STUDIA MATHEMATICA, T. XXIII. (1963)

Spaces of continuous functions (VI)
(Localization of multiplicative linear functionals)

by

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To the memory
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Introduction

Various mathematicians (M. H. Stone, S. Kakutani, M. Krein and S. Krein, I. Gelfand and M. Naimark, and others) have established certain criteria for a space X to be equivalent (in an suitable sense) to the space $C(\Omega)$ of all real- or complex-valued functions on a compact Hausdorff space Ω. They investigated general properties of such spaces and created an important method in functional analysis. It enables us to reduce the proofs of several properties of certain Banach spaces to the case of spaces $C(\Omega)$. The space Ω, corresponding to such a Banach space X, is determined by X uniquely (up to homeomorphism). If X is an algebra satisfying certain conditions, then Ω may be defined as the set of all non-zero linear multiplicative functionals over X; if X is a Banach lattice, then real lattice homomorphisms are considered. If X is merely a Banach space, some other constructions of Ω are known (cf. e. g. [5]; in this case points of Ω are represented by some sets of elements of X or X^*). Thus, in general the definition of Ω depends on the specific structure of X. Assuming that a representation of X is established, the points of Ω may be identified with certain linear functionals (namely with those corresponding to evaluations at points of Ω), so we may always consider points of Ω as linear functionals over X. No name for elements of Ω seems to be completely satisfactory. There are many terms in use; some of them (like "non-zero real-valued homomorphism") refer to a special structure of X and may not be acceptable if we express X in terms of another theory. The term "point functional" is good, but we prefer to reserve it for functionals which are really values at some points. E. g. if X is the space $C^*(T)$ of all bounded real-valued continuous functions on a completely regular space T, then Ω may be identified with the Stone-Čech compactification βT. We have $\beta T = T \cup (\beta T \setminus T)$; functionals of Ω corresponding to points of T are considered as point functionals, but those of $\beta T \setminus T$ are not. We wish to have a short generic term to be used for convenience of reference, which does not assume any special structure of X; we introduce the term *spot functional* for any element of Ω.

Given Ω, an element x of X is represented by the function $x(\xi) = \xi(x)$ defined for $\xi \in \Omega$. This method, used first by Banach and Mazur ([1], p. 185) in their representation of abstract Banach spaces (defined axiomatically) as concrete Banach spaces (subspaces of certain spaces of continuous functions), has become one of the most important methods of functional analysis.

In case the space X is a space of functions defined on a topological space T, the following natural question arises: What are the relations between elements of Ω and points of T? The purpose of this paper [1] is an answer to this question. The basic notion is that of a localization point.

A point t of T will be called a *localization point* of a linear functional ξ over X if, for any neighborhood U of t and for any pair x, y of functions of X, the condition

$$(*) \qquad x(t) = y(t) \quad \text{for all } t \in U$$

implies $\xi(x) = \xi(y)$. If this is the case, then $\xi(x)$ depends only on the local behavior of x at t and is actually determined by the germ of x at t in the sheaf generated by X.

This definition may be introduced for any class of functions, but it seems to be particularly appropriate for spaces admitting a $C(\Omega)$ representation.

The definition of a localization point admits an obvious generalization to the case when X is a quotient space obtained from a class of functions Y by neglecting their values on sets belonging to a fixed ideal

[1] Thesis presented at Adam Mickiewicz University in Poznań on November 11, 1959.

$\mathcal{R}$ of subsets of T. $\mathcal{R}$ represents an ideal of "negligible" sets (like sets of measure 0, or sets of the first category); in this case equation $(*)$ is meant *$\mathcal{R}$-almost everywhere.* The case of a quotient space is of course more general because if $\mathcal{R}$ is the trivial ideal $\mathcal{O}$ consisting of the empty set only, then we get the preceding case.

The notion of localization is closely related to that of a generalized limit at a point t; by a *generalized limit* at t we mean here a linear functional ξ over X such that

$$\varliminf_{u\to t} x(u) \leqslant \xi(x) \leqslant \varlimsup_{u\to t} x(u)$$

holds for all $x \in X$. If we consider $X = Y/\mathcal{R}$, then both $\varliminf$ and $\varlimsup$ are *$\mathcal{R}$-essential limits* defined in a way analogous to that in measure theory.

In this paper Y is supposed to be a linear subring of the space $m(T)$ of all bounded real-valued functions on T and the constant function 1 is supposed to be in Y. The quotient space $Y/\mathcal{R}$ is a normed space with the norm

$$\|x\| = \sup_{T}{}_{\mathcal{R}} |x(t)|$$

where $\sup_{\mathcal{R}}$ denotes the *$\mathcal{R}$-essential supremum.* If Y is closed in $m(T)$ with respect to the uniform convergence, then $Y/\mathcal{R}$ is equivalent (as a Banach space, ring and lattice) to a space $C(\Omega)$.

$L(t)$ denotes the set of all spot functionals over $Y/\mathcal{R}$ which are localized at t. The First Localization Theorem states that $L(t)$ is always non-void and

$$\sup\{\xi(x) : \xi \in L(t)\} = \varlimsup_{u\to t}{}^{*}_{\mathcal{R}} x(u)$$

for all $x \in Y/\mathcal{R}$, where $\varlimsup^*_{\mathcal{R}}$ denotes the $\mathcal{R}$-essential limes superior at t in which the case $u = t$ is admitted (we assume that $u \neq t$ in the definition of $\varlimsup_{\mathcal{R}} x(u)$; for the details see 1.2).

The Second Localization Theorem states that if T is compact, then

$$\Omega = \bigcup_{t\in T} L(t)$$

which means that every spot functional has at least one point of localization. It need not be unique (e. g. if Y consists only of constant functions, then $L(t) = \Omega$ for each $t \in T$). In order that $L(t_1) \cap L(t_2) = \emptyset$ it is necessary and sufficient that the space $Y/\mathcal{R}$ separate the points t_1 and t_2 in the following sense: there exist neighborhoods U_1 and U_2 of t_1 and t_2, respectively, and a function $x \in Y/\mathcal{R}$ such that $x =_{\mathcal{R}} 0$ on U_1 and $x =_{\mathcal{R}} 1$ on U_2, where $=_{\mathcal{R}}$ means equality $\mathcal{R}$-almost everywhere. If T is compact, then every spot functional has a unique localization point if and only if $Y/\mathcal{R}$ contains $C(T)$ in the sense that every continuous function on T is $\mathcal{R}$-equivalent to some function of Y.

The Third Localization Theorem deals with criteria in order that the set $L(T) = \bigcup\{L(t) : t \in T\}$ be dense in Ω; it turns out that this depends on the following localization property of $\mathscr{R}$: If a subset A of T belongs to $\mathscr{R}$ locally at each of its points, then $A \in \mathscr{R}$.

Studying the localization of spot functionals answers similar questions concerning localization of arbitrary linear functionals over $Y/\mathscr{R}$, for every such functional ξ localized at t can be uniquely represented in the form

$$\xi(x) = \int_{L(t)} \eta(x)\mu(d\eta) \qquad (x \in Y/\mathscr{R})$$

where μ is a Radon measure on Ω; every generalized limit at t can be weakly approximated by convex combinations of spot functionals localized at this point.

Localization theorems can be used for investigation of topological properties of Ω; e. g. in some cases they yield existence or non-existence of countable dense sets.

Most of the results of the paper have been published (without proofs and under stronger assumptions) in [16] (cf. also [15]).

The author is indebted to Professors A. Alexiewicz, W. Orlicz, C. Ryll-Nardzewski and R. Sikorski for many counsels and suggestions given during preparation of the paper.

Chapter I. Quotient spaces $Y/\mathscr{R}$

1.1. Notation. We shall consider a triple $\langle T, Y, \mathscr{R}\rangle$ in which T is any topological space (we assume only that finite sets are closed), Y is is a linear class of bounded real-valued functions on T containing the constant functions and closed with respect to the finite lattice (²) operations (i. e. $x \in Y$, $y \in Y$ imply $x \vee y \in Y$ where $(x \vee y)(t) = \max[x(t), y(t)]$), and $\mathscr{R}$ is an ideal of boundary (³) subsets of T (i. e. $A \in \mathscr{R}$ and $B \subset A$ imply $B \in \mathscr{R}$; $A \in \mathscr{R}$ and $B \in \mathscr{R}$ imply $A \cup B \in \mathscr{R}$; no open non-void subset of T belongs to $\mathscr{R}$).

We will assume these conditions throughout the whole paper.

$\mathscr{R}$ will be called a *σ-ideal* if $A_n \in \mathscr{R}$ $(n = 1, 2, \ldots)$ imply $\bigcup A_n \in \mathscr{R}$.

(²) If Y is any closed subspace of $m(T)$ and contains 1, then it is a sublattice if and only if it is a subring; if it is not closed, then both notions need not coincide.

(³) Since local $\mathscr{R}$-essential properties are investigated, we have to assume that neighborhoods of points are not null set. Let us notice that if $\mathscr{R}$ is any ideal with the localization property (cf. 5.1. and [18]), then the union G of all open subsets of $\mathscr{R}$ belongs to $\mathscr{R}$ and the space $Y(\mathscr{R})$ is equivalent to $Y_1/\mathscr{R}_1$ where $Y_1 = \{y \in m(T \setminus G):$ $y = \mathrm{Rest}_{T\setminus G}\, x,\ x \in Y\}$ and $\mathscr{R}_1 = \mathrm{Rest}_{T\setminus G}\mathscr{R}$.

We shall use the following notation:

t, t_0, u, v etc. will denote points of T (we shall write occasionally t instead of $\{t\}$, e. g. $A \setminus t$ will denote the difference $A \setminus \{t\}$ of the set A and the one-point set $\{t\}$. Letters x, y, z will stand for real-valued functions on T and capital Roman letters $A, B, \ldots$ for subsets of T. $\mathrm{Rest}_A\, x$ will denote the function x restricted to A.

$\mathscr{G}$ will denote the family of all open subsets of T and $\mathscr{G}(t) = \{G \in \mathscr{G} :$ $t \in \mathscr{G}\}$ — that of neighborhoods of t.

α, β being real numbers, $\alpha \vee \beta$ will denote $\max(\alpha, \beta)$.

χ_A = the characteristic function of A;

N = the set of positive integers;

$m(E)$ = the space of all bounded real-valued functions on E;

$\mathscr{O}$ = the trivial ideal consisting of the empty set only;

$\mathrm{Rest}_A \mathscr{R} = \{A \cap B : B \in \mathscr{R}\}$ = the restriction of $\mathscr{R}$ to A;

$\Gamma(\alpha) = \{\gamma : \gamma \leqslant \alpha\}$ and $\Gamma_0(\alpha) = \{\gamma : \gamma < \alpha\}$ are sets of ordinals with the order topology; ω_α = the smallest ordinal of power $\aleph_\alpha$; $\omega = \omega_0$.

$T_{\mathscr{R}}$ will denote the set of all points t such that $\{t\} \in \mathscr{R}$, i. e. the union of all sets of $\mathscr{R}$. Further notation is introduced in next sections.

1.2. $\mathscr{R}$-essential supremum and limit. The notion of the essential supremum used in measure theory may be mutatis mutandis formulated for any ideal $\mathscr{R}$, though some care is necessary if the ideal is not a σ-ideal because some definitions equivalent for σ-ideals are not equivalent for arbitrary ideals. We begin by listing some of such admissible definitions of the *$\mathscr{R}$-essential supremum* and *infimum* over a set, denoted by $\sup_{\mathscr{R}, A} x$ or $\sup_{\mathscr{R}}\{x(t) : t \in A\}$ and $\inf_{\mathscr{R}, A} x$ or $\inf_{\mathscr{R}}\{x(t) : t \in A\}$, respectively. In a similar way we define the *$\mathscr{R}$-essential limes superior* and *limes inferior* at a point t of T. Actually, we shall use two notions: $\overline{\lim}_{\mathscr{R}}\, x(t)$ will denote such a limit if $t = u$ is excluded and $\overline{\lim}^*_{\mathscr{R}}\, x(t)$ will denote the $\mathscr{R}$-essential limes superior in which $t = u$ is admitted. If $u \in T_{\mathscr{R}}$, then both notions coincide; if $u \notin T_{\mathscr{R}}$, then

$$\overline{\lim_{t\to u}}{}^*_{\mathscr{R}}\, x(t) = \max[x(u), \overline{\lim_{t\to u}}{}_{\mathscr{R}}\, x(t)].$$

Two functions x and y will be called *equal $\mathscr{R}$-almost everywhere on a set* A and we shall write $x_A =_{\mathscr{R}} y$ if for every $\varepsilon > 0$ the set $\{t \in A : |x(t) - y(t)| \geqslant \varepsilon\}$ belongs to $\mathscr{R}$. This is not the same as to say that $\{t \in A : x(t) \neq y(t)\} \in \mathscr{R}$, unless $\mathscr{R}$ is a σ-ideal and the second notion is not appropriate. Similarly, we define $x \leqslant_{\mathscr{R}} y$ on A if $x \vee y =_{\mathscr{R}} y$ on A.

The formal definitions of the $\mathscr{R}$-essential supremum and limits are as follows: Given any $x \in m(T)$, let

$$M_\alpha(x) = \{t \in T : x(t) < \alpha\},$$

$$\sup_{A}{}_{\mathcal{R}}\, x = \inf\{a : A \cap M_a(x) \in \mathcal{R}\}, \qquad \inf_{A}{}_{\mathcal{R}} x = -[\sup_{A}{}_{\mathcal{R}}(-x)]$$

(if $A \in \mathcal{R}$, then $\sup_{A}{}_{\mathcal{R}} x = -\infty$ and $\inf_{A}{}_{\mathcal{R}} x = \infty$). Further,

$$\overline{\lim_{u\to t}}{}_{\mathcal{R}}\, x(u) = \begin{cases} x(t) & \text{if } t \text{ is isolated in } T \\ \inf\{\sup_{G\setminus t}{}_{\mathcal{R}}\, x : G \in \mathcal{G}(t)\} & \text{otherwise,} \end{cases}$$

$$\overline{\lim_{u\to t}}{}_{\mathcal{R}}\, x(u) = \begin{cases} x(t) & \text{if } t \text{ is isolated in } T, \\ \inf\{\sup_{G}{}_{\mathcal{R}} x : G \in \mathcal{G}(t) & \text{otherwise,} \end{cases}$$

$$\overline{\lim_{u\to t}}\, x(u) = \overline{\lim_{u\to t}}{}_{\mathcal{O}}\, x(u), \qquad \overline{\lim_{u\to t}}{}^{*} x(u) = \max[\overline{\lim_{u\to t}} x(u),\ x(t)],$$

$$\underline{\lim_{u\to t}}{}_{\mathcal{R}}\, x(u) = -\overline{\lim_{u\to t}}{}_{\mathcal{R}}[-x(u)], \qquad \underline{\lim_{u\to t}}{}^{*}_{\mathcal{R}}\, x(u) = -\overline{\lim_{u\to t}}{}^{*}_{\mathcal{R}}[-x(u)].$$

1.2.1. *The following relations hold for any* $A \notin \mathcal{R}$ and $x \in m(T)$:

(1) $\sup_{A}{}_{\mathcal{R}} x = \inf_{B\in\mathcal{R}} \sup_{A\setminus B} x$,

(2) $\overline{\lim_{u\to t}}{}_{\mathcal{R}}\, x(u) = \inf_{B\in\mathcal{R}} \overline{\lim_{u\to t}}\, x(u)$,

(3) $x \leqslant_{\mathcal{R}} y$ *on* A *if and only if* $\{t \in A : x(t) > y(t)+\varepsilon\} \in \mathcal{R}$ *for each* $\varepsilon > 0$,

(4) *if* x *is continuous and* G *is open, then* $\sup_{G}{}_{\mathcal{R}} x = \sup_{G} x$,

(5) $\sup_{A}{}_{\mathcal{R}} x = \inf\{\sup_{A} y : x =_{\mathcal{R}} y$ on $A\}$,

(6) $\inf_{A} x \leqslant \inf_{A}{}_{\mathcal{R}} x \leqslant \sup_{A}{}_{\mathcal{R}} x \leqslant \sup_{A} x = \sup_{A}{}_{\mathcal{O}} x$,

(7) $\sup_{A}{}_{\mathcal{R}}(x+y) \leqslant \sup_{A}{}_{\mathcal{R}} x + \sup_{A}{}_{\mathcal{R}} y$,

(8) *if* $x \leqslant_{\mathcal{R}} y$ *on* A, *then* $\sup_{A}{}_{\mathcal{R}} x \leqslant \sup_{A}{}_{\mathcal{R}} y$,

(9) *if* $B \in \mathcal{R}$, *then* $\sup_{A}{}_{\mathcal{R}} x = \sup_{A\cup B}{}_{\mathcal{R}} x = \sup_{A\setminus B}{}_{\mathcal{R}} x$,

(10) *if* $z(t) = \min[x(t), \sup_{A}{}_{\mathcal{R}} x]$, *then* $z =_{\mathcal{R}} x$ on A,

(11) *if* $\sup_{A}{}_{\mathcal{R}} x = a$, *then* $x \leqslant_{\mathcal{R}} a$ *on* A,

(12) *if* $x \geqslant_{\mathcal{R}} 0$, $y \geqslant_{\mathcal{R}} 0$, *then* $\sup_{A}{}_{\mathcal{R}}(x\vee y) = (\sup_{A}{}_{\mathcal{R}} x)\vee(\sup_{A}{}_{\mathcal{R}} y)$,

(13) *if* $x \wedge y =_{\mathcal{R}} 0$ *on* A, *then* $\sup_{A}{}_{\mathcal{R}}|x+y| = \sup_{A}{}_{\mathcal{R}}|x-y|$,

(14) *if* x *is a simple function and* $x =_{\mathcal{R}} 0$ *on* A, *then* $\{t \in A : x(t) \neq 0\} \in \mathcal{R}$,

(15) $\overline{\lim_{u\to t}}{}_{\mathcal{R}}[x(u)+y(u)] \leqslant \overline{\lim_{u\to t}}{}_{\mathcal{R}}\, x(u) + \overline{\lim_{u\to t}}{}_{\mathcal{R}}\, y(u)$,

(16) *if* $\overline{\lim_{u\to t}}{}_{\mathcal{R}}|y(u)| = 0$, *then* $\overline{\lim_{u\to t}}{}_{\mathcal{R}}[x(u)+y(u)] = \overline{\lim_{u\to t}}{}_{\mathcal{R}}\, x(u)$,

(17) *if* $x \geqslant_{\mathcal{R}} y$ *and* $y \geqslant_{\mathcal{R}} x$, *then* $x =_{\mathcal{R}} y$,

(18) *if* $x \leqslant_{\mathcal{R}} y$, $y \leqslant_{\mathcal{R}} z$, *then* $x \leqslant_{\mathcal{R}} z$.

Proof. We shall prove (2), (6) and (18) only. (18) follows from (3) and from the inclusion

$$\{t : x(t) > z(t)+\varepsilon\} \subset \{t : x(t) > y(t)+\tfrac{1}{2}\varepsilon\} \cup \{t : y(t) > z(t)+\tfrac{1}{2}\varepsilon\};$$

(6) follows from the fact that if $\inf_{A}{}_{\mathcal{R}} x > a_1 > a_2 > \sup_{A}{}_{\mathcal{R}} x$, then $\{t \in A : x(t) < a_1\} \in \mathcal{R}$ and $\{t \in A : x(t) > a_2\} \in \mathcal{R}$, whence $A \in \mathcal{R}$. In order to show (2), let us assume that t is not isolated and choose $\varepsilon > 0$. There exists $G \in \mathcal{G}(t)$ such that $\overline{\lim_{u\to t}}{}_{\mathcal{R}}\, x(u) > \sup_{\mathcal{R}}\{x(u) : u \in G,\ u \neq t\} - \varepsilon$. There exists, in turn, a $B \in \mathcal{R}$ such that $\sup_{\mathcal{R}}\{x(u) : u \in G,\ u \neq t\} > \sup\{x(u) : u \in G\setminus B,\ u \neq t\} - \varepsilon \geqslant \overline{\lim_{\substack{u\to t\\ u\notin B}}}\, x(u) - \varepsilon$ whence $\overline{\lim_{u\to t}}{}_{\mathcal{R}}\, x(u) > \inf_{B\in\mathcal{R}} \overline{\lim_{\substack{u\to t\\ u\in B}}} x(u) - 2\varepsilon$. In order to prove the converse inequality, let us choose $A \in \mathcal{R}$ such that $\inf_{B\in\mathcal{R}} \overline{\lim_{\substack{u\to t\\ u\notin B}}} x(u) > \overline{\lim_{\substack{u\to t\\ u\notin A}}} x(u) - \varepsilon$ and choose $G \in \mathcal{G}(t)$ such that $\overline{\lim_{\substack{u\to t\\ u\notin A}}}\, x(u) > \sup\{x(u) : u \in G\setminus A,\ u \neq t\} - \varepsilon$. Then $\inf_{B\in\mathcal{R}} \overline{\lim_{\substack{u\to t\\ u\notin B}}} x(u) > \sup\{x(u) : u \in G\setminus A,\ u \neq t\} - 2\varepsilon \geqslant \sup_{\mathcal{R}}\{x(u) : u \in G\setminus A,\ u \neq t\} - 2\varepsilon \geqslant \overline{\lim}_{\mathcal{R}}\, x(u) - 2\varepsilon$.

Thus, (2) is proved.

Since the inequalities above are analogous to those in the usual case (when $\mathcal{R} = \mathcal{O}$ or $\mathcal{R}$ is the ideal of sets of measure 0), we shall use them without any referring.

1.2.2. *Let* A *be compact,* $z \in m(T)$ *and* $\sup_{A}{}_{\mathcal{R}} z(t) = 1$; *then there exists* $t_0 \in A$ *such that* $\overline{\lim_{u\to t_0}}{}^{*}_{\mathcal{R}}\, x(u) = 1$.

Proof. If $\overline{\lim}{}^{*}_{\mathcal{R}}\, x$ were smaller than 1 for all $t \in A$, then, for every $u \in A$, there would exist an open set G_u and $\delta_u > 0$ such that $u \in G_u$ and $x \leqslant_{\mathcal{R}} 1-\delta_u$ on G_u. Since $\bigcup G_u \supset A$, there would exist points $u_1, \ldots, u_n$ such that $A \subset G_{u_1} \cup \ldots \cup G_{u_n}$ whence $z \leqslant_{\mathcal{R}} 1 - \min \delta_k$ and $\sup_{A}{}_{\mathcal{R}} z < 1$.

1.3. The space $Y/\mathcal{R}$. The relation $x \sim y$ *if* $x =_{\mathcal{R}} y$ *on* T is obviously symmetric, transitive and reflexive, whence it determines a quotient space $Y/\mathcal{R}$. The coset corresponding to a function $x(\cdot)$ will be denoted by $x/\mathcal{R}$ or shortly by x; conversely, if x is any coset of $Y/\mathcal{R}$, $x(\cdot)$ will denote any function belonging to the coset and $x(t)$ will denote the value of a fixed representative $x(\cdot)$ of the coset x at t. Relation $x \geqslant y$ will mean that $x(\cdot) \geqslant_{\mathcal{R}} y(\cdot)$.

The set $J = \{x \in Y : x \sim 0\}$ is a linear lattice ideal, i. e. $y \in J$, $x \in J$ imply $x+y \in J$, and $x \in J$, $|y| \leqslant |x|$ imply $y \in J$. Consequently, the quotient space $Y/\mathscr{R} = Y/J$ is a vector lattice (cf. Birkhoff [2], p. 222). Moreover, J is closed in Y with respect to the uniform convergence.

Next, let us write $\|x\| = \sup_{\mathscr{R}}\{|x(u| : u \in T\}$.

It is a pseudonorm on Y and J is exactly the set $\{y \in Y : \|y\| = 0\}$, whence $Y/\mathscr{R}$ is a normed space. Let us notice that condition (5) of 1.2.1 means that $\sup_{\mathscr{R}}|x|$ is equal to $\|x\| = \inf\{\sup_T |y(t)| : x-y \in J\}$, i. e. to the standard quotient norm in a normed linear space considered modulo a closed linear subset. If Y is closed in $m(T)$ with respect to the uniform convergence, then $Y/\mathscr{R}$ is complete, whence it is a Banach lattice satisfying the condition $\|x\vee y\| = \|x\| \vee \|y\|$ for $x \geqslant 0$, $y \geqslant 0$. The cosets $x\vee y$ and $x \wedge y$ contain the functions $\max[x(t), y(t)]$ and $\min[x(t), y(t)]$, respectively, so there is no confusion in using symbols $x \vee y$, $(x\vee y)(t)$ and so on; similarly $x \geqslant y$ if and only if $x(\cdot) \geqslant_{\mathscr{R}} y(\cdot)$.

Thus, if Y is closed in $m(T)$, $Y/\mathscr{R}$ is an M-space in the sense of Kakutani [7] and $l/\mathscr{R}$ is the unit (4); if $Y/\mathscr{R}$ is not complete, then its completion is an M-space. Similarly, if Y is a subring of $m(T)$, then J is a ring ideal in Y, $Y/\mathscr{R}$ is a ring and the coset xy corresponds to $x(\cdot)y(\cdot)$.

Ξ will denote the Banach space conjugate to $Y/\mathscr{R}$; it is a Banach lattice with the usual order: $\xi \geqslant 0$ if $\xi(x) \geqslant 0$ for all $x \geqslant 0$. $\sigma(\Xi, Y/\mathscr{R})$ will denote the *-weak topology of Ξ with neighborhoods of the form

$$V(\xi_0; x_1, \ldots, x_n, \varepsilon) = \bigcap_{k=1}^{n} \{\xi \in \Omega : |\xi_0(x_k) - \xi(x_k)| < \varepsilon\}.$$

Throughout the paper Ω or $\Omega(Y, \mathscr{R})$ will denote the set of all linear functionals on $Y/\mathscr{R}$ satisfying the following conditions:

(i) $\xi \geqslant 0$,

(ii) $\xi(1) = 1$ (*equivalently*, $\|\xi\| = 1$),

(iii) $x \wedge y = 0$ *implies* $\xi(x)\xi(y) = 0$.

It is well known that if Y is also a ring, then Ω coincides with the set of all multiplicative linear functionals on $Y/\mathscr{R}$ excluding the trivial functional 0. Ω is compact in the topology $\sigma(\Xi, Y/\mathscr{R})$; in the sequel, all topological notions concerning Ω will refer to this topology. The functionals of Ω satisfy the conditions $\xi(x\vee y) = \xi(x) \vee \xi(y)$, $\xi(x\wedge y) = \xi(x) \wedge \xi(y)$, $|\xi(x)| = \xi(|x|)$, and conversely, each of these conditions is sufficient in order that a linear functional on $Y/\mathscr{R}$ of norm 1 belong to Ω.

The set Ω does not differ if we consider the completion of the space $Y/\mathscr{R}$; indeed, all linear functionals can be uniquely extended to the completion and the *-weak topology is finer for the completion, but on the compact set Ω both topologies are equivalent.

(4) We shall denote $1/\mathscr{R}$ simply by 1 if it does not cause any confusion. If $x \in Y/\mathscr{R}$ then $0 < x < 1$ will stand for $0 <_{\mathscr{R}} x <_{\mathscr{R}} 1$ on T.

The paper is founded on the following theorem due to S. Kakutani [7] and to M. Krein and S. Krein [9]:

1.3.1. *The map $x \to x(\cdot)$, where $x(\xi) = \xi(x)$ for $\xi \in \Omega$, establishes a one-one, linear, isometric, lattice-isomorphic and ring-isomorphic correspondence between the elements of $Y/\mathscr{R}$ and elements of $C(\Omega)$ and transforms $Y/\mathscr{R}$ onto a (strongly) dense subset of $C(\Omega)$; if $Y/\mathscr{R}$ is complete, then it is mapped onto $C(\Omega)$.*

The functionals of Ω will be called *spot functionals*.

Now, we recall two well-known theorems:

1.3.2. (Stone's Continuous Image Theorem). *Given two compact Hausdorff spaces Ω_1 and Ω_2, in order that there exist a continuous map from Ω_1 onto Ω_2 it is necessary and sufficient that $C(\Omega_2)$ be linearly, isometrically and ring-isomorphically embedded in $C(\Omega_1)$ so that the unit of $C(\Omega_2)$ is mapped onto the unit of $C(\Omega_1)$. If it is so, the mapping $\Omega_1 \to \Omega_2$ may be established as the restriction of spot functionals from $C(\Omega_1)$ to $C(\Omega_2)$. The passing from the map $X_2 \to X_1$ to $\Omega_1 \to \Omega_2$ is a contravariant functor.*

1.3.3. *Given any set $\Lambda \subset \Omega$, in order that Λ be dense in Ω it is necessary and sufficient that $\sup\{\xi(x) : \xi \in \Lambda\} = \|x\|$ hold for every $x \in Y/\mathscr{R}$.*

If Y_1, Y_2 are two linear subrings of $m(T)$ and $Y_2 \subset Y_1$, then $Y_2/\mathscr{R} \subset Y_1/\mathscr{R}$ and the units in both spaces coincide. Hence, by 1.3.2, we infer

1.3.4. *If $Y_2 \subset Y_1$, then there exists a natural continuous map σ from $\Omega(Y_1, \mathscr{R})$ onto $\Omega(Y_2, \mathscr{R})$, established by restriction of spot functionals from $Y_1/\mathscr{R}$ to $Y_2/\mathscr{R}$*

Another situation occurs when we consider a fixed Y with various $\mathscr{R}$.

1.3.5. *If $\mathscr{R}_1 \subset \mathscr{R}_2$, then $\Omega(Y, \mathscr{R}_1) \underset{\text{top}}{\subset} \Omega(Y, \mathscr{R}_2)$.*

Proof. Let $X = Y/\mathscr{R}$, $J = \{x : x \in X,\ x =_{\mathscr{R}_2} 0\}$, then X/J is equivalent to $Y/\mathscr{R}_2$. Any functional $\xi \in \Omega(Y, \mathscr{R}_2)$ may be considered as a real non-zero lattice-homomorphism on X/J; let $\xi_J(x) = \xi(x+J)$ for $x \in X$, $x + J \in X/J$. Then $\xi_J \in \Omega(Y, \mathscr{R}_1)$ and the map $\xi \to \xi_J$ is one-one bicontinuous from $\Omega(Y, \mathscr{R}_2)$ onto the set $\{\eta \in \Omega(Y, \mathscr{R}_1) : z \in J \Rightarrow \eta(z) = 0\}$.

1.3.7. Evidently, the spaces Y and $Y/\mathscr{R}$ do not determine $\mathscr{R}$ uniquely, but there exists a *minimal* ideal $\mathscr{R}_Y$ such that $Y/\mathscr{R} = Y/\mathscr{R}_Y$, which means that $x =_{\mathscr{R}} y$ is equivalent to $x =_{\mathscr{R}_Y} y$ for all $x, y \in Y$ and if $Y/\mathscr{R} = Y/\mathscr{M}$, then $\mathscr{M} \supset \mathscr{R}_Y$ for any ideal $\mathscr{M}$ of boundary subsets of T. Namely, $\mathscr{R}_Y$ is the intersection of all such ideals $\mathscr{M}$.

1.4. Examples. We shall consider two typical classes of spaces $Y/\mathscr{R}$.

1.4.1. Let $\mathscr{A}$ be any algebra of subsets of T (the union and complement of any two members of $\mathscr{A}$ are assumed to belong to $\mathscr{A}$), let $\mathscr{R} \subset \mathscr{A}$ be any ideal of boundary subsets and let Y be the class of all simple functions on $\mathscr{A}$ i. e. linear combinations of characteristic functions of sets of $\mathscr{A}$. Denote $Y/\mathscr{R}$ by $A(T, \mathscr{A}, \mathscr{R})$. It is well known (cf. Kakutani [7]) that $\Omega(Y, \mathscr{R})$ *is the Stone space of the quotient Boolean algebra* $\mathscr{A}/\mathscr{R}$, i. e. Ω is a 0-dimensional compact Hausdorff space such that the algebra of all open-closed subsets of Ω is isomorphic to the algebra of cosets of the form

$$A/\mathscr{R} = \{B \in \mathscr{A} : (A\setminus B) \cup (B\setminus A) \in \mathscr{R}\}.$$

1.4.2. Let $\mathscr{R}$ be any σ-ideal of boundary subsets of T and Y be the class of bounded functions with the set of point of discontinuity belonging to $\mathscr{R}$. The spaces $\mathscr{H}(T, \mathscr{R}) = Y/\mathscr{R}$ are considered in [18]. A typical such space is the class of Riemann integrable functions considered almost everywhere and the corresponding set Ω is the Stone space of the quotient Boolean algebra of Jordan measurable subsets considered up to sets of measure 0.

1.5. Point functionals. Let $t_0 \in T\setminus T_{\mathscr{R}}$. Then condition $x =_{\mathscr{R}} y$ implies $x(t_0) = y(t_0)$. Thus, the symbol $x(t_0)$ is well defined for all $x \in Y/\mathscr{R}$, and the functional $\xi_{t_0}(x) = x(t_0)$ belongs to Ω:, such functionals will be called *point functionals*. Obviously, if $t \in T_{\mathscr{R}}$, the symbol ξ_t (as a functional on $Y/\mathscr{R}$) is meaningless.

1.5.1. *If* $\mathscr{R} = \mathcal{O}$, *then set the* Ω_0 *of all point functionals is dense in* $\Omega = \Omega(Y, \mathcal{O})$.

Indeed, $\|x\| = \sup\{|x(t)| : t \in T\} = \sup\{|\xi(x)| : \xi \in \Omega_0\}$ and we apply 1.3.3.

Now, we shall say that *the values* $x(t_0)$ *of functions* $x \in Y/\mathscr{R}$ *at a point* $t_0 \in T\setminus T_{\mathscr{R}}$ *do not depend on the behavior of* x *on a subset* A *of* T if there exist functions $x, y \in Y/\mathscr{R}$ such that $x =_{\mathscr{R}} y$ on A and $x(t_0) \neq y(t_0)$. A point $t_0 \in T\setminus T_{\mathscr{R}}$ will be called *free* (with respect to $Y/\mathscr{R}$) if the values of functions of $Y/\mathscr{R}$ at t_0 do not depend on their behavior on $T\setminus t_0$ (i. e. if there exists a function $z \in Y/\mathscr{R}$ such that $z =_{\mathscr{R}} 0$ on $T\setminus t_0$ and $z(t_0) = 1$).

1.5.2. *If* t_0 *is free, then* ξ_{t_0} *is isolated in* Ω.

Indeed, then $Y/\mathscr{R} \equiv E^1 \times Y/\mathscr{R}_1$ where $\mathscr{R}_1 = \mathscr{R} \vee \{t\}$, and we apply a theorem of S. Eilenberg ([5], p. 579). On the other hand, if ξ_t is isolated, then it need not be free (e. g. let $Y = \{x \in m(T) : x(t_1) = x(t_2)\}$, then $\xi_{t_1} = \xi_{t_2}$ is isolates and neither point is free).

Chapter II. Points of localization and generalized limits

2.1. Definitions and examples. A point t of T is said to be a *localization point* of a linear functional ξ over $Y/\mathscr{R}$ if the following condition is satisfied: If x, y are any two elements of $Y/\mathscr{R}$ and $x =_{\mathscr{R}} y$ on a neighborhood of t, then $\xi(x) = \xi(y)$. $\Xi(t)$ will denote the set of all linear functionals over $Y/\mathscr{R}$ localized at t; $L(t)$ will denote $\Xi(t) \cap \Omega$ and $L(A) = \bigcup\{L(t) : t \in A\}$.

A functional ξ over $Y/\mathscr{R}$ is said to be a *generalized limit at a point* t of T if the inequalities

$$\underline{\lim}_{u\to t}{}_{\mathscr{R}}\, x(u) \leqslant \xi(x) \leqslant \overline{\lim}_{u\to t}{}_{\mathscr{R}}\, x(u)$$

hold for every $x \in Y/\mathscr{R}$. $K(t)$ will denote the set of all generalized limits at t and $L_0(t) = K(t) \cap \Omega$. Obviously, $K(t) \subset \Xi(t)$, $L_0(t) \subset L(t)$. If $\xi \in K(t)$, then $\|\xi\| = 1$ and $\xi \geqslant 0$.

2.1.1. *The sets* $\Xi(t)$ *and* $K(t)$ *are closed in* Ξ *with respect to the *-weak topology* $\sigma(\Xi, Y/\mathscr{R})$; *consequently,* $L(t)$ *and* $L_0(t)$ *are compact subsets of* Ω. $\Xi(t)$ *is linear and* $K(t)$ *is convex for each* $t \in T$.

We omit the easy proof. The point functional ξ_t corresponding to a point $t \in T\setminus T_{\mathscr{R}}$ (cf. 1.5) always belongs to $L(t)$. It may belong to $L_0(t)$ (e. g. in the case $Y = C(T)$, T compact, $\mathscr{R} = \mathcal{O}$), but it may also belong to $L(t)\setminus L_0(t)$, e. g. if t is a free point (cf. 1.5.2).

If $T = N$ and if Y is the class of convergent sequences $\{x(n)\}$, then, for each $n \in N$, $L(n)$ consists of one point functional ξ_n; $\xi_\infty(x) = \lim x(n)$ has no localization point in N (it would have one if we considered the one-point compactification $\Gamma(\omega)$ of $N = \Gamma_0(\omega)$).

If $T = \Gamma(\omega)$, $Y = m(T)$ and if $\mathscr{R}$ consists of two sets $\{\omega\}$ and $\emptyset$, then $L(\omega)$ consists of 2^c generalized limits (cf. Čech [3], p. 831, M. H. Stone [22], Mazur [13], Kelley [8]).

2.2. Equivalent conditions. Let us write

$$I(t, a) = \bigcup_{U\in\mathscr{G}(t)} \{y ; y =_{\mathscr{R}} a \text{ on } U\},$$

$$I_0(t, a) = \bigcup_{U\in\mathscr{G}(t)} \{y : y =_{\mathscr{R}} a \text{ on } U\setminus t\}.$$

2.2.1. *Let* $\xi \in \Omega$. *Each of the following conditions is necessary and sufficient in order that* t *be a localization point of* ξ:

(1) *for each* $y \in Y/\mathscr{R}$, $y \in I(t, 0)$ *implies* $\xi(y) = 0$,
(2) *for each* $y \in Y/\mathscr{R}$, $y \in I(t, 1)$ *implies* $\xi(y) = 1$,
(3) *for each* $y \in Y/\mathscr{R}$, $y \in I(t, 0)$ *and* $0 \leqslant y \leqslant 1$ *imply* $\xi(y) = 0$,
(4) *for each* $y \in Y/\mathscr{R}$, $y \in I(t, 1)$ *and* $0 \leqslant y \leqslant 1$ *imply* $\xi(y) = 1$.

If we replace $I(t, a)$ *by* $I_0(t, a)$, *we obtain characterizations of the functionals of* $L_0(t)$.

Proof. We prove only that $(4) \Rightarrow (1)$. Suppose that $y =_{\mathscr{R}} 0$ on U and $U \in \mathscr{G}(t)$. Let $z = 1-(|y|\wedge 1)$. Then $z =_{\mathscr{R}} 1$ on U and $0 \leqslant z \leqslant 1$ whence, by (1),

$$0 = 1-\xi(z) = \xi(1)-\xi(z) = \xi(1-z) = \xi(|y|\wedge 1) = \xi(|y|)\wedge 1$$

and $0 = \xi(|y|) = |\xi(y)|$.

2.2.2. *Let $\xi \in \Xi$. Then $\xi \in \Xi(t)$ if and only if*

$$|\xi(y)| \leqslant \|\xi\| \sup_{U}{}_{\mathscr{R}} |y|$$

holds for any $y \in Y/\mathscr{R}$ and $U \in \mathscr{G}(t)$.

Proof. Necessity. Let $\xi \in \Xi(t)$, $y \in Y/\mathscr{R}$, $U \in \mathscr{G}(t)$ and let $z = \sup_{\mathscr{R}} \{|y(t)| : t \in U\}$ (z is a constant function on T). Then $z \wedge |y| = |y|$ on U, whence

$$|\xi(y)| \leqslant |\xi|(|y|) = |\xi|(z \wedge |y|) \leqslant |\xi|(z) = \|\xi\| \sup_{U}{}_{\mathscr{R}} |y|,$$

where $|\xi| = \xi \vee (-\xi)$ in the lattice Ξ.

Sufficiency. Let $y =_{\mathscr{R}} 0$ on U, then $0 \leqslant |\xi(y)| \leqslant \|\xi\| \sup_{U}{}_{\mathscr{R}} |y| = 0$.

2.2.3. *Let $\xi \in \Xi$, $\xi \geqslant 0$, $\|\xi\| = 1$. Then the following conditions are equivalent:*

(1) $\xi \in \Xi(t)$,

(2) *$U \in \mathscr{G}(t)$ and $x \leqslant_{\mathscr{R}} y$ on U imply $\xi(x) \leqslant \xi(y)$,*

(3) *$\inf_{U}{}_{\mathscr{R}} y \leqslant \xi(y) \leqslant \sup_{U}{}_{\mathscr{R}} y$ for each $y \in Y/\mathscr{R}$ and $U \in \mathscr{G}(t)$,*

(4) *$\xi(y) \leqslant \underline{\lim}^*_{\mathscr{R}\, u \to t}\, y(u)$ for each $y \in Y/\mathscr{R}$,*

(5) *$\underline{\lim}^*_{\mathscr{R}\, u \to t}\, y(u) \leqslant \xi(y) \leqslant \overline{\lim}^*_{\mathscr{R}\, u \to t}\, y(u)$ for each $y \in Y/\mathscr{R}$,*

(6) *$\lim^*_{\mathscr{R}\, u \to t} |x(u) - a| = 0$ implies $\xi(x) = a$.*

The proof is analogous to that of 2.2.2.

2.2.4. *Let $\xi \in \Xi$, $\xi \geqslant 0$, $\|\xi\| = 1$ and let t be an accumulation point of T. Then the following conditions are equivalent:*

(1) $\xi \in K(t)$,

(2) *$U \in \mathscr{G}(t)$ and $x =_{\mathscr{R}} y$ on $U \setminus t$ imply $\xi(x) = \xi(y)$,*

(3) *$U \in \mathscr{G}(t)$ and $x \geqslant_{\mathscr{R}} 0$ on $U \setminus t$ imply $\xi(x) \geqslant 0$,*

(4) *$U \in \mathscr{G}(t)$ implies $\xi(x) \leqslant \sup_{\mathscr{R}} \{x(u) : u \in U\}$.*

(5) *$\xi(x) \leqslant \overline{\lim}_{\mathscr{R}\, u \to t}\, x$ for any $x \in Y/\mathscr{R}$.*

Proof. We prove only that (2) $\Rightarrow$ (3). Let $x \geqslant_{\mathscr{R}} 0$ on $U \setminus t$ and let $z = x \vee 0$, then $z =_{\mathscr{R}} x$ on $U \setminus t$, whence $0 \leqslant \xi(z) = \xi(x)$.

2.3. First Localization Theorem. It concerns existence of localized functionals and generalized limits and relations between them.

THEOREM 1. *The sets $L(t)$ and $L_0(t)$ are not empty, and*

(i) $\sup \{\xi(x) : \xi \in L(t)\} = \overline{\lim}^*_{\mathscr{R}\, u \to t}\, x(u)$,

(ii) $\sup \{\xi(x) : \xi \in L_0(t)\} = \overline{\lim}_{\mathscr{R}\, u \to t}\, x(u)$

hold for all $x \in Y/\mathscr{R}$. Analogous statements hold for inf *and* $\underline{\lim}_{\mathscr{R}}$. *The sets $L(t)$ and $L_0(t)$ do not coincide if and only if $\overline{\lim}^*_{\mathscr{R}\, u \to t}\, x(u) \neq \overline{\lim}_{\mathscr{R}\, u \to t}\, x(u)$ for some $x \in Y/\mathscr{R}$. If $L(t) \neq L_0(t)$, then $t \notin T_{\mathscr{R}}$ and $L(t) \setminus L_0(t)$ consists of exactly one functional $\xi_t(x) = x(t)$. Thus, $L_0(t)$ is always open and closed with respect to $L(t)$.*

Proof. First we prove that $L_0(t)$ is non-empty, whence so is $L(t)$ as $L(t) \supset L_0(t)$. By 2.2.1, $L_0(t) = \bigcap \{\Phi(x) : x \in Z\}$, where $\Phi(x) = \{\xi \in \Omega : \xi(x) = 1\}$ and $Z = \{x \in Y/\mathscr{R} : 0 \leqslant x \leqslant 1\}$, $x \in I_0(t, 1)$. Since all sets $\Phi(x)$ are non-empty and compact in Ω and since $\Phi(x_1) \cap \ldots \cap \Phi(x_n) = \Phi(x_1 \wedge \ldots \wedge x_n) \neq \emptyset$, L_0 is non-empty, too. Now, let us consider the following cases:

1° $t \in T_{\mathscr{R}}$, then clearly $L(t) = L_0(t)$.

2° t is isolated in T, then also $L(t) = L_0(t)$.

3° $t \in T \setminus T_{\mathscr{R}}$ and is not isolated. Let $\mathscr{R}_1$ be the ideal generated by $\mathscr{R}$ and t. Let $P(x) = x/\mathscr{R}_1$ for $x \in Y/\mathscr{R}$ (to each coset $x(\cdot)/\mathscr{R}$ we assign the coset $x(\cdot)/\mathscr{R}_1$), let $Q(x) = (P(x), x(t)) \in Y/\mathscr{R}_1 \times E^1$. Then $P : Y/\mathscr{R} \to Y/\mathscr{R}_1$, is onto and $Q : Y/\mathscr{R} \to Y/\mathscr{R}_1 \times E^1$ need not be onto (it depends on the behavior of functions of Y; cf. 1.5.2). Thus, the adjoint P^* maps the conjugate of $Y/\mathscr{R}_1$ into Ξ and maps the set $L_1(t)$ of all spot functionals over $Y/\mathscr{R}_1$ localized at t onto $L_0(t)$. Since P is onto, the deficiency of $Q(Y/\mathscr{R})$ in $Y/\mathscr{R}_1 \times E^1$ is at most one (because Q composed with the natural projection π of $Y/\mathscr{R}_1 \times E^1$ on $Y/\mathscr{R}_1$ is onto), hence the image of the conjugate of $Y/\mathscr{R}_1$ under Q^* has also deficiency at most one. Since $P = \pi Q$, $P^* = Q^* \pi^*$, whence the only functional in $L(t)$ which is not in the image of $L_1(t)$ is ξ_t.

The proof of equation (ii) will consist of a number of steps; (i) is a trivial consequence of (ii) and of the foregoing argument.

Let $Z_1 = \{x \in Y/\mathscr{R} : 0 \leqslant x \leqslant 1, \overline{\lim}_{\mathscr{R}\, u \to t}\, x(u) = 1\}$. We shall prove that for each $y \in Z_1$ there exists $\xi_0 \in L_0(t)$ such that $\xi_0(y) = 1$. Fix $y_0 \in Z_1$. Let $\mathscr{M}$ be the family of all sets Z contained in Z_1 with the following properties: $y_0 \in Z$, and if $z_1, \ldots, z_n \in Z$, then $z_1 \wedge \ldots \wedge z_n \in Z$. $\mathscr{M}$ is non-empty because the one-point set $\{y_0\}$ belongs to $\mathscr{M}$. Standard application of the maximum principle ([5]) yields existence of a set Z_0 maximal in $\mathscr{M}$, i. e. such that $Z_0 \in \mathscr{M}$ and conditions $Z \subset Z_0$ and $Z \in \mathscr{M}$ imply $Z = Z_0$. The family of sets $\Phi(x)$ with $x \in Z_0$ has the finite intersection property whence there exists ξ_0 in all sets $\Phi(x)$, $x \in Z_0$, simultaneously.

We now prove that $\xi_0 \in L_0(t)$. Let $U \in \mathscr{G}(t)$, $z \in Y/\mathscr{R}$, $0 \leqslant z \leqslant 1$ and $z =_{\mathscr{R}} 1$ on $U \setminus t$. Let $Z_2 = Z_0 \cup \{z\} \cup \{z \wedge x : x \in Z_0\}$. Clearly, if $x \in Z_0$, then $z \wedge y_0 \wedge x \in Z_1$, whence $Z_2 \subset Z_1$. Since the minimum of any finite family of elements of Z_2 belongs to Z_2, Z_2 belongs to $\mathscr{M}$. Since Z_0 is maximal, Z_2 and Z_0 coincide. Thus, we have shown that $z \in Z_0$ whence $\xi_0 \in \Phi(z)$, i. e. $\xi_0(z) = 1$. By 2.2.1, $\xi_0 \in L_0(t)$.

([5]) Cf. Hausdorff [6], p. 140, and Kuratowski [11].

In the sequel $\overline{\lim}_{\mathscr{R}}$ will refer to our fixed point t.

Suppose that x is any element of $Y/\mathscr{R}$ and $a = \sup\{\xi(x) : \xi \in L_0(t)\}$. We are going to show that $a = \overline{\lim}_{\mathscr{R}}\, x(u)$. The inequality $a \leqslant \overline{\lim}_{\mathscr{R}}\, x(u)$ is obvious, and we suppose that the equality did not occur. Let $y = x \wedge a$; then $\overline{\lim}_{\mathscr{R}}\, y = a$. Evidently, if we prove that $\overline{\lim}_{\mathscr{R}} |x-y| = 0$, then we shall be through, so suppose that $\beta = \overline{\lim}_{\mathscr{R}} |x-y| > 0$ and consider $z = |\beta^{-1}(x-y)| \wedge 1$. Since $0 \leqslant z \leqslant 1$ and $\overline{\lim}_{\mathscr{R}}\, z = 1$, by the preceding part of the proof, there exists $\xi_0 \in L_0(t)$ such that $\xi_0(z) = 1$. This yields a contradiction

$$1 = \xi_0(z) = [\beta^{-1}\xi_0(|x-y|)] \wedge 1 = [\beta^{-1}|\xi_0(x) - \xi_0(y)|] \wedge 1 = 0,$$

as $\xi_0(y) = \xi_0(x) \wedge a = \xi_0(x)$, and this ends the proof of Theorem 1.

It yields existence theorems for generalized limits (in the case $\mathscr{R} = \mathcal{O}$ such theorems have been proved by Mazur [13] and Sikorski [19]).

2.3.1. *Let $A \subset G \subset T$ and let G be open. If ξ belongs to the closure (in Ω) of $L(A)$, if $z \in Y/\mathscr{R}$ and if $z \leqslant_{\mathscr{R}} a$ on G, then $\xi(z) \leqslant a$.*

We omit the proof.

2.3.2. *Let $\xi \in \Omega$. Then $\xi \in L(t)$ if and only if the conditions $z \in Y/\mathscr{R}$, $0 \leqslant z \leqslant 1$ and $\xi(z) = 1$ imply $\overline{\lim}^*_{\mathscr{R}\, u\to t}\, z(u) = 1$.*

Proof. Necessity follows from Theorem 1. In order to prove the sufficiency, assume that ξ satisfies this condition and $x \in Y/\mathscr{R}$, $0 \leqslant x \leqslant 1$ and $x =_{\mathscr{R}} 0$ on U, $U \in \mathscr{G}(t)$. If $\xi(x)$ were positive, we would consider $y = [(\xi x)^{-1}x] \wedge 1$ and get $0 \leqslant y \leqslant 1$ and $\xi(y) = 1$, whence $\overline{\lim}^*_{\mathscr{R}}\, y(u) = 1$ and $\overline{\lim}^*_{\mathscr{R}}\, x(u) \geqslant \xi(x) > 0$ contradicting the choice of x.

Some applications of Theorem 1 are shown in [18]; e. g. it yields easily the proof of existence of a countable dense set in the Stone space of the Boolean algebra of Borel subsets of $[0, 1]$ modulo sets of the first category.

Chapter III. Uniqueness of a localization point

3.1. Separation of T by $Y/\mathscr{R}$. Uniqueness problem for the point of localization can be stated in the following way: What are the necessary and sufficient conditions in order that $L(u) \cap L(v) = 0$ for $u \neq v$? We shall also consider a related problem: When does $L(u) \neq L(v)$ hold? In both cases some separation properties are involved.

The space $Y/\mathscr{R}$ will be said to *separate points u and v* of T if there exists z in $Y/\mathscr{R}$ such that

$$(**)\qquad \overline{\lim}^*_{\mathscr{R}\, t\to u}\, x(t) < \underline{\lim}^*_{\mathscr{R}\, t\to v}\, x(t).$$

In the case e. g. when $u \notin T_{\mathscr{R}}$ and $v \in T_{\mathscr{R}}$ this means existence of $y \in Y/\mathscr{R}$ such that

$$\overline{\lim}_{\mathscr{R}\, t\to u}\, y(t) < \underline{\lim}_{\mathscr{R}\, t\to v}\, y(t) \leqslant y(v)$$

and so on. $Y/\mathscr{R}$ is said to *separate T* if it separates each pair of distinct points.

3.1.1. *$Y/\mathscr{R}$ separates u and v if and only if there exist $z \in Y/\mathscr{R}$, $U \in \mathscr{G}(u)$ and $V \in \mathscr{G}(v)$ such that $0 \leqslant z \leqslant 1$, $z =_{\mathscr{R}} 0$ on U and $z =_{\mathscr{R}} 1$ on V.*

Proof. Sufficiency being obvious, let us assume (**). Let $c_1 = \underline{\lim}^*_{\mathscr{R}\, t\to v}\, x(t)$, $c_2 = \overline{\lim}^*_{\mathscr{R}\, t\to u}\, x(t)$ and $b = \frac{1}{3}(c_1 - c_2)$. Then, by the definition of $\overline{\lim}_{\mathscr{R}}$ and $\underline{\lim}_{\mathscr{R}}$, there exist neighborhoods U and V of u and v, respectively, such that $x \geqslant_{\mathscr{R}} c_1 - b$ on V and $x \leqslant_{\mathscr{R}} c_2 + b$ on U. Let $y = b^{-1}(x - c_2 + b)$ and $z = (1 \wedge y) \vee 0$. Then $y \leqslant_{\mathscr{R}} 0$ on U, $y \geqslant_{\mathscr{R}} 1$ on V, $0 \leqslant z \leqslant 1$ and $z =_{\mathscr{R}} 0$ on U and $z =_{\mathscr{R}} 0$ on V.

3.1.2. *Given u and v in T, $L(u) \cap L(v) = \emptyset$ if and only if $Y/\mathscr{R}$ separates u and v.*

Proof. Sufficiency follows from 3.1.1; assume that $L(u) \cap L(v) = \emptyset$. By 2.1.1, there exists a continuous function f on Ω such that $0 \leqslant f \leqslant 1$, $f = 0$ on $L(u)$ and $f = 1$ on $L(v)$. By 1.3.1, f corresponds to some x in $Y/\mathscr{R}$ and $\xi(x) = 0$ for $\xi \in L(u)$ and $\xi(x) = 1$ for $\xi \in L(v)$, whence $Y/\mathscr{R}$ separates u and v by Theorem 1.

3.1.3. *If $Y/\mathscr{R}$ separates T, then T is a Hausdorff space.*

3.1.4. *If $\mathscr{R}_1 \supset \mathscr{R}_2$ and $Y/\mathscr{R}_1$ separates T, then so does $Y/\mathscr{R}_2$. If $Y_1 \subset Y_2$ and $Y_1/\mathscr{R}$ separates T, then so does $Y_2/\mathscr{R}$.*

The proofs of 3.1.3 and 3.1.4 are trivial.

3.2. Partial separation. $Y/\mathscr{R}$ is said to *separate partially u and v* if there exists $x \in Y/\mathscr{R}$ such that

$$\overline{\lim}^*_{\mathscr{R}\, t\to u}\, x(t) \neq \overline{\lim}^*_{\mathscr{R}\, t\to v}\, x(t).$$

If $u \notin T_{\mathscr{R}}$ and $v \notin T_{\mathscr{R}}$, this separation may be obtained by different values at u and v, or by different values of $\overline{\lim}_{\mathscr{R}}$ etc.

3.2.1. *$L(u) \neq L(v)$ if and only if $Y/\mathscr{R}$ separates partially u and v.*

We omit the easy proof.

Similar conditions may be stated for uniqueness of $L_0(t)$.

3.3. Examples. If Y contains only the constant functions on T, we have no separation and $L(t) = \Omega$ for all $t \in T$ (Ω has one element only).

If T is completely regular, then $C^*(T)$ separates T; if T is a Hausdorff space, then $m(T)$ separates T.

3.3.1. Let $T = [0, 1]$, let Y be the class of all functions such that $\{t : x(t) \neq 0\}$ is countable and let $Y = \{x + \text{const} : x \in nY\}$. Then $Y = Y/\emptyset$ does not separate any pair, but it does separate all pairs partially.

3.3.2. Let $T_1 = \Gamma_\omega$ and $T_2 = \Gamma_{\omega\cdot 2}$ and let $\varphi : T_1 \to T_2$ be defined as follows: $\varphi(\omega) = \omega$, $\varphi(0) = \omega\cdot 2$, $\varphi(2m) = m-1$ for $1 \leqslant m < \omega$ and $\varphi(2m-1) = \omega + m$. Let $Y = \{x[\varphi(t)] : x \in C(T_2)\}$. If $y \in Y \subset C(T_1)$ and $y = 0$ in a neighborhood of ω, then $y(0) = 0$ as well, so the points 0 and ω are not separated by $Y = Y/\emptyset$, but they are partially separated and $L(0) \subset L(\omega)$.

3.3.3. Let $T = [0, 1]$, let Y be the class of all functions on T which are uniformly continuous on the open interval $(0, 1)$ and such that $x(0) = \lim_{t\to 1} x(t)$, $x(1) = \lim_{t \to 0} x(t)$.

Let $\mathscr{R} = \emptyset$. Then $L(0) = L(1)$ and both sets consist exactly of two functionals $\xi_0(x) = x(0)$ and $\xi_1(x) = x(1)$. Thus, existence of $x \in Y/\mathscr{R}$ such that $x(u) \neq x(v)$ does not imply the partial separation.

3.3.4. Modify example 3.3.3 assuming that $\mathscr{R}$ is spanned on 0 and 1. Then $Y/\mathscr{R}$ separates T. This shows that in the definition of separation we may not require that there exists a representative $x(\cdot) \in Y$ of a coset x such that $x(u) \leqslant \overline{\lim}_{\mathscr{R}, t\to u}\, x(t) < \underline{\lim}_{\mathscr{R}, t\to v}\, x(t) \leqslant x(v)$.

3.3.5. There exist $\langle T, Y, \mathscr{R}\rangle$ and t such that t is isolated in T, $Y/\mathscr{R}$ separates T, $L(t) = \{\xi_t\}$ and ξ_t is not isolated in Ω.

Indeed, let T be $\Gamma_0(\omega_1)$ together with an isolated point t, let $\mathscr{R} = \emptyset$ and Y be the class of all functions continuous on T with $x(t) = \lim_{a\to\omega_1} x(a)$.

3.4. Continuity of the localization point. If $Y/\mathscr{R}$ separates T, then each $\xi \in \Omega$ has at most one localization point (by 3.1.2); let us denote it by $l(\xi)$. Thus, it is a function defined on a subset of Ω and is onto T, and $L(B) = l^{-1}(B)$ for each $B \subset T$.

3.4.1. *If T is completely regular and $C^*(T) \subset Y$, then $l\,(T) \to T$ is continuous* (l is well defined by preceding remarks).

Proof. We have to show that $L(F)$ is closed in $L(T)$ whenever F is closed in T (of course, $L(T)$ need not be closed in Ω). Let $\xi \in L(T) \setminus L(F)$. Then $t = l(\xi) \in T \setminus F$, whence there exists $z \in C^*(T)$ such that $0 \leqslant z \leqslant 1$, $z = 1$ on a neighborhood U of t and $z = 0$ on an open set G containing F. Hence $\xi(z) = 1$ and $\eta(z) = 0$ for $\eta \in L(F)$, whence ξ is not in the *-weak closure of $L(F)$.

3.4.2. *Suppose that Y_1 and Y_2 are two subrings of $m(T)$ and $Y_2 \subset Y_1$, and let σ: $\Omega(Y_1, \mathscr{R}) \to \Omega(Y_2, \mathscr{R})$ be as in 1.3.4. If t is a localization point*

of a functional $\xi \in \Omega(Y_1 \mathscr{R})$; then it is a localization point of $\sigma(\xi)$. If T is completely regular and $C^*(T) \subset Y_2$, then the converse is also true.

Proof. First part is obvious. In order to prove the converse, let us assume that t is the localization point of $\sigma(\xi)$, $x \in Y_2/\mathscr{R}$, $0 \leqslant x \leqslant 1$, $U \in \mathscr{G}(t)$ and $x =_{\mathscr{R}} 0$ on U. There exists $y \in C^*(T)$ such that $y = 1$ on $T \setminus U$ and $y = 0$ on V for some $V \in \mathscr{G}(t)$. Hence $y \geqslant_{\mathscr{R}} x$ on T and $y \in Y_2$ whence

$$0 \leqslant \xi(x) \leqslant \xi(y) = (\sigma\xi)(y) = 0,$$

and we apply 2.2.1.

Chapter IV. Localization when T is compact

4.1. Second Localization Theorem. We shall prove that if T is compact (not necessarily Hausdorff), then $L(T) = \Omega$ which means that every spot functional on $Y/\mathscr{R}$ has at least one localization point. If $T = T_{\mathscr{R}}$, then this means that any spot functional is a generalized limit at a point. If $Y/\mathscr{R}$ contains enough continuous functions, then the converse theorem is also true, i. e. $\Omega = L(T)$ implies the compactness of T.

Before the formulation of the Second Localization Theorem we prove the main part of it in a very simple and suggestive case when we consider the space L_∞ of essentially bounded measurable functions.

Let $T = [0, 1]$, let $\mathscr{R}$ be the ideal of all sets of Lebesgue measure 0 and let Y be the class of bounded measurable functions. Then $Y/\mathscr{R} = L_\infty$. For any $\xi \in \Omega$ consider two intervals $\Delta_1 = [0, \frac{1}{2}]$ and $\Delta_2 = [\frac{1}{2}, 1]$ and their characteristic functions x_1 and x_2. Since $x_1 + x_2 = 1$ and $x_1 \cdot x_2 = 0$ (almost everywhere), $\xi(x_1) + \xi(x_2) = 1$ and $\xi(x_1)\xi(x_2) = 0$. Hence either $\xi(x_1) = 0$ and $\xi(x_2) = 1$ or $\xi(x_1) = 1$ and $\xi(x_2) = 0$. Consider this interval for which ξ is equal to 1 and continue the argument. We get a decreasing sequence of closed intervals $\Delta^{(1)}, \Delta^{(2)}, \ldots$ such that $\xi(\chi_{\Delta^{(n)}}) = 1$. It is easy to check that the common point of these intervals is the localization point of ξ.

THEOREM 2. *Let T be compact. Then*:

(i) *$L(T) = \Omega$, i. e. each $\xi \in \Omega$ has at least one localization point*;

(ii) *If $Y/\mathscr{R}$ is complete and separates T, then it contains $C(T)$ in the following sense: For each function z continuous on T there exists $y \in Y$ such that $z =_{\mathscr{R}} y$ on T*;

(iii) *If $Y/\mathscr{R}$ separates T, then the map l* (cf. 3.4) *is a continuous map of Ω onto T corresponding in a natural way to Stone's continuous map of the structure space of $Y/\mathscr{R}$ onto that of $C(T)$.*

We shall prove this theorem in a number of steps.

4.1.1. *If $F \subset T$ and F is compact, so is $L(F)$.*

Proof. Let F be compact; we have to show that $\overline{L(F)}$ is closed in Ω. Let $\xi \in \overline{L(F)}$, $Z = \{z \in Y/\mathscr{R} : 0 \leqslant z \leqslant 1,\ \xi(z) = 1\}$ and $A(z) = \{t \in F : \overline{\lim}^*_{\mathscr{R}\, u\to t} z(u) = 1\}$. By upper semicontinuity of $\overline{\lim}^*_{\mathscr{R}}$, all sets $A(z)$ are closed for $z \in Z$. We shall show that $A(z) \neq \emptyset$ for $z \in Z$. If some $A(z)$ were empty, then for each $t \in F$ there would exist $\varepsilon_t > 0$ and a neighborhood U_t of t such that $z \leqslant_{\mathscr{R}} 1-\varepsilon_t$ on U_t. Let $U_{t_1}, \ldots, U_{t_n}$ be a finite subcovering of F; then $z \leqslant_{\mathscr{R}} 1-\varepsilon$ on $G = U_{t_1} \cup \ldots \cup U_{t_n}$, where $\varepsilon = \min \varepsilon_{t_i}$, whence $\xi(z) \leqslant 1-\varepsilon$ by 2.3.1. This contradicts $z \in Z$. Hence $A(z) \neq \emptyset$ for any $z \in Z$. Since $A(z_1) \cap \ldots \cap A(z_n) \supset A(z_1 \wedge \ldots \wedge z_n)$ and $z_1 \wedge \ldots \wedge z_n \in Z$ whenever $z_1 \in Z, \ldots, z_n \in Z$, the family $\{A(z)\}, z \in Z$, has a non-empty intersection. By 2.3.2, any point of this intersection is a localization point of ξ, whence $\xi \in L(F)$. This concludes the proof of 4.1.1.

Now, if T is compact, then by 1.2.2 and by Theorem 1, $L(T)$ is dense in Ω, whence $L(T) = \Omega$ by 4.1.1; thus, condition (i) of Theorem 2 has been proved. The other two will be proved in 4.2.

4.1.2. *Let T be completely regular and let $C^*(T) \subset Y$. If $L(T) = \Omega$, then T is compact.*

This is a trivial consequence of 3.4.1. Of course, the condition $L(T) = \Omega$ itself does not imply the compactness of T (e. g. if Y consists of constant functions only).

4.2. Approximation of continuous functions by functions of $Y/\mathscr{R}$. If T is completely regular and $C^*(T) \subset Y$, then $Y/\mathscr{R}$ separates T; the converse theorem is obviously false (e. g. T locally compact and Y consisting of functions vanishing at infinity). We shall show, however, that if T is compact, then this converse is true in a certain sense, stated in condition (ii) of Theorem 2. It is a generalization of the Stone-Weierstrass theorem which follows from (ii) by substituting $\mathscr{R} = \emptyset$.

4.2.1. *Let T be compact and $Y/\mathscr{R}$ separate T. For each pair F_1 and F_2 of disjoint closed subsets of T there exists $z \in Y$ such that $0 \leqslant z \leqslant 1$, $z =_{\mathscr{R}} 0$ on F_1 and $z =_{\mathscr{R}} 1$ on F_2.*

Proof. Fix $v \in F_2$. For each $u \in F_1$ there exist (by 3.1.1) open neighborhoods U_t and V_t of u and v, respectively, and $z_t \in Y$ such that $0 \leqslant z_t(w) \leqslant 1$ for all $w \in T$, $z_t =_{\mathscr{R}} 0$ on U_t and $z_t =_{\mathscr{R}} 1$ on V_t. Take a finite covering $U_{t_1}, \ldots, U_{t_n}$ of F_1 and consider $V = V_{t_1} \cap \ldots \cap V_{t_n}$ and $y = z_{t_1} \wedge \ldots \wedge z_{t_n}$. Then $V \in \mathscr{G}(v)$, $y \in Y$, $y =_{\mathscr{R}} 0$ on F_1 and $y =_{\mathscr{R}} 1$ on V. Let $y_v = y$.

Now, consider all possible $v \in F_2$; a similar argument yields our lemma.

From lemma 4.2.1 we deduce condition (ii) applying classical arguments and completeness of $Y/\mathscr{R}$.

4.2.2. *There exist a compact space T and $Y/\mathscr{R}$ separating T such that $C(T)$ is not contained in Y.*

In other words, in condition (ii) of Theorem 2 we cannot replace $z =_{\mathscr{R}} y$ by $z = y$, cf. example 3.3.4.

Condition (ii) of Theorem 2 means that each $z \in C(T)$ determines a coset of $Y/\mathscr{R}$ and in this sense we have a natural embedding of $C(T)$ into $Y/\mathscr{R}$ which is a linear isometry, a ring and lattice isomorphism and transforms 1 of $C(T)$ onto 1 of $Y/\mathscr{R}$. Thus, by 1.3.2, there exists a continuous map $\varphi \colon \Omega \to T$ defined as follows: Take $\xi \in \Omega$; then $\xi_0 = \mathrm{Rest}_{C(T)} \xi$ is a non-zero multiplicative linear functional on $C(T)$ whence $\xi_0(x) = x(t)$ for all $x \in C(T)$ and a certain $t \in T$; define $t = \varphi(\xi)$. Let $l(\xi)$ be the localization point of ξ. By 3.4.2 applied to the subrings $Y_1 = Y$ and $Y_2 = C(T)$ (cf. 3.1.3), we infer that $\varphi(\xi) = l(\xi)$ and this concludes the proof of Theorem 2.

By condition (iii) of Theorem 2, the decomposition

$$\Omega = \bigcup_{t \in T} L(t)$$

is semicontinuous. It need not be continuous, as the following example shows: Let $T = \{e^{i\varphi} : 0 \leqslant \varphi < 2\pi\}$, let Y be the class of all functions y on T which are continuous at $z \neq 1$ and have one-side limits at $z = 1$, let $\mathscr{R}$ be the ideal of subsets of $\{1\}$. Then Ω is homeomorphic to $[0, 1]$ and $l \colon \Omega \to T$ is the identification of the ends 0 and 1; since l is not open, the decomposition $\Omega = \bigcup L(t)$ is not continuous.

4.3. Generalized compactification. Given a triple $\langle T, Y, \mathscr{R}\rangle$, we may ask if it can be embedded into another one $\langle T_1, Y_1, \mathscr{R}_1\rangle$ such that T_1 is a compactification of T and $Y/\mathscr{R}$ is equivalent to $Y_1/\mathscr{R}_1$ in a natural way; by Theorem 2 every spot functional over $Y_1/\mathscr{R}_1$ woud have a localization point in T_1, whence every spot functional over $Y/\mathscr{R}$ would be considered as localized at a point of T_1 (but not necessarily at a point of T). The following construction generalizes the Stone-Čech one.

4.3.1. Let T be completely regular and $C^*(T) \subset Y$. Let $T_1 = \beta T$ and

$$x_1(t) = \begin{cases} x(t) & \text{for} \quad t \in T, \\ x(t_0) & \text{for} \quad t \in T_1 \setminus T, \end{cases}$$

where t_0 is any fixed point of T. Then the class $Y_1 = \{x_1 : x \in Y\}$ and the ideal $\mathscr{R}_1$ of all sets of the form $R \cup A$ with $R \in \mathscr{R}$ and $A \subset T_1 \setminus T$ satisfy conditions of 1.1. Clearly, $Y/\mathscr{R}$ and $Y_1/\mathscr{R}_1$ are equivalent in a natural way, $Y_1/\mathscr{R}_1$ separates T_1 and each spot functional on $Y/\mathscr{R}$ corresponds to a unique spot functional over $Y_1/\mathscr{R}_1$ which has a unique localization point in T_1.

If Y does not contain all functions of $C^*(T)$ but if $Y \cap C^*(T)$ separates points from closed sets in T, then we may consider another compactification as T_1. E. g. if T is locally compact we may consider any functional of $\Omega\setminus L(T)$ as localized at infinity. For related questions, cf. Edwards [4].

4.3.2. *Let $T = \bigcup_{n=1}^{\infty} T_n$ be completely regular, T_n be compact and let $C^*(T)\subset Y$. A spot functional ξ over $Y/\mathscr{R}$ belongs to $L(T)$ if and only if the conditions $x_n \in C^*(T)$, $\|x_n\| \leqslant 1$ and $\sup\{|x_n(t)| : t \in T_k\} \to 0$ as $n\to 0$ $(k = 1, 2, \ldots)$ imply $\xi(x_n)\to 0$.*

Proof. Necessity. If t is the localization point of ξ and $t\in T$, then $\xi(x) = x(t)$ for all $x\in C^*(T)$ (by Theorem 1), whence we infer our condition.

Sufficiency. If the condition above is satisfied, then there exists a Radon measure μ on βT concentrated on T such that $\xi(x) = \int x d\mu$ for all $x \in C^*(T)$ (cf. e. g. [17]); at the same time, ξ being a spot functional, $\xi(x) = x(t)$ for some $t \in \beta T$ whence uniqueness of μ yields $t\in T$.

4.4. Application to Boolean algebras. We shall show that Theorem 2 yields a proof that the Stone space of the Boolean algebra of Lebesgue measurable subsets of $[0,1]$ modulo sets of measure zero does not contain a countable dense set.

4.4.1. *Let T be an uncountable compact Hausdorff space and μ an atomless positive Radon measure on T which does not vanish on any open non-void set. Let A be the algebra of Borel sets and $\mathscr{R}$ be the ideal of sets of measure 0. Then no countable set is dense in $\Omega = \Omega[A(T, \mathscr{A}, \mathscr{R})]$* (cf. 1.4.1).

Proof. Assume that $\mu(T) = 1$ and $\eta_1, \eta_2, \ldots$ belong to Ω. We are going to show that this sequence is not dense in Ω. For each n there exists t_n such that $\eta_n \in L(t_n)$ and there exists $G_n \in \mathscr{G}(t_n)$ such that $\mu(G_n) < 2^{-n-1}$. Let x be the characteristic function of $E = T\setminus\bigcup_{n=1}^{\infty} G_n$. Thus

$$\mu(E) = 1-\mu(\bigcup G_n) \geqslant 1-\sum\mu(G_n) > \tfrac{1}{2},$$

whence $\|x\| = 1$. Since $x = 0$ on each G_n, $\eta_n(x) = 0$ for all n. By 1.3.3 η_n is not dense in Ω.

4.5. Applications to investigation of Ω. In most important cases, if $Y/\mathscr{R}$ is not separable, then the power of Ω is greater than that of continuum and the topological structure of Ω is fairly complicated. In some cases Theorem 2 helps us to establish properties of Ω. Let us consider some examples.

4.5.1. Let $T = [0,1]$, let Y be the class of bounded functions with one-side limits at each point of T and let $\mathscr{R} = \emptyset$. The corresponding set Ω can be decomposed into $\Omega = \bigcup\{L(t) : t\in T\}$ and each set $L(t)$ consists of three functionals

$$\xi_t(x) = x(t), \qquad \eta_t(x) = x(t+), \qquad \zeta_t(x) = x(t-)$$

whenever $0 < t < 1$ and of two functionals if $t = 0$ or $t = 1$. Hence Ω is homeomorphic to the ordered set $2+3\lambda+2$ (where λ is the order type of the real line and multiplication denotes the lexicographical product) with the order topology. Ω is a compact non-metrisable Hausdorff space with the first axiom of countability.

If $\mathscr{R}_1$ is the ideal of finite sets, then an analogous argument shows that $\Omega(Y/\mathscr{R}_1)$ is homeomorphic to $1+2\lambda+1$ (space obtained from $[0,1]$ by "splitting each point t into two halves t_+ and t_-"). If $\mathscr{R}_2$ is the ideal of sets of measure 0, then $Y/\mathscr{R}_1 = Y/\mathscr{R}_2$ and we may consider $Y/\mathscr{R}_2$ as a subring of $Z/\mathscr{R}_2 = L_\infty$ where Z is the class of bounded measurable functions. Clearly the set $L(t)$ of all functionals of $\Omega(Z/\mathscr{R}_2)$ is split into two parts $L(t+)$ and $L(t-)$ obtained as the counter-images $\varphi(t+)$ and $\varphi(t-)$ where φ is the natural map of $\Omega(Z/\mathscr{R}_2)$ onto $\Omega(Y/\mathscr{R}_2)$ (cf. 1.3.2 and 1.3.4).

4.5.2. If T is the square $0\leqslant u\leqslant 1$, $0\leqslant v \leqslant 1$, if Y is the class of bounded measurable functions on T and $\mathscr{R}$ is the ideal of sets of measure 0, then for each spot functional ξ over $L_\infty(T)$ there corresponds a localization point t and "localization direction" a in the following sense: If A is any triangle of vertex t and such that all points $t+\lambda a$ belong to the interior of A for $0<\lambda<\lambda_0$, λ_0 being a positive number, then $x =_{\mathscr{R}} y$ on A implies $\xi(x) = \xi(y)$.

Chapter V. Sets $L(A)$ with $A\subset T$

5.1. Third Localization Theorem. The set $L(T)$, i. e. the set of all spot functionals localized somewhere in T, need not exhaust Ω. By Proposition 4.3.1, some spot functionals may have localization points in $\beta T\setminus T$, but density of T in βT enable us to expect that $L(T)$ must be dense in Ω. We shall show examples that it need not be the case even if $C^*(T)\subset Y$, but density of $L(T)$ holds under some assumptions concerning $\mathscr{R}$.

$\mathscr{R}$ will be said to have the *localization property* if for any set A not belonging to $\mathscr{R}$ there exists a point $t\in A$ such that $G\cap A\notin\mathscr{R}$ for every $G\in\mathscr{G}(t)$; in other words:

$$A\cap A^{\mathscr{R}} = \emptyset \text{ implies } A\in\mathscr{R},$$

where $A^{\mathscr{R}}$ is the set of points at which A does not belong locally to $\mathscr{R}$ (cf. [12], p. 34, and [18]).

$\mathscr{R}$ will be said to have the *weak localization property* if for any set A not belonging to $\mathscr{R}$ there exists a point $t\in T$ such that $G\cap A\notin\mathscr{R}$ for every $G\in\mathscr{G}(t)$; in other words:

$$A^{\mathscr{R}} = \emptyset \text{ implies } A\in\mathscr{R}.$$

The former property is essentially stronger than the latter (cf. [18]).

THEOREM 3. *Let B be a subset of T such that $T\setminus B\in\mathcal{R}$ and $\mathcal{R}_B=\mathrm{Rest}_B\mathcal{R}$ has the weak localization property with respect to B, then $L(B)$ is dense in Ω.*

In particular, if $\mathcal{R}$ has the weak localization property (with respect to T), then $L(T)$ is dense in Ω.

Proof. Let $z\in Y/\mathcal{R}$. Then, by Theorem 1 and Lemma 2 of [18],

$$\|z\| = \sup_{T}{}_{\mathcal{R}}|x| = \sup_{B}{}_{\mathcal{R}}|x| = \sup_{B}{}_{\mathcal{R}_B}|x|$$

$$= \sup_{t\in B}\overline{\lim_{u\to t}}{}^{*}_{\mathcal{R}_B}|x(u)| = \sup_{t\in B}\sup_{\xi\in L(t)}|\xi(x)|$$

$$= \sup\{\xi(|x|):\ \xi\in L(B)\}.$$

Hence $L(B)$ is dense in Ω by 1.3.3.

Now, we shall state some corollaries and remarks.

5.1.1. *If T is a separable metric space and $\mathcal{R}$ is any σ-ideal, then for each $R\in\mathcal{R}$ the set $L(T\setminus R)$ is dense in Ω.*

5.1.2. *If ξ is isolated in Ω and $\mathcal{R}$ has the weak localization property, then $\xi\in L(t)$ for some $t\in T$.*

5.1.3. *If $\mathcal{R}$ does not have the weak localization property, then there exists Y such that $L(T)$ is not dense in Ω.*

Proof. Let $Y=m(T)$. By Lemma 2 of [18], there exists $x\in m(T)$ such that $\|x\|=1$, $0\leqslant x\leqslant 1$, and $\overline{\lim_{u\to t}}{}^{*}_{\mathcal{R}}x(u)=0$ for each $t\in T$; apply 1.3.3 and Theorem 1.

Proposition 5.1.3 does not mean that if $\mathcal{R}$ does not have the weak localization property, then $L(T)$ is not dense in Ω for any class Y satisfying conditions of 1.1; indeed, consider $Y=C^*(T)$.

5.1.4. *There exist a compact Hausdorff space T, an ideal $\mathcal{R}$ with the weak localization property, a class Y and a set $R\in\mathcal{R}$ such that $L(T\setminus R)$ is not dense in Ω.*

Proof. Let λ be the order type of $(0,1)$ and let H be the lexicographical product $\lambda\times\omega_1$ (i. e. $(\alpha,\beta)<(\alpha',\beta')$ if $\beta<\beta'$ or $\beta=\beta'$ and $\alpha<\alpha'$). Then H is a locally compact Hausdorff space in the order topology(6); let T be the one-point compactification of H. Let $\mathcal{R}$ be the ideal of countable sets in T and $Y=m(T)$. Then $L(H)$ is not dense in $L(T)=\Omega$, because if A contains exactly one point of each set $\lambda\times\{\alpha\}$, $\alpha\in\Gamma_0(\omega_1)$, and $x=\chi_A$, then $\|x\|=1$ and $\overline{\lim}_{\mathcal{R}}x=0$ at every point of H, whence $\xi(x)=0$ for every $\xi\in L(H)$.

(6) This is a space obtained from $\Gamma_0(\omega_1)$ by replacing each point by an open interval and is called *long line*.

5.2. Isolated points in Ω. The conversion of 1.5.2 is not always true; an isolated point of Ω need not be a point functional (e. g. if T is an uncountable set, $\mathcal{R}$ the ideal of countable sets and Y contains only constant functions). Under the assumptions of Theorem 3, a partial converse is true.

5.2.1. *Suppose that $\mathcal{R}$ has the localization property and $Y/\mathcal{R}$ separates T. Then any spot functional ξ isolated in Ω is a point functional.*

Proof. By Theorem 3, there exists a unique $t\in T$ such that $\xi\in L(t)$, because $L(T)$ is dense in Ω and $\{\xi\}$ is open in Ω. If t is isolated in T, there is nothing to prove, so let us suppose that t is not isolated in T; we are going to show that $\xi\notin L_0(t)$. There exists $z\in Y/\mathcal{R}$ such that $\xi(z)=1$ and $\eta(z)=0$ for $\eta\neq\xi$, $\eta\in\Omega$. Hence $\sup\{|\xi(z)|:\xi\in L(u)\}=0$ for any $u\neq t$ and any $z\in Y/\mathcal{R}$, whence $\overline{\lim_{v\to u}}{}^{*}_{\mathcal{R}}|z(v)|=0$ (by Theorem 1). Since $\mathcal{R}$ has the localization property, $z=_{\mathcal{R}}0$ on $T\setminus t$ whence $\xi(z)>\overline{\lim_{u\to t}}{}_{\mathcal{R}}z(u)$ and $\xi\notin L_0(t)$.

5.2.2. *If $Y/\mathcal{R}$ separates T, if $\mathcal{R}$ has the localization property and if $T_{\mathcal{R}}=T$, then Ω is dense-in-itself.*

It follows from 5.2.1.

5.3. Topological properties of sets $L(A)$. We shall show that very few can be said about the common properties of all sets $L(T)$ or $L(t)$.

5.3.1. *$L(T)$ need not be a Borel subset of Ω.*

Proof. Let T be a dense non-Borel subset of $[0,1]$, let $\mathcal{R}=\emptyset$ and let Y be the class of functions uniformly continuous on T. Then $\Omega=[0,1]$ (up to homeomorphism) and $L(T)=T$.

5.3.2. *If F is any compact Hausdorff space, then there exist $\langle T,Y,\mathcal{R}\rangle$ and $t\in T$ such that $L(t)$ is homeomorphic to F.*

Proof. Let $F_1,F_2,\ldots$ be a sequence of copies of F and $\varphi_n\colon F\to F_n$ be homeomorphisms onto. Let T be the one-point compactification of the disjoint union $F_1\cup F_2\cup\ldots$ with t being the point at infinity.

Let

$$Y=\bigcup_{x\in C(F)}\{z\in m(T):\ z[\varphi_n(t)]=x(t)\ \text{for}\ n=1,2,\ldots\}$$

and let $\mathcal{R}=\{\{t\},\emptyset\}$.

Chapter VI. Integral representation of localized functionals

6.1. Representation of functionals of $\Xi(t)$ by integrals over $L(t)$. This section will be devoted to localization of non-multiplicative functionals.

6.1.1. *For each $t \in T$, the set $\Xi(t)$ is a complete sublattice of Ξ, i. e. if $\xi_\varkappa \in \Xi(t)$ for all $\varkappa$ and $\xi = \bigvee_\varkappa \xi_\varkappa$ exists, then $\xi \in \Xi(t)$.*

Proof. By a theorem of F. Riesz ([14], p. 179),

$$\xi(x) = \sup\{\xi_{\varkappa_1}(x_1)+\ldots+\xi_{\varkappa_n}(x_n): x_1+\ldots+x_n = x,\ x_i \geqslant 0\}$$

for any $x \geqslant 0$. Suppose that $0 \leqslant x \leqslant 1$ and $x =_{\mathscr{R}} 0$ on U, $U \in \mathscr{G}(t)$. Then $x_i =_{\mathscr{R}} 0$ on U whenever $x_1+\ldots+x_n = x$ and $x_i \geqslant 0$, whence $\xi(x_i) = 0$ for $i = 1, \ldots, n$ and $\xi(x) = 0$.

6.1.2. *A functional ξ over $Y/\mathscr{R}$ of the form*

$$\xi(x) = \int \eta(x)\mu(d\eta),$$

where μ is a Radon measure on Ω (cf. 1.3.1), has a localization point at t if and only if μ is concentrated on $L(t)$, i. e. if

$$\xi(x) = \int_{L(t)} \eta(x)\mu(d\eta)$$

holds for each $x \in Y/\mathscr{R}$.

ξ is a generalized limit at t if and only if $\mu \geqslant 0$ and $\|\mu\| = 1 = \mu[L(t)] = \mu[L_0(t)]$.

*The set $K(t)$ of such limits is the smallest *-weakly compact convex set containing $L_0(t)$ and $\Xi(t)$ is the *-weak closure of the linear span of $L(t)$.*

Proof. Sufficiency of such representations follows from 2.1.1 as the integral is a *-weak accumulation point of linear or convex combinations, respectively.

To prove necessity it is enough to show that $K(t)$ is the *-weak closure of conv $L_0(t)$ (actually points of $L_0(t)$ are extreme in $K(t)$) and apply theorems on *-weak closedness of sets of measures; for $\Xi(t)$ the proof is analogous.

Suppose that $\eta \notin \mathrm{Cl\ conv}\ L_0(t)$, then by a theorem of Krein and Šmulyan [10], there exists $x_0 \in Y/\mathscr{R}$ such that $\eta(x_0) > 1$ and $\zeta(x) \leqslant 1$ for all $\zeta \in L_0(t)$. Hence, by Theorem 1, $\overline{\lim}_{\mathscr{R}\, u\to t}\, x_0(u) \leqslant 1$ and η cannot be a generalized limit by 2.2.4.

6.2. Localization of functionals represented by finitely additive measures. Every linear functional ξ over $\Lambda(T, \mathscr{A}, \mathscr{R})$ (cf. 1.4.1) can be represented as $\xi(x) = \int x(t)\nu(dt)$ where ν is a finitely additive set function of bounded variation defined on $\mathscr{A}/\mathscr{R}$, i. e. ν is defined on $\mathscr{A}$ and vanishes on $\mathscr{R}$. ξ is a spot functional if and only if $\nu(A \cap B) = \nu(A)\nu(B)$ for any two sets A, B of $\mathscr{A}$ (equivalently, if $\nu(A)$ is a zero-one set function) and $\nu(T) = 1$ (cf. Šmulyan [21], Yosida and Hewitt [23]).

According to Yosida and Hewitt ([23], p. 48), a finitely additive set function ν on $\mathscr{A}/\mathscr{R}$ is called *purely finitely additive* if no non-zero σ-additive measure μ can satisfy $0 \leqslant \mu \leqslant |\nu|$.

6.2.1. *If $\xi \in \Omega$, then the corresponding ν is either σ-additive or purely finitely additive.*

Indeed, if ξ is a spot functional, then the lattice ideal generated by ξ in Ξ is one-dimensional.

6.2.2. *Let t be a G_δ-point of T and let $A_1, A_2, \ldots$ be a sequence of sets of $\mathscr{A}$ such that $\bigcap A_n = \{t\}$ and t is an interior point of each A_n. Let $\xi \in L(t)$ and $\xi(x) = \int x\,d\nu$. Then if it is σ-additive, $\nu(t) = 1$; if $t \in T_{\mathscr{R}}$, then ν is purely finitely additive.*

Proof. First part follows from 2.2.1, as

$$\nu(t) = \lim \nu(A_n) = \lim \xi(\chi_{A_n}) = 1.$$

The second part follows from 6.2.1.

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Reçu par la Rédaction le 16. 9. 1962

STUDIA MATHEMATICA, T. XXIII. (1963)

On the differentiability of weak solutions of certain non-elliptic equations II

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Introduction

In the first part of this paper (which will be quoted here as [I]) a theorem was given concerning the periodic weak solutions of certain partial differential equations of non-elliptic type. The differentiability properties of such solutions were described with the aid of some Hilbert spaces, which have been defined in the first chapter of [I]. In the present paper we are going to prove some further properties of these Hilbert spaces and to study the differentiability of weak solutions of the mentioned equations under some special boundary conditions.

We recall some definitions and notations of [I]. Let Ω be the product of two domains: $\overset{1}{\Omega}$ of the space E^R, and $\overset{2}{\Omega}$ of the space E^S ($R+S = N$), and let $C_2^\infty(\Omega)$ be the class of all complex-valued functions which are infinitely differentiable in Ω and whose all the derivatives are square summable in Ω. We denote by B a linear subset of the class $C_2^\infty(\Omega)$ containing the class $C_0^\infty(\Omega)$ which has the following properties:

1° for each function $\varphi \in C_0^\infty(\overset{1}{\Omega})$ or $\psi \in C_0^\infty(\overset{2}{\Omega})$, and for each $u \in B$, the functions φu and ψu are also in B,

2° for each $u \in B$ all the derivatives of u are also in B.

$B_{0,-}$ denotes the subset of the class B consisting of all functions $u(x, y)$ which vanish for $x \in \overset{1}{\Omega} - K$ and $y \in \overset{2}{\Omega}$, when K is a compact contained in $\overset{1}{\Omega}$ (depending on u). $B_{-,0}$ has the same meaning when the roles of x and y are interchanged. Let especially Ω be the N-dimensional cube; so B_p denotes the class of all functions infinitely differentiable in E^N which are periodic with Ω as period-parallelogram.

We have defined in [I] the two-indices norms for $u \in C_2^\infty(\Omega)$ as follows:

$$\|u\|_{0,k}^2 = \sum_{0 \leqslant |\beta| \leqslant k} \|D_y^\beta u\|_{L^2(\Omega)}^2,$$