$ $(0 \leqslant |\beta| \leqslant n)$, we obtain

$$\|D_y^{2\beta} u\|_{H_{-m,k-2n}(\Omega,B_p)} \leqslant \|D_y^\gamma u\|_{H_{-m,0}(\Omega,B_p)} \qquad (u \in B_p)$$

sor $|\gamma| \geqslant k-2n+2|\beta|$, and this yields the estimate

$$\|\Delta_y^n u\|_{H_{-m,k-2n}(\Omega,B_p)} \leqslant c \sum_{0\leqslant|\gamma|\leqslant k} \|D_y^\gamma u\|_{H_{-m,0}(\Omega,B_p)} \qquad (u \in B_p). \tag{3}$$

Applying now lemma 13 [I] to the operator Δ_y^n and the Schwarz inequality to the sum on the right of (3) we get

$$\|u\|^2_{H_{-m,k}(\Omega,B_p)} \leqslant c \sum_{0\leqslant|\beta|\leqslant k} \|D_y^\beta u\|_{H_{-m,0}(\Omega,B_p)} \qquad (u \in B_p). \tag{4}$$

Inequalities (2) and (4) give us the equivalency of the norms $\| \ \|_{H_{-m,k}(\Omega,B_p)}$ and $||| \ |||_{-m,k,(\Omega,B_p)}$ in the class B_p. From inequality (1) and lemma 3 (chapter 2 of this paper) it follows that the above reasoning remains true in the case when $\overset{1}{\Omega}$ is the cube and $\overset{2}{\Omega}$ and arbitrary domain, if the class of all periodic functions is replaced by the class $B_{p,0}$ (it will be only supposed that the integer n is such that $n \leqslant k \leqslant 2n$). So we can also conclude that the norms $\| \ \|_{H_{-m,k}(\Omega,B_{p,0})}$ and $||| \ |||_{m,k,(\Omega,B_{p,0})}$ are equivalent in the class $B_{p,0}$. In these two cases the definition of the spaces $H_{p,q}$ is symmetric in x and y.

1.2. It is easy to show [9] that the convergence in the space $D(\Omega)$ is stronger than the convergence in each norm $\| \ \|_{p,q}$. Therefore the restriction of each functional on $H_{p,q}(\Omega, B)$ to the set $C_0^\infty(\Omega)$ is a distribution in the sense of L. Schwartz [9], and thus each space $H_{p,q}(\Omega, B)$ treated as the adjoined space to $H_{-p,-q}(\Omega, B)$ can be mapped in the space $D'(\Omega)$ of all distributions. As the set $C_0^\infty(\Omega)$ is in general not dense in B with respect to the norm $\| \ \|_{-p,-q}$, the same distribution can be defined by two different functionals on $H_{-p,-q}(\Omega, B)$, and so this mapping is in general not one-to-one. We shall study it more precisely. Let $\dot{H}_{-p,-q}(\Omega, B)$ be the closure of $C_0^\infty(\Omega)$ in the space $H_{-p,-q}(\Omega, B)$, and $H^{\perp}_{-p,-q}(\Omega, B)$ its orthogonal completion. The adjoined space can be decomposed as follows:

$$H_{p,q}(\Omega, B) = H^d_{p,q}(\Omega, B) \oplus \tilde{H}_{p,q}(\Omega, B),$$

where $(v, u) = 0$ for $v \in H^{\perp}_{-p,-q}(\Omega, B)$, $u \in H^d_{p,q}(\Omega, B)$ or, for $v \in \dot{H}_{-p,-q}(\Omega, B)$, $u \in \tilde{H}_{p,q}(\Omega, B)$. So each element u of $H_{p,q}(\Omega, B)$ can be uniquely presented in the form $u = u^d + \tilde{u}$, where u^d (the distributional part of u) belongs to $H^d_{p,q}(\Omega, B)$ and $\tilde{u}$ is in $\tilde{H}_{p,q}(\Omega, B)$. Consider the mapping I of $H^d_{p,q}(\Omega, B)$ into $D'(\Omega)$ defined by the relation

$$(\varphi, u^d) \underset{\varphi \in C_0^\infty(\Omega)}{\equiv} \langle \varphi, Iu^d \rangle \qquad (u \in H_{p,q}(\Omega, B)).$$

The following theorem describes the differentiability properties of the distribution Iu^d:

Theorem 1. 1° *For each p, q the mapping I is one-to-one and completely continuous.*

2° *For non-negative p, q the subspace $H^d_{p,q}(\Omega, B)$ is identical with the whole space $H_{p,q}(\Omega, B)$, and for each $u \in H_{p,q}(\Omega, B)$ the distribution Iu agrees, in the sense of the theory of distributions, with the adjoined-valued function $\bar{u}$; the distribution Iu has all the derivatives $D_x^\alpha D_y^\beta Iu$ $(0 \leqslant |\alpha| \leqslant p$, $0 \leqslant |\beta| \leqslant q)$ belonging to $L^2(\Omega)$.*

3° *For non-positive p, q the distribution Iu^d has the form*

$$Iu^d = \sum_{\substack{0\leqslant|\alpha|\leqslant -p \\ 0\leqslant|\beta|\leqslant -q}} D_x^\alpha D_y^\beta v_{\alpha\beta}, \tag{5}$$

where $v_{\alpha\beta}$ are in $L^2(\Omega)$.

4° *Let p, q have different signs, namely $p \geqslant 0$ and $q \leqslant 0$. Thus for each $u \in H_{p,q}(\Omega, B)$ all the derivatives $D_x^\alpha Iu^d$ $(0 \leqslant |\alpha| \leqslant p)$ can be presented in the form*

$$D_x^\alpha Iu^d = \sum_{0\leqslant|\beta|\leqslant -q} D_y^\beta w_{\alpha\beta},$$

where $w_{\alpha\beta}$ belongs to $L^2(\Omega)$.

In condition 4° the roles of x and y may be interchanged (so for an arbitrary class B we get the symmetric characterization of the distributional parts $H^d_{p,q}(\Omega, B)$ in the casse $p, q < 0$, although the definition of the whole space $H^d_{p,p}(\Omega, B)$ is a symmetric in x and y).

Proof. Condition 1° follows immediately from the definition of the mapping I and from the weak compactness of the sphere in the Hilbert space. Condition 2° is just as easy to prove, if it is remarked that I maps the strong derivatives of u on the distributional derivatives of Iu. To prove condition 3° we apply the Riesz theorem to the space $H_{-p,-q}(\Omega, B)$. So for $u \in H_{p,q}(\Omega, B)$ there exist some $v \in H_{-p,-q}(\Omega, B)$ such that for $\varphi \in C_0^\infty(\Omega)$

$$(\varphi, u^d) = \sum_{\substack{0 \leqslant |\alpha| \leqslant -p \\ 0 \leqslant |\beta| \leqslant -q}} (D_x^\alpha D_y^\beta \varphi, D_x^\alpha D_y^\beta v). \tag{6}$$

Equation (5) is obtained from (6) when we put $v_{\alpha\beta} = (-1)^{|\alpha|+|\beta|} D_x^\alpha D_y^\beta \bar{v}$. We shall now prove condition 4°. Let $p \leqslant 0$, $q \geqslant 0$, and $u \in H_{p,q}(\Omega, B)$. From Riesz theorem applied to the space $H_{-p,-q}(\Omega, B)$ follows the existence of such $w \in H_{-p,-q}(\Omega, B)$ that for $\varphi \in C_0^\infty(\Omega)$

$$(\varphi, u^d) = \sum_{0 \leqslant |\alpha| \leqslant -p} (D_x^\alpha \varphi, D_x^\alpha w)_{0,-q}, \tag{7}$$

where D_x^α denotes the strong derivation in the norm $\| \|_{0,-q}$. So $D_x^\alpha w$ belongs to $H_{0,-q}(\Omega, B)$, and theorem 1 [I] enables us to present inequality (7) in the form

$$(\varphi, u^d) = \sum_{0 \leqslant |\alpha| \leqslant -p} (D_x^\alpha \varphi, z_\alpha) \quad (\varphi \in C_0^\infty(\Omega)), \tag{8}$$

where $z_\alpha \in H_{0,q}(\Omega, B)$. Putting in (8) $w_\alpha = (-1)^{|\alpha|} z_\alpha$ we get

$$Iu^d = \sum_{0 \leqslant |\alpha| \leqslant -p} D_x^\alpha I w_\alpha \tag{9}$$

with $w_\alpha \in H_{0,q}(\Omega, B)$ and from this and from condition 2° it follows that 4° is true in the case $p \leqslant 0, q \geqslant 0$. Let now be $p \geqslant 0$, $q \leqslant 0$. So for $u \in H_{p,q}(\Omega, B)$ the strong derivative in the norm $\| \|_{0,q}$, denoted by $D_x^\alpha u$, is an element of $H_{0,q}(\Omega, B)$ and condition 3° gives us

$$I(D_x^\alpha u)^d = \sum_{0 \leqslant |\beta| \leqslant -q} D_y^\beta w_{\alpha\beta}$$

with $w_{\alpha\beta} \in L^2(\Omega)$; condition 4° is thus proved, because it can be easily verified that $I(D_x^\alpha u)^d = D_x^\alpha I u^d$ (the derivative on the right taken in the distributional sense), q. e. d.

2. Some further applications of the spaces $H_{p,q}(\Omega, B)$ to the study of differential operators of class Λ

2.1. In the present section we are going to define the product of an element of some space $H_{p,q}(\Omega, B)$ with an infinitely differentiable function. The following two lemmas prove that under some supplementary assumptions the multiplication by such a function is a continuous operation.

Lemma 1. *Let f be a complex-valued function of class $C^\infty(\Omega)$ with all the derivatives bounded in Ω. We suppose that for $v \in B$ all the products $vD^\alpha f$ and $v\overline{D^\alpha f}$ ($|\alpha| = 0, 1, \ldots$) are also in B. So*

$$\|fu\|_{p,q} \leqslant c\|u\|_{p,q} \tag{10}$$

for $u \in B$.

Proof. In the case $p, q \geqslant 0$ it is sufficient to estimate the expression $\|D_x^\alpha D_y^\beta (fu)\|_{L^2(\Omega)}$ with $0 \leqslant |\alpha| \leqslant p$, $0 \leqslant |\beta| \leqslant q$. The derivation of the product fu gives

$$\|D_x^\alpha D_y^\beta (fu)\|_{L^2(\Omega)} \leqslant \sum \|D_x^{\alpha'} D_y^{\beta'} f D_x^{\alpha''} D_y^{\beta''} u\|_{L^2(\Omega)}, \tag{11}$$

where $|\alpha''|$ and $|\beta''|$ do not exceed p and q, respectively. Thus we obtain the estimation

$$\|D_x^\alpha D_y^\beta (fu)\|_{L^2(\Omega)} \leqslant \sup_{\substack{(x,y)\in\Omega \\ 0 \leqslant |\alpha| \leqslant p \\ 0 \leqslant |\beta| \leqslant q}} |D_x^\alpha D_y^\beta f(x,y)| \sum \|D_x^{\alpha''} D_y^{\beta''} u\|_{L^2(\Omega)}, \tag{12}$$

and from this follows (10), when the Schwarz inequality is applied to the sum on the right-hand side of (12).

In the case $p, q \leqslant 0$ the application of these results to the inequality

$$|(fu, v)| \leqslant \|u\|_{p,q} \|\bar{f}v\|_{-p,-q} \quad (u, v \in B)$$

yields

$$\|fu\|_{p,q} = \sup_{v \in B} \frac{|(fu, v)|}{\|v\|_{-p,-q}} \leqslant c\|u\|_{p,q}.$$

To derive (10) for $p \geqslant 0$, $q \leqslant 0$ consider at first the expression $\|D_x^\alpha (fu)\|_{0,q}$ $(0 \leqslant |\alpha| \leqslant p)$. It can be presented in the form

$$\sup_{v \in B} \frac{|\sum (D_x^{\alpha'} f D_x^{\alpha''} u, v)|}{\|v\|_{0,-q}} \quad (0 \leqslant |\alpha'|, |\alpha''| \leqslant p)$$

and so it is evident that

$$\|D_x^\alpha (fu)\|_{0,q} \leqslant \sum \|D_x^{\alpha'} f D_x^{\alpha''} u\|_{0,q} \quad (0 \leqslant |\alpha'|, |\alpha''| \leqslant p). \tag{13}$$

From inequality (10) with $p=0$, $q\leqslant 0$ it follows that the sum on the right-hand side of (13) is not greater than

(14) $$c\sum_{0\leqslant|\gamma|\leqslant p}\|D_x^\gamma u\|_{0,q}.$$

Applying the Schwarz inequality to the sum in (14) we get the estimate (10).

The proof of (10) for $p\leqslant 0$, $q\geqslant 0$, is similar as in the case $p,q\leqslant 0$.

LEMMA 2. *Let us consider two domains $\Omega=\overset{1}{\Omega}\times\overset{2}{\Omega}$ and $\Omega'=\overset{1}{\Omega}{}'\times\overset{2}{\Omega}$, $\overset{1}{\Omega}{}'$ being a subdomain of $\overset{1}{\Omega}$, and denote by B, B' the corresponding classes of smooth functions satisfying the assumptions formulated in the introduction. When φ is an infinitely differentiable function of x with compact support contained in $\overset{1}{\Omega}{}'$, then*

(15) $$\|\varphi u\|_{H_{p,q}(\Omega,B)}\leqslant c\|u\|_{H_{p,q}(\Omega',B')}\quad (u\in B'),$$

(16) $$\|\varphi u\|_{H_{p,q}(\Omega',B')}\leqslant c\|u\|_{H_{p,q}(\Omega,B)}\quad (u\in B).$$

The roles of x and y may be interchanged.

Proof. For $p,q\geqslant 0$ both inequalities (15), (16) are an immediate consequence of lemma 1, when we consider the equation

$$\|\varphi u\|_{H_{p,q}(\Omega,B)}=\|\varphi u\|_{H_{p,q}(\Omega',B')}\quad (p,q\geqslant 0)$$

for $u\in B$ or $u\in B'$. To consider the case $p,q\leqslant 0$ we estimate first the expression $(\varphi u,v)_{L^2(\Omega)}$ with $u\in B'$ and $v\in B$:

$$|(\varphi u,v)_{L^2(\Omega)}|=|(u,\bar{\varphi}v)_{L^2(\Omega')}|\leqslant\|u\|_{H_{p,q}(\Omega',B')}\|\bar{\varphi}v\|_{H_{-p,-q}(\Omega',B')}.$$

Applying inequality (16) for $p,q\geqslant 0$ we get

$$|(\varphi u,v)_{L^2(\Omega)}|\leqslant c\|u\|_{H_{p,q}(\Omega',B')}\|v\|_{H_{-p,-q}(\Omega,B)},$$

and this yields

$$\sup_{v\in B}\frac{|(\varphi u,v)_{L^2(\Omega)}|}{\|v\|_{H_{-p,-q}(\Omega,B)}}\leqslant c\|u\|_{H_{p,q}(\Omega',B')}.$$

Thus (15) is proved in the case $p,q\leqslant 0$. The proof of (16) is similar. The proof in the remaining cases, when p and q have different signs is equally simple. For $p\geqslant 0$, $q\leqslant 0$, we apply inequality (13) in the space $H_{p,q}(\Omega,B)$ or $H_{p,q}(\Omega',B')$ and inequalities (15) or (16) with $p=0$, $q\leqslant 0$. In the case $p\leqslant 0$, $q\geqslant 0$, the proof is analogous as in the case $p,q\leqslant 0$, when we apply inequalities (15) and (16) for $p\geqslant 0$ and $q\leqslant 0$, q. e. d.

2.2. Let M be the differential operator in Ω defined for sufficiently smooth u by formula

(17) $$Mu=\sum_{\substack{0\leqslant|\alpha|\leqslant k\\0\leqslant|\beta|\leqslant l}}d_{\alpha\beta}D_x^\alpha D_y^\beta u$$

and let $d_{\alpha\beta}$ be the complex-valued functions of x and y satisfying all the assumptions concerning the function f in lemma 1. So from lemma 4 [I] and lemma 1 of this paper follows the inequality

(18) $$\|Mu\|_{H_{p-k,q-l}(\Omega,B)}\leqslant c\|u\|_{H_{p,q}(\Omega,B)},$$

where

$$u\in\begin{cases}B & \text{for}\quad p\geqslant k,\ q\geqslant l,\\ B_{0,-} & \text{for}\quad p<k,\ q\geqslant l,\\ B_{-,0} & \text{for}\quad p\geqslant k,\ q<l,\\ C_0^\infty(\Omega) & \text{for}\quad p<k,\ q<l.\end{cases}$$

In the periodic case (Ω being the cube and $B=B_p$) inequality (18) holds for all $u\in B_p$, when we suppose that the coefficients $d_{\alpha\beta}$ belong to B_p. So from lemmas 13 and 14 [I] together with (18) follows

THEOREM 2. *Let Ω be the N-dimensional cube and L an operator of class Λ with coefficients belonging to B_p. We suppose that inequality (0) is true and denote by $\tilde{L}$ the closure of the operator L treated as the mapping from $H_{p,q}(\Omega,B_p)$ into $H_{p-2m,q-2n}(\Omega,B_p)$. Then $\tilde{L}$ mapps continuously the space $H_{p,q}(\Omega,B_p)$ on the whole space $H_{p-2m,q-2n}(\Omega,B_p)$ and possesses the continuous inverse $\tilde{L}^{-1}$.*

Let u be the periodic weak solution of the equation

(19) $$Lu=v,$$

where L is an operator of class Λ with coefficients belonging to B_p satisfying inequality (0) and $v\in H_{p,q}(\Omega,B_p)$. By theorem 3 [I] u is an element of the space $H_{p+2m,q+2n}(\Omega,B_p)$ and $u_1\overset{\mathrm{df}}{=}\tilde{L}^{-1}v_1$ must be equal to u, because it is also the periodic weak solution of (19) with the same right member v. So $\tilde{L}u=v$ and we see that each periodic weak solution of (19) is also a strong one in the following sense: there exists some sequence $\{u_n\}\subset B_p$ such that $\|u_n-u\|_{p+2m,q+2n}\to 0$ and $\|Lu_n-v\|_{p,q}\to 0$ as $n\to\infty$.

2.3. Let Ω be a product of two domains: $\overset{1}{\Omega}$ of the space E^R and $\overset{2}{\Omega}$ of the space E^S $(R+S=N)$, and let B be a class of smooth functions satisfying all the assumptions formulated in the introduction. Consider a differential operator L defined in Ω and satisfying the assumption:

for all $\varphi \in B$ the function $L^{+}\varphi$ is also in B. We say that an element u of a space $H_{p_0,q_0}(\Omega, B)$ is a weak solution of equation (19) with respect to the class B (v being an element of a space $H_{p,q}(\Omega, B)$) if the identity

$$(L^{+}\varphi, u) = (\varphi, v) \tag{20}$$

holds for all $\varphi \in B$. In [I] we have examined the periodic case, Ω being the N-dimensional cube and B the class of all periodic functions with Ω as period-parallelogram. Now we are going to study some other examples of boundary conditions and the corresponding weak solutions of equation (19).

Let first $\overset{1}{\Omega}$ be the R-dimensional cube and consider the class $B_{p,0}$ defined in 1.1. By means of Fourier expansion we can construct for each function $\varphi \in B_{p,0}$ a function φ_1, also lying in $B_{p,0}$ and such that $\Delta_x^r \varphi_1 = \varphi$ (r being a fixed natural number). So the same arguments as were used in the proof of lemma 13 [I] together with inequality (1) yield

LEMMA 3. *Let L be an operator of class Λ with the coefficients satisfying the conditions* (C_1) *and* (C_2) (see section 1.1) *and suppose, that inequality* (0) *is true. So*

$$\|u\|_{H_{p,q}(\Omega, B_{p,0})} \leqslant c\|Lu\|_{H_{p-2m,q-2n}(\Omega, B_{p,0})} \qquad (u \in B_{p,0}) \tag{21}$$

for $q \geqslant n$.

The same reasonning as was used in the proof of lemma 14 [I] gives

LEMMA 4. *Denote by $\Gamma_{p,q}$ the closure in the space $H_{p,q}(\Omega, B_{p,0})$ of the linear set $\{L\varphi\colon \varphi \in B_{p,0}\}$. Then, under the assumptions of lemma 3, the subspace $\Gamma_{p,q}$ is identical with the whole space $H_{p,q}(\Omega, B_{p,0})$ for $p \leqslant -m$, $q \leqslant -n$ in the case $n > 0$, and for arbitrary p, q in the case $n = 0$.*

By means of these lemmas the following theorems can be stated:

THEOREM 3. *Let L be an operator of class Λ with $m, n > 0$ satisfying the assumptions of lemma 3. Then*

1° *for each $v \in H_{p,q}(\Omega, B_{p,0})$ (with $q \leqslant -n$) there exists a weak solution u of equation* (19) *with respect to the class $B_{p,0}$ lying in $H_{p+2m,q+2n}(\Omega, B_{p,0})$; if $v \in H_{p,-n}(\Omega, B_{p,0})$ with $p \geqslant -m$, this weak solution is unique;*

2° *when $q \geqslant -n$, the closure $\tilde{L}$ of the operator L (treated as the mapping from $H_{p+2m,q+2n}(\Omega, B_{p,0})$ into $H_{p,q}(\Omega, B_{p,0})$) maps continuously the space $H_{p+2m,q+2n}(\Omega, B_{p,0})$ on the subspace $\Gamma_{p,q}$ of the space $H_{p,q}(\Omega, B_{p,0})$ and possesses a continuous inverse; if especially $p \leqslant -m$, then $\Gamma_{p,-n}$ is identical with the whole space $H_{p,-n}(\Omega, B_{p,0})$.*

Proof. To prove 1° let us consider the linear functional $l(\psi) \overset{\mathrm{df}}{=} (\varphi, v)$ ($\varphi \in B_{p,0}$), where $\psi = L^{+}\varphi$; it is well defined on the set $\Gamma_{-p-2m,-q-2n}$ because of inequality (21) and continuous in the norm $\|\ \|_{H_{-p-2m,-q-2n}(\Omega, B_{p,0})}$. So by Banach-Hahn theorem it can be prolonged on the whole space $H_{-p-2m,-q-2n}(\Omega, B_{p,0})$, therefore by theorem 1 [I] there exists some $u \in H_{p+2m,q+2n}(\Omega, B_{p,0})$ such that $(L^{+}\varphi, u) = (\varphi, v)$ for all $\varphi \in B_{p,0}$. In the case $p \geqslant -m$, $q \geqslant -n$ lemma 4 assures the unicity.

2° follows immediately from lemma 3 and 4 together with inequality (18) applied to the operator L (according to (1) and to our assumptions concerning the coefficients of L, inequality (18) holds for all $u \in B_{p,0}$ without any restriction concerning the support). The proof is complete.

In the same way can be proved

THEOREM 4. *Let L be an operator of class Λ with $n = 0$ (so it is elliptic in $\overset{1}{\Omega}$ with the coefficients depending on the parameter $y \in \overset{2}{\Omega}$) and suppose that the assumptions of lemma 3 hold. Then*

1° *for each $v \in H_{p,q}(\Omega, B_{p,0})$ with $q \leqslant 0$ there exists only one weak solution of equation* (19) *with respect to the class $B_{p,0}$; this weak solution is in $H_{p+2m,q}(\Omega, B_{p,0})$,*

2° *if $q \geqslant 0$, the operator $\tilde{L}$ maps continuously the space $H_{p+2m,q}(\Omega, B_{p,0})$ on the whole space $H_{p,q}(\Omega, B_{p,0})$ and has continuous inverse.*

According to lemmas 7 and 8 [I] the above theorems assure the differentiability in the classical sense of the weak solutions of equation (19) with respect to the class $B_{p,0}$, provided that p, q are positive sufficiently large numbers and $\overset{2}{\Omega}$ satisfies the assumptions formulated in section 2.1 [I].

2.4. In this section we shall apply the differentiability theorems proved above to the study of weak solutions of equation (19) in the case $n = 0$ with more general boundary conditions. Following the method used by Lax [7] we are going to prove the differentiability properties of the product Φu, Φ being an infinitely differentiable function of the variable x with compact support contained in $\overset{1}{\Omega}$. This will give us some information about the differentiability properties of u in $\Delta \times \overset{2}{\Omega}$, Δ being an arbitrary (proper) subdomain of $\overset{1}{\Omega}$.

THEOREM 5. *Let $\Omega = \overset{1}{\Omega} \times \overset{2}{\Omega}$, $\overset{1}{\Omega}$ being a bounded domain of the Euclidean space E^R contained in the R-dimensional cube $\overset{1}{\Omega}_0$ and $\overset{2}{\Omega}$ being an arbitrary domain of the Euclidean space E^S. Let B be the class of smooth functions in Ω satisfying the assumptions of the introduction and the condition $B = B_{-,0}$. We consider in the domain Ω a differential operator L satisfying the following suppositions:*

1° *L is an operator of class Λ with $n = 0$ (so it is elliptic in $\overset{1}{\Omega}$ with the coefficients depending on the parameter $y \in \overset{2}{\Omega}$);*

2° *for all $\varphi \in B$ the function $L^{+}\varphi \in B$;*

3° *L can be extended to an operator L_0 defined in the whole domain $\Omega_0 \overset{\mathrm{df}}{=} \overset{1}{\Omega}_0 \times \overset{2}{\Omega}$ is such a way that*

(a) *L_0 is of class Λ in the domain Ω_0 with $n = 0$,*

(b) *the coefficients of L_0 satisfy conditions* (C_1) *and* (C_2) (see section 1.1),

(c) *inequality* (0), *adapted to the operator L_0 and the domain Ω_0, holds for $u \in B_{p,0}$* (1).

Let v be an element of $H_{p,q}(\Omega, B)$ and suppose that $u \in \bigcup_{p \leqslant 0} H_{p,q}(\Omega, B)$ is a weak solution of equation (19) *with respect to the class B. So for each $\Phi \in C_0^\infty(\overset{1}{\Omega})$ the product Φu is in $H_{p+2m,q}(\Omega_0, B_{p,0})$.*

Proof. We suppose first $q \leqslant 0$. Let Φ_1 be a function of class $C_0^\infty(\overset{1}{\Omega})$ equal identically to 1 on a compact Δ_1 containing the support of Φ. Identity (20) with $\varphi = \Phi_1 \psi$, ψ being an arbitrary function of the class $B_{p,0}$, yields

$$(22) \qquad \big(L_0^+(\Phi_1\psi), u\big)_{(\Omega,B)} = (\Phi_1\psi, v)_{(\Omega,B)} \qquad (\psi \in B_{p,0}).$$

Applying the differential operator L_0^+ to the product $\Phi_1\psi$ we get

$$(23) \qquad L_0^+(\Phi_1\psi) = \Phi_1 L_0^+\psi - N_1\psi,$$

where

$$(24) \qquad N_1\psi = \sum_{0 \leqslant |\alpha| \leqslant 2m-1} \varphi_\alpha d_\alpha D_x^\alpha \psi$$

with $\varphi_\alpha \in C_0^\infty(\overset{1}{\Omega})$ and d_α satisfying conditions (C_1) and (C_2) (see section 1.1). From lemma 2 it follows that the products $\Phi_1 u$ and $\Phi_1 v$ are well defined; $\Phi_1 u \in H_{p_0,q}(\Omega_0, B_{p,0})$ with some sufficiently negative p_0 and $\Phi_1 v \in H_{p,q}(\Omega_0, B_{p,0})$. So from (22) and (23) we get

$$(25) \qquad (L_0^+\psi, \Phi_1 u)_{(\Omega_0,B_{p,0})} = (N_1\psi, u)_{(\Omega,B)} + (\psi, \Phi_1 v)_{(\Omega_0,B_{p,0})} \qquad (\psi \in B_{p,0}).$$

In virtue of lemmas 1 and 2 the first member on the right-hand side can be estimated as follows:

$$|(N_1\psi, u)_{(\Omega,B)}| \leqslant \|N_1\psi\|_{H_{-p_0,-q}(\Omega,B)} \|u\|_{H_{p_0,q}(\Omega,B)}$$
$$\leqslant c \sum_{0 \leqslant |\alpha| \leqslant 2m-1} \|D_x^\alpha \psi\|_{H_{-p_0,-q}(\Omega_0,B_{p,0})} \|u\|_{H_{p_0,q}(\Omega,B)}.$$

So, from inequality (1),

$$(26) \qquad |(N_1\psi, u)_{(\Omega,B)}| \leqslant c\|u\|_{H_{p_0,q}(\Omega,B)} \|\psi\|_{H_{-p_0+2m-1,-q}(\Omega_0,B_{p,0})}.$$

(1) This is certainly fulfilled, when the real parts of the coefficients $a_{\alpha\alpha}$ ($|\alpha| = m$) and b_{00} of operator L have a sufficiently large positive lower bound and when $\overset{2}{\Omega}$ satisfies all the assumptions of section 2.1 [I].

Thus from theorem 1 [I] follows the existence of an element $w_1 \in H_{p_0-2m+1,q}(\Omega_0, B_{p,0})$ such that

$$(27) \qquad (N_1\psi, u)_{(\Omega,B)} = (\psi, w_1)_{(\Omega_0,B_{p,0})} \qquad (\psi \in B_{p,0}),$$

and (25) together with (27) yields

$$(28) \qquad (L_0^+\psi, \Phi_1 u)_{(\Omega_0,B_{p,0})} = (\psi, w_1 + \Phi_1 v)_{(\Omega_0,B_{p,0})} \qquad (\psi \in B_{p,0}).$$

Applying the first part of theorem 4 we conclude from (28) that $\Phi_1 u \in H_{p_0+1,q}(\Omega_0, B_{p,0})$ (2). Let now Φ_2 be a function of class $C_0^\infty(\overset{1}{\Omega})$ vanishing on the complementary of Δ_1 and equal identically to 1 on some compact Δ_2 containing the support of Φ. Putting in (20) $\varphi = \Phi_2\psi$ ($\psi \in B_{p,0}$), we get, in the same manner as above,

$$(L_0^+\psi, \Phi_2 u)_{(\Omega_0,B_{p,0})} = (N_2\psi, u)_{(\Omega,B)} + (\psi, \Phi_2 v)_{(\Omega_0,B_{p,0})} \qquad (\psi \in B_{p,0}).$$

But

$$(N_2\psi, u)_{(\Omega,B)} = (N_2\psi, \Phi_1 u)_{(\Omega_0,B_{p,0})} \qquad (\psi \in B_{p,0})$$

and similar arguments as used above show that

$$(L_0^+\psi, \Phi_2 u)_{(\Omega_0,B_{p,0})} = (\psi, w_2 + \Phi_2 v)_{(\Omega_0,B_{p,0})} \qquad (\psi \in B_{p,0})$$

with $w_2 \in H_{p_0-2m+2,q}(\Omega_0, B_{p,0})$; thus $\Phi_2 u \in H_{p_0+2,q}(\Omega_0, B_{p,0})$.

Repeating k times the described procedure (where $k = p + 2m - p_0$) we obtain $\Phi u \in H_{p+2m,q}(\Omega_0, B_{p,0})$ when we put $\Phi = \Phi_k$. So the theorem is proved in the case $q \leqslant 0$. It can be also proved in the case $q > 0$ by the use of the same procedure, if we apply the second part of theorem 4 and notice that each strong solution of equation (19) is also a weak one.

Similar reasoning as in the proof of theorem 5, together with theorem 3 [I], yields

THEOREM 6. *Let $\overset{1}{\Omega}$ be a bounded domain of the Euclidean space E^R contained in the R-dimensional cube $\overset{1}{\Omega}_0$ and let $\overset{2}{\Omega}$ be the S-dimensional cube; we write $\Omega = \overset{1}{\Omega} \times \overset{2}{\Omega}$. Consider the class B of smooth functions in Ω satisfying the assumptions made in* [I] *and containing all the products $\Phi\psi$, where $\Phi \in C_0^\infty(\overset{1}{\Omega})$ and ψ is a function infinitely differentiable in E^N and*

(2) It can be assumed without loss of generality that $p_0 < p + 2m - 1$.

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periodic with $\Omega_0 \overset{\mathrm{df}}{=} \overset{1}{\Omega}_0 \times \overset{2}{\Omega}$ *as period-parallelogram. We suppose that each function* φ *belonging to* B *satisfies the conditions*

(C_4) *for arbitrary fixed* $x \in \overset{1}{\Omega}$ *the function* $\varphi(x, -)$ *is periodic with* $\overset{2}{\Omega}$ *being the period-parallelogram;*

(C_5) *when* φ *is prolonged on the domain* $\overset{1}{\Omega} \times E^S$ *by the condition of periodicity,* $\varphi \in C^\infty(\overset{1}{\Omega} \times E^S)$.

Let L *be a differential operator defined in* Ω *and satisfying the assumptions:*

1° L *is an operator of class* Λ *with* $n = 0$ *and with the coefficients satysfying conditions* (C_4) *and* (C_5);

2° *for all* $\varphi \in B$ *the function* $L^+\varphi \in B$;

3° L *can be extended to an operator* L_0 *defined in the whole domain* Ω_0 *in such a way that*

(a) L_0 *is of class* Λ *in the domain* Ω_0 *with* $n = 0$,

(b) *the coefficients of* L_0 *belong to* B_p,

(c) *inequality* (0), *adapted to the operator* L_0 *and the domain* Ω_0, *holds for* $u \in B_p$.

If v *is an element of* $H_{p,q}(\Omega, B)$ *and* $u \in \bigcup_{p \leqslant q} H_{p,q}(\Omega, B)$ *is a weak solution of equation* (19) *with respect to the class* B, *then for each function* $\Phi \in C_0^\infty(\overset{1}{\Omega})$ *the product* Φu *is in* $H_{p+2m,q}(\Omega_0, B_p)$.

Let u be the weak solution of equation (19), considered in the last two theorems, and let $\overset{1}{\Omega}{}'$ be an arbitrary subdomain of $\overset{1}{\Omega}$; write $\Omega' \overset{\mathrm{df}}{=} \overset{1}{\Omega}{}' \times \overset{2}{\Omega}$. If Φ is a function of class $C_0^\infty(\overset{1}{\Omega})$ equal identically to 1 on $\overset{1}{\Omega}{}'$ and $\{u_n\}$ a sequence of smooth functions approximating u in the norm $\| \|_{H_{p_0,q}(\Omega,B)}$, then

$$(\varphi, \Phi u_n) = (\varphi, u_n) \tag{29}$$

for all $\varphi \in C_0^\infty(\Omega')$ and (29) yields in the limit

$$(\varphi, \Phi u) = (\varphi, u) \quad (\varphi \in C_0^\infty(\Omega')). \tag{30}$$

From identity (30) it follows that the distributions $I(\Phi u)^d$ and Iu^d are equal in the domain Ω', and therefore have the same differentiability properties (described by theorem 1). If especially $p+2m$ and q are non-negative and sufficiently large, and $\overset{2}{\Omega}$ satisfies the assumptions of section 2.1 [I], then in view of lemma 7 [I] the distribution Iu^d considered in the domain Ω' is identical (in the distributional sense) with some function having a certain number of continuous derivatives in the classical sense.

2.5. In the present section we are going to study some more examples of boundary conditions and to prove the differentiability theorem for weak solutions of equation (19) in the case when the operator L is of a special form.

Consider a domain $\Omega = \overset{1}{\Omega} \times \overset{2}{\Omega}$, $\overset{1}{\Omega}$ being the cube defined by the inequalities $0 < x_i < 1$ $(i = 1, \ldots, R)$ and $\overset{2}{\Omega}$ being an arbitrary domain of the space E^S. Let $\overset{1}{\Omega}_0$ be the cube $-1 < x_i < 1$ $(i = 1, \ldots, R)$ and write $\Omega_0 \overset{\mathrm{df}}{=} \overset{1}{\Omega}_0 \times \overset{2}{\Omega}$. If u is a function defined almost everywhere in Ω, we put, for $x \in \overset{1}{\Omega}_0$ and $y \in \overset{2}{\Omega}$: $(\tau u)(x_1, \ldots, x_R, y_1, \ldots, y_S) \overset{\mathrm{df}}{=} u(|x_1|, \ldots, |x_R|, y, \ldots, \ldots, y_S)$. Let $B^{(1)}$ be a class of smooth functions in Ω satisfying the assumptions of the introduction, and suppose that each function $u \in B^{(1)}$ satisfies the following conditions:

(C_6) each derivative of u is, for fixed $y \in \overset{2}{\Omega}$, the continuous function of x in the closed domain $\overline{\overset{1}{\Omega}}$;

(C_7) if $\alpha = (\alpha_1, \ldots, \alpha_R)$ with $\alpha_j \neq 0$ $(1 \leqslant j \leqslant R)$, then the derivative $D_x^\alpha u$ vanishes for $x_j = 0$ or $x_j = 1$ when $0 \leqslant x_k \leqslant 1$ $(k = 1, \ldots \ldots, j-1, j+1, \ldots, R)$ and $y \in \Omega$.

Each function u belonging to $B^{(1)}$ can be extended on the domain Ω_0 by means of the operation τ, and τu is also a smooth function on account of condition (C_7); we denote by $\tau B^{(1)}$ the class of all such extended functions and consider the class $B^{(2)}$ of smooth functions in Ω_0 satisfying the assumptions of the introduction and such that $\tau B^{(1)} \subset B^{(2)}$. It can easily be verified that

$$(\tau u, \tau v)_{(\Omega_0, B^{(2)})} = 2^R (u, v)_{(\Omega, B^{(1)})} \quad (u, v \in B^{(1)}) \tag{31}$$

and

$$\|\tau u\|_{H_{p,q}(\Omega_0, B^{(2)})} \begin{cases} = 2^{R/2}\|u\|_{H_{p,q}(\Omega, B^{(1)})} & \text{for} \quad p, q \geqslant 0, \\ \geqslant 2^{R/2}\|u\|_{H_{p,q}(\Omega, B^{(1)})} & \text{for} \quad p, q \leqslant 0 \end{cases} \quad (u \in B^{(1)}). \tag{32}$$

We shall study the weak solutions of equation (19), L being an operator of class Λ defined in Ω with the coefficients satisfying conditions (C_6) and (C_7) and the following additional assumptions:

1° $a_{\alpha\alpha'\beta\beta'}$ vanish identically except the case $\alpha + \alpha'$ being of the form 2γ $(\gamma = (\gamma_1, \ldots, \gamma_R)$, γ_j non-negative integers);

2° write $\alpha + \alpha' \overset{\mathrm{df}}{=} \delta$ (δ being not of the form 2γ) and let $\delta_{i_1}, \ldots, \delta_{i_P}$ be all the odd numbers in the sequence $(\delta_1, \ldots, \delta_R)$; so $b_{\alpha\alpha'\beta\beta'}$ vanish if $x_{i_j} = 0$ or $x_{i_j} = 1$ for some j $(1 \leqslant j \leqslant P)$, when $0 \leqslant x_k \leqslant 1$ $(k = 1, \ldots, i_j-1, i_j+1, \ldots, R)$ and $y \in \overset{2}{\Omega}$.

The above assumptions are satisfied for example if L is an operator of class Λ having the form (for sufficiently differentiable u)

$$Lu = \sum_{\substack{|\gamma|=m \\ |\beta|=|\beta'|=n}} (-1)^{|\gamma|+|\beta|} D_x^\gamma D_y^\beta (a_{\gamma\gamma\beta\beta'} D_x^\gamma D_y^{\beta'} u) + \sum_{\substack{0\leqslant|\gamma|\leqslant m \\ 0\leqslant|\beta|,|\beta'|\leqslant n \\ 2|\gamma|+|\beta|+|\beta'|<2(m+n)}} (-1)^{|\gamma|+|\beta|} D_x^\gamma D_y^\beta (b_{\gamma\gamma\beta\beta'} D_x^\gamma D_y^{\beta'} u)$$

with the coefficients $a_{\gamma\gamma\beta\beta'}$ and $b_{\gamma\gamma\beta\beta'}$ belonging to $B^{(1)}$. Consider now the differential operator M defined in Ω_0 as follows:

$$Mu = \sum_{\substack{0\leqslant|\alpha|,|\alpha'|\leqslant m \\ 0\leqslant|\beta|,|\beta'|\leqslant n}} (-1)^{|\alpha|+|\beta|} D_x^\alpha D_v^\beta (c_{\alpha\alpha'\beta\beta'} D_x^{\alpha'} D_y^{\beta'} u),$$

where

$$(33)\qquad c_{\alpha\alpha'\beta\beta'} = \begin{cases} \tau a_{\alpha\alpha'\beta\beta'} & \text{for } |\alpha| = |\alpha'| = m,\ |\beta| = |\beta'| = n, \\ \tau b_{\alpha\alpha'\beta\beta'} & \text{for } |\alpha|+|\alpha'|+|\beta|+|\beta'| < 2(m+n),\quad \alpha+\alpha' = 2\gamma, \end{cases}$$

and in the remaining case, when $|\alpha|+|\alpha'|+|\beta|+|\beta'| < 2(m+n)$ but $\alpha+\alpha'$ is not of the form 2γ, we put

$$(34)\qquad c_{\alpha\alpha'\beta\beta'}(x_1, \ldots, x_R, y_1, \ldots, y_S) = \begin{cases} b_{\alpha\alpha'\beta\beta'}(|x_1|, \ldots, |x_R|, y_1, \ldots, y_S) & \text{if } x_{i_1}\cdot\ldots\cdot x_{i_P} \geqslant 0, \\ -b_{\alpha\alpha'\beta\beta'}(|x_1|, \ldots, |x_R|, y_1, \ldots, y_S) & \text{if } x_{i_1}\cdot\ldots\cdot x_{i_P} < 0. \end{cases}$$

Notice that, on account of the assumptions concerning the coefficients of L, all the functions $c_{\alpha\alpha'\beta\beta'}$ are smooth in Ω_0. From the definition of the operator M it follows also that it belongs to the class Λ in Ω_0 and is related to the operator L by the identity

$$(35)\qquad \tau L^+\varphi = M^+\tau\varphi \qquad (\varphi \in B^{(1)}).$$

Suppose now that for all $\varphi \in B^{(1)}$ also $L^+\varphi \in B^{(1)}$, and consider a weak solution u (belonging to some space $H_{p_0,q_0}(\Omega, B^{(1)})$) of equation (19) with respect to the class $B^{(1)}$, v lying in the space $H_{p,q}(\Omega, B^{(1)})$. We say that equation (19) is *weakly extensible on* Ω_0 with respect to the class $B^{(2)}$, if the following conditions are fulfilled:

1° there exists a sequence $\{u_n\}$ ($\{v_n\}$) $\subset B$ tending to u (v) with respect to the norm $\|\ \|_{H_{p_0,q_0}(\Omega,B^{(1)})}$ ($\|\ \|_{H_{p,q}(\Omega,B^{(1)})}$) such that $\{\tau u_n\}$ $\{(\tau v_n\}$) is fundamental in the norm $\|\ \|_{H_{p_0,q_0}(\Omega_0,B^{(2)})}$ ($\|\ \|_{H_{p,q}(\Omega_0,B^{(2)})}$) and for all sequences $\{u_n\}$ ($\{v_n\}$) with the above property the sequences $\{\tau u_n\}$ ($\{\tau v_n\}$) are equivalent with respect to the norm $\|\ \|_{H_{p_0,q_0}(\Omega_0,B^{(2)})}$ ($\|\ \|_{H_{p,q}(\Omega_0,B^{(2)})}$); the corresponding limit shall be denoted by $\tau_{p_0,q_0}u$ ($\tau_{p,q}v$);

2° the identity

$$(M^+\psi, \tau_{p_0,q_0}u)_{(\Omega_0,B^{(2)})} = (\psi, \tau_{p,q}v)_{(\Omega_0,B^{(2)})}$$

holds for all $\psi \in B^{(2)}$.

Condition 1° is always satisfied when p_0, q_0 and p, q are non-negative numbers, thus the right-hand side member of equation (19) and the considered weak solution are functions square-summable in Ω. From the construction of the operator M it follows, in view of (31) and (35), that

$$(M^+\tau\varphi, \tau_{p_0,q_0}u)_{(\Omega_0,B^{(2)})} = (\tau\varphi, \tau_{p,q}v)_{(\Omega_0,B^{(2)})}$$

for $\varphi \in B^{(1)}$, thus condition 2° is a little stronger than the supposition, u being the weak solution of (19) with respect to $B^{(1)}$.

Let especially the domain Ω and the class $B^{(1)}$ be chosen so that $\tau B^{(1)}$ is a subset of the class B_p or $B_{p,0}$; thus theorem 3 [I] and theorems 3 and 4 of this paper can be adapted to the study of weak solutions of equation (19) and so we get the following theorems:

THEOREM 7. *Let Ω be the N-dimensional cube and let $B^{(1)}$ satisfy conditions* (C_4) *and* (C_5). *We suppose also that the coefficients of L satisfy conditions* (C_4), (C_5) *and that inequality* (0) *holds for the operator M when $u \in B_p$* (³). *If under all the assumptions of the present section equation* (19) *is weakly extensible on Ω_0 with respect to the class B_p, then $\tau_{p_0,q_0}u$ is in $H_{p+2m,q+2n}(\Omega_0, B_p)$.*

THEOREM 8. *Suppose that all the assumptions of the present section concerning the domain Ω, the class $B^{(1)}$, and the operator L, hold and that $B^{(1)} = B^{(1)}_{-,0}$. We suppose also that inequality* (0) *holds for the operator M with $u \in B_{p,0}$, and that equation* (19) *is weakly extensible on Ω_0 with respect to the class $B_{p,0}$. Thus if $v \in H_{p,q}(\Omega, B^{(1)})$ with $p \geqslant -m$, $q = -n$, then $\tau_{p_0q_0}u$ is in $H_{p+2m,q+2n}(\Omega_0, B_{p,q})$. If especially $n = 0$, this result holds for all p, q, without any restriction.*

It is easy to show that if p_0 and q_0 are non-negative, then $\tau_{p_0,q_0}u$ is equal almost everywhere in Ω_0 to the function τu. If also $p+2m$ and $q+2n$ are non-negative and sufficiently large and $\overset{2}{\Omega}$ satisfies the assumptions of section 2.1 [I], then from lemma 7 [I] it follows that τu is equal almost everywhere in Ω_0 to some function having a certain number of continuous derivatives and thus u has the same differentiability properties in Ω.

2.6. Finally, we give an example of the boundary value problem, to which the method described in the preceding section can be applied.

(³) This supposition is certainly fulfilled when the functions $\mathrm{Re}\, b_{\alpha\alpha00}$ ($|\alpha| = m$), $\mathrm{Re}\, b_{00\beta\beta}$ ($|\beta| = n$), and $\mathrm{Re}\, b_{0000}$ have a sufficiently large lower bound.

Let $\overset{1}{\Omega}$ be the open interval $0 < x < 1$ and $\overset{2}{\Omega}$ the square defined by the inequalities $0 < y_1 < 2\pi$, $0 < y_2 < 2\pi$ (so Ω is a three-dimensional cube, $R = 1$ and $S = 2$). Let us consider in Ω the differential operator L defined as follows:

$$Lu = \frac{\partial^2}{\partial x \partial y_1}\left(a_1 \frac{\partial^2 u}{\partial x \partial y_1}\right) + \frac{\partial^2}{\partial x \partial y_2}\left(a_2 \frac{\partial^2 u}{\partial x \partial y_2}\right) - $$

$$- \frac{\partial}{\partial x}\left(b_1 \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y_1}\left(b_2 \frac{\partial u}{\partial y_1}\right) - \frac{\partial}{\partial y_2}\left(b_3 \frac{\partial u}{\partial y_2}\right) + b_4 u$$

with

$$a_1 = \big(\alpha(x)+\eta\big)(2+\sin y_1), \qquad a_2 = \big(\alpha(x)+\eta\big)(2+\sin y_2),$$

$$b_1 = \alpha(x)+c, \qquad b_2 = 1+c+\sin y_1,$$

$$b_3 = 1+c+\sin y_2, \qquad b_4 = c+i\cos y_1 \cos y_2,$$

where

$$\alpha(x) = \begin{cases} \exp\left(\dfrac{1}{x(x-1)+\varepsilon(\varepsilon-1)}\right) & \text{for} \quad \varepsilon < x < 1-\varepsilon, \\ 0 & \text{for} \quad 0 < x \leqslant \varepsilon \text{ or } 1-\varepsilon \leqslant x < 1, \end{cases}$$

η being an arbitrary positive constant, ε some positive number less than $\frac{1}{3}$ and c a positive constant exceeding the number t_1 given by lemma 10 [I]. So L is an operator of class Λ with $m = n = 1$ and satisfies inequality (0).

Let $B^{(1)}$ be the class of all infinitely differentiable functions φ defined in Ω which satisfy conditions (C_4), (C_5), and the following additional condition: there exists a compact K (depending on φ) contained in $\overset{1}{\Omega}$ such that φ is not depending on x in $(\overset{1}{\Omega}-K)\times\overset{2}{\Omega}$. It is clear that the class $B^{(1)}$ defined in this way and the coefficients of the operator L satisfy all the assumptions of section 2.5, so that theorem 7 is applicable to the boundary value problem under consideration. Let v be an arbitrary function belonging to $L^2(\Omega)$ (so $p = q = 0$), and let us consider a weak solution u of the equation

$$Lu = v$$

also belonging to $L^2(\Omega)$ and having the property that equation (19) is weakly extensible on Ω_0 (defined by the inequalities $-1 < x < 1$, $0 < y_1 < 2\pi$, $0 < y_2 < 2\pi$) with respect to the class B_p. From theorem 7 it follows that τu is in $H_{2,2}(\Omega_0, B_p)$, and lemma 7 [I] yields that τu can be identified with some function lying in $P^{1,0}(\Omega_0)$. This implies that the function u can be treated as belonging to $P^{1,0}(\Omega)$ (so u is continuous in $\overline{\Omega}$ and has the derivative of the first order with respect to x also continuous in $\overline{\Omega}$).

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