Determinants in Banach spaces

by

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§ 1. Terminology and notation. X is a Banach space, \( X^* \) is the dual of \( X \). \( x, y, z \) denote elements of \( X \), and \( \xi, \eta, \zeta \) — elements of \( E \). \( \xi x \) is the value of \( \xi \) at \( x \). \( E \) is the Banach algebra of all bounded endomorphisms in \( X \) (called also operators), with the unit \( 1 \). If \( y = A x \) is a bounded endomorphism in \( X \), then \( \eta = fA \) denotes the adjoint endomorphism in \( E \). Endomorphisms (operators) will often be interpreted as bilinear functionals \( \xi Ax = \xi (A x) = (A \xi) x \). For fixed \( x_0, \xi_n \) the symbol \( x_0, \xi_n \) denotes the one-dimensional operator \( Ax = x_0, \xi_n x \).

\( T : E \) is said to be quasinuclear if there exists a bounded linear functional \( F \) on \( E \) such that \( \xi TX = F(\xi x) \). Then \( T \) is denoted by \( T_F \), and \( F \) is called quasinucleas of \( T \). E.g. if \( x_0, \xi_n \) are fixed, then \( F(A) = \xi_n A x_0 \) (or \( A e E \) is a functional on \( E \), denoted by \( \xi_n x_0 \), which is a quasinucleas of \( x_0, \xi_n \). All functionals in the closure of the set of all finite sums \( \sum z_i x_i \) are called nuclear. If \( F \) is nuclear, then \( T = T_F \) is called nuclear and \( F \) is the nucleus of \( T \). For any quasinucleas \( F, F(I) \) is called trace of \( F \) and denoted by \( Tr E \). The space \( Q \) of all quasinucleas is a Banach algebra with multiplication: \( F_G(A) = \xi F_I(A x) \). The canonical mapping \( F \to \xi \) is a ring homomorphism. We write sometimes \( F_{e_0}(\xi x) \) instead of \( F(T) \).

§ 2. The determinant system for an \( A \in E \) is an infinite sequence

\[
D_1, D_2, D_3, \ldots
\]

such that: (1) \( D_1 \) is a scalar; (2) for \( n > 0 \), \( D_n(\xi_1, \ldots, \xi_n) \) is a \( 2n \)-linear functional on \( E^* \times X^* \), skew symmetric in variables \( \xi_1, \ldots, \xi_n \) and skew symmetric in \( x_1, \ldots, x_n \); (3) \( D_2(\xi_1, \ldots, \xi_n) \) interpreted as a function of \( \xi_1 \) and \( x_1 \) only, is a bilinear functional on \( S^* \times X \) of the form \( \xi_1 C x_1 \), where \( C \in E \); (4) for an integer \( r \), \( D_r \) does not vanish identically (the smallest \( r \)
§ 5. Effective analytic formulae for a determinant system. For any quasinucleus $F$, let

\[ D_n(F) = \sum_{m=1}^{\infty} \frac{1}{m!} D_{n,m}(F) \quad (n = 0, 1, 2, \ldots), \]

where

\[ D_{n,m}(F) = D_{n,m}(F)[\xi_1, \ldots, \xi_n]_{\sigma_1, \ldots, \sigma_m} \]

\[ = \int_{F_{n+1}(\pi_{n+1})^{-1}} \ldots \int_{F_{n+1}(\pi_{n+1})^{-1}} \xi_{n+1} \ldots \xi_n \sigma_{n+1} \ldots \sigma_m \]

The sequence (called the determinant system of $F$)

\[ D_k(F), D_k(D_k(F), D_k(D_k(F), \ldots) \]

is a determinant system for $A = I + T_F$. Moreover,

\[ D_{n,m}(F) = \begin{bmatrix} s_1 & m-1 & 0 & 0 \\ s_2 & s_1 & m-2 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_{n-2} & \ldots & s_1 \\ s_n & s_{n-1} & \ldots & s_2 \end{bmatrix} \]

\[ T_\alpha^m = \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \\ \Delta_{n,m}(F) & \ldots & \Delta_{n,m}(F) \end{bmatrix} \quad (m = 1, 2, \ldots), \]

where $s_n = \text{Tr}(F^{\alpha}) = F(T_F^{\alpha-1})$, and $T_\alpha^m$ is the $2n$-linear functional

\[ T_\alpha^m(\xi_1, \ldots, \xi_\alpha, \ldots, \xi_n) = \sum_{i_1, \ldots, i_n=1}^{m} \det(\xi_1 T_{i_1}^{\alpha}, \ldots, \xi_n T_{i_n}^{\alpha}) \]

is a determinant system for $A$. 

§ 6. First fundamental theorem. If $A \in B$ has a determinant system, then $A$ is Fredholm. This is the order $r$ is 0, then $A^{-1} = D_1(A)$. If $r > 0$ and $\sigma_1, \ldots, \sigma_r, \pi_1, \ldots, \pi_r$ are fixed elements such that $\delta = D_1(\eta_1, \ldots, \eta_r) \neq 0$, let $\zeta_1, \ldots, \zeta_r, \sigma_1, \ldots, \sigma_r$ and $B \in E$ be such that, for all $\xi, \sigma$:

\[ \begin{align*}
\zeta B x &= \delta^{-1} D_1(\eta_1, \ldots, \eta_r, x, y_1, \ldots, y_r, \sigma_1, \ldots, \sigma_r) \\
\zeta x &= D_1(\eta_1, \ldots, \eta_r, y_1, \ldots, y_r, \sigma_1, \pi_1, \ldots, \sigma_r) \\
\zeta x &= D_1(\eta_1, \ldots, \eta_r, x_1, \ldots, x_r, \sigma_1, \pi_1, \ldots, \sigma_r) \\
\zeta x &= D_1(\eta_1, \ldots, \eta_r, y_1, \ldots, y_r, \sigma_1, \pi_1, \ldots, \sigma_r) 
\end{align*} \]

Then $\zeta_1, \ldots, \zeta_r$ are linearly independent in $E$, and so are $\eta_1, \ldots, \eta_r$. The linear equation $Ax = x$ has a solution $x = 0$ if $\zeta_1 \eta_1 = 0$ for $i = 1, \ldots, r$; then $x = B x_0 + \alpha_1 x_1 + \cdots + \alpha_r x_r$ is the general form of the solution. The adjoint equation $J A \xi = \xi$ has a solution $\xi = 0$ if $\zeta \xi = 0$ for $i = 1, \ldots, r$; then $\zeta = \xi B = \xi_1 \xi_2 + \cdots + \xi_r \xi_1$ is the general form of the solution.

§ 4. The second fundamental theorem. If $A$ is Fredholm, then $A$ has a determinant system. The determinant system of $A$ is determined by $A$ uniquely up to a scalar factor $\neq 0$.

§ 5. Examples of determinant systems. (a) Let $X$ be the $m$-dimensional space, and $A$ an endomorphism in $X$, determined by a square matrix $(a_{ij})$. Let $D_0 = \det(a_{ij})$. For $0 < n < m$, the set of all algebraic minors obtained from $(a_{ij})$ by omitting $n$ rows and $n$ columns is an $m$-covariant and $m$-contravariant tensor, i.e., a $2n$-linear functional $D_n(\eta_1, \ldots, \eta_n)$ on $\mathbb{R}^m \times X$. Let $D_0(\eta_1, \ldots, \eta_n) = \det(\xi a_i)$, and $D_n = 0$ for $n > m$. Then $D_0, D_1, D_2, \ldots$ is a determinant system for $A$.

(b) If $X$ is any Banach space, and $A \in E$ has the inverse $A^{-1}$, then

\[ \begin{align*}
D_0 &= 1 \\
D_0(\xi_1, \ldots, \xi_n) &= \det(\xi_1 A^{-1}, x)
\end{align*} \]

is a determinant system for $A$. 

\[ \begin{align*}
D_n(\xi_1, \ldots, \xi_n) &= \sum_{i_1, \ldots, i_n=1}^{m} \det(\xi_1 T_{i_1}^{\alpha}, \ldots, \xi_n T_{i_n}^{\alpha}) \\
&= \sum_{i_1, \ldots, i_n=1}^{m} \det(\xi_1 T_{i_1}^{\alpha}, \ldots, \xi_n T_{i_n}^{\alpha})
\end{align*} \]
§ 7. Some identities for the determinant system (4). Let

\[ D_\delta(F; x_1, \ldots, x_n) = \lim_{\varepsilon \to 0} \frac{D_\delta(F + \varepsilon F_0) - D_\delta(F)}{\varepsilon}, \]

and, by induction

\[ D_\delta^n(F; x_1, \ldots, x_n) = \lim_{\varepsilon \to 0} \frac{D_\delta^{n-1}(F + \varepsilon F_{n-1}) - D_\delta(F; x_1, \ldots, x_{n-1})}{\varepsilon}. \]

We have

\[ D_\delta(F) \left( \xi; x_1, \ldots, x_n \right) = D_\delta^n(F; \xi; x_1, \ldots, x_n). \]

\[ D_\delta(F) \] is the only analytic solution (in \( Q \)) of the differential equation

\[ D_\delta(F; F_1 + F_2) = D_\delta(F) \cdot D_\delta(F_2) \]

with the initial condition \( D_\delta(0) = 1 \).

For \( |F| < 1 \),

\[ D_\delta(F) = \exp \text{Tr} \log(J + F), \]

where \( J \) is the abstract unit added to the algebra \( Q \). For all \( F_1, F_2, F_3 \in \mathfrak{Q}, \)

\[ D_\delta(F_1 + F_2 + F_3) = D_\delta(F_1)D_\delta(F_2)D_\delta(F_3) \]

(\text{theorem on multiplication of determinants}), and

\[ D_\delta(F) = D_\delta(F) \cdot D_\delta^*, \]

where \( D_\delta^* \) is defined by (1) with \( A = I + T_F \).

§ 8. The case where \( X \) is the one-dimensional space. Then formulas (2), (3) yield the algebraic determinant system described in § 5 (b).

§ 9. The case where \( X, \mathfrak{F} \) are spaces of measurable functions defined on a set \( F \) with a measure \( \mu, \xi \in \mathfrak{F}(x, t) \) be a function such that the functional \( F(x)(K) = \int \mathfrak{F}(x, t) x(t) d\mu(t) \) is continuous on the class of integral operators \( K: Kx(s) = \mathfrak{F}(x, t) x(t) d\mu(t) \). Let \( F \) be any extension of \( F_\delta \) over the whole \( E \). Then (4) is the determinant system for the integral equation

\[ x(s) = \int F(x, t) x(t) d\mu(t) = x_\delta(s), \]

and \( T_F \) is the integral endomorphism with the kernel \( \mathfrak{F}(s, t) \). The determinant system (4) does not coincide, in the case \( X = C \), with the original Fredholm determinant and subdeterminants. The Fredholm determinant system coincides with the sequence \( (D_\delta(F)) \), where

\[ D_\delta(F) \left( \xi_1, \ldots, \xi_n ; x_1, \ldots, x_n \right) = D_\delta(F) \left( \xi_1 T, \ldots, \xi_n T ; x_1, \ldots, x_n \right) = D_\delta(F) \left( \xi_1 T x_1, \ldots, \xi_n T x_n \right) \]

and \( T = T_F \). The same integral formulas for (13) can be written as in the case investigated by Fredholm.

§ 10. The case of infinite square matrices. Substituting in \( \xi = x \) the set of positive integers with a trivial measure, we get a generalization Koch's theory of determinants and subdeterminants of infinite square matrices.

§ 11. The non-uniqueness effect. Observe that the canonical mapping \( F \mapsto T_F \) is not one-to-one; consequently the determinant system (4) for \( A \) is uniquely determined by \( F \), but if it is not uniquely determined by \( A = I + T_F \). In many concrete cases we know that the canonical mapping is one-to-one on the class of all nuclei. Then, if we restrict ourselves to examine only the operators \( A = I + T \), where \( T \) is nuclear, we can uniquely assign, to every \( A \) of this form, a determinantal system, viz. the system (4) where \( F \) is the only nucleus of \( T \). In the general case we cannot prove that the canonical mapping is one-to-one on the set of all nuclei. The problem whether the canonical mapping is one-to-one on the set of all nuclei is equivalent to the problem whether every compact endomorphism is a uniform limit of a sequence of finitely dimensional operators.

Observe that, for every quasinucleus \( F \), the sequence

\[ (D_\delta(F)) \exp (-T_F + 4T_F^2) \]

is also a determinant system for \( A = I + T_F \) and is uniquely determined by \( A \) only! The sequence

\[ (D_\delta(F)) \exp (-T_F) \]

has the same property, provided \( T_F \) is a uniform limit of finitely dimensional operators. However, in the case where \( X \) is a finitely dimensional space, neither (15) nor (14) coincides with the algebraic determinant system \( \Sigma 5(a) \).

§ 12. The Carleman determinant system in \( L^1(F, \mu) \). If the kernel \( \mathfrak{F}(s, t) \) of the integral equation (12) is such that \( \int \mathfrak{F}(s, t) x(t) d\mu(t) \) is in \( C \), the determinant system (4) does not exist, in general. However, the expressions (15) remain sensible and give a whole determinant system for (12).
Формула механических кубатур называют обычно приближенной формулой

\[ \int_{\Omega} \varphi(x) \, dx = \sum_{k=1}^{N} \lambda_k \varphi(x_k), \]

где \( \Omega \) — некоторая область в мером пространства, точки \( x_k \) суть такие точки внутри этой области, а коэффициенты \( \lambda_k \) — задания системы чисел. Ошибка формулы зависит от функции \( \varphi \). Для различных классов функций эту ошибку можно оценивать по разному. В пространствах \( C^m \) (\( m \geq 1 \)) и \( W^{m,p} \) (\( m > n/p \)), как это следует из теорем вложении, функционал

\[ (l, \varphi) = \int_{\Omega} \varphi(x) \varphi(x_k) \]

является дифференцируемым. Максимум такого функционала на единичной сфере в \( W^{m,p} \) может быть найден эффективно. Ниже будет показано, как это произойдет. Искомое выражение для максимума можно поставить задачу о нахождении

\[ \min \{ \max \{ l, \varphi \} \} \]

в \( C(x_0, C) \) для всех \( x_0 \) в \( \mathbb{R}^n \).

т. е. о построении оптимальной формулы механических кубатур с заданным числом точек, что представляет собой задачу о нахождении экстремума функции конечного числа переменных.

Напомню, что формула механических кубатур определяется нормой в \( W^{m,p} \) в смысле

\[ \| \phi \|_{W^{m,p}} = \| \phi \|_{L^p} + \| \partial_j^m \phi \|_{L^1}, \]

где \( S^{m-1} \) пространство многочленов степени \( m-1 \), \( L^1 \) фактор, пространство

\[ L^1 \hookrightarrow W^{m-1,p} \]