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Reflexivity and summability*

by

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A theorem of Banach and Saks [1] asserts that any bounded sequence in $L_p(0,1)$ or $l_p\,(p>1)$ has a subsequence whose $(C,1)$ means converge strongly. Kakutani [4] later showed that for weakly convergent sequences in a uniformly convex Banach space the same conclusion holds. Since uniformly convex spaces are now known to be reflexive [5, 7, 8], "weakly convergent" here may be replaced by "bounded". Schreier [9] has shown that the theorem of Banach and Saks cannot be extended to $C[0,1]$. Thus we observe that for a subclass of the reflexive Banach spaces the conclusion of the Banach-Saks theorem holds and for some non-reflexive Banach spaces the theorem does not hold. We shall show (Corollary, Theorem 2) that Banach spaces for which the Banach-Saks the orem hold are reflexive. Two natural questions which arise are:

1) Are there reflexive Banach spaces which are not uniformly convex and for which the Banach-Saks theorem holds?

2) Is reflexivity equivalent to a summability property?

In § 1 we give a class of spaces for which the Banach-Saks theorem holds which includes the isomorphs of uniformly convex spaces. By exhibiting examples we show the inclusion to be proper. In § 2 we give an affirmative answer to the second question. In § 3 we give a summability property which, in a Banach space with a Schauder base, implies that it is boundedly complete.

1. Consider the class of Banach spaces satisfying the property:

(*) There is a $\theta\in(0,1)$ such that in every sequence $\{x_n\}$, $\|x_n\|\leqslant 1$, which converges weakly to zero, there is a pair x_{n_1}, x_{n_2} such that $\|x_{n_1}+x_{n_2}\|<2\theta$.

Kakutani has shown ([4], p. 191), that every uniformly convex space satisfies another formulation of this property. The equivalence of these formulations is easily seen. It is clear from the proof of Kakutani that

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this property implies that a weakly convergent sequence has a subsequence whose $(C,1)$ means converge strongly. Thus, in the class of reflexive Banach spaces satisfying $(*)$, which includes the uniformly convex spaces, the Banach-Saks theorem holds.

Let B_i be the i-dimensional space of points $b_i = (b_{i1}, \ldots, b_{ii})$ and let $\|b_i\| = \sup_{j\leqslant i} |b_{ij}|$. For $p>1$, let $P^p\{B_i\}$ be the class of sequences $b = \{b_i\}$, $b_i \in B_i$, and $\|b\| = \big(\sum_{i=1}^{\infty} \|b_i\|^p\big)^{1/p} < \infty$. Day [2] has shown that the Banach space $P^p\{B_i\}$ is reflexive, but not isomorphic to any uniformly convex space.

THEOREM 1. *The spaces $P^p\{B_i\}$ have property* $(*)$.

Proof. Suppose $\{b^n\}$ is a sequence in the unit ball of $P^p\{B_i\}$ which converges weakly to zero.

Given $\varepsilon>0$, there exists an N such that $\sum_{i>N} \|b_i^1\|^p < \varepsilon^p$. Since weak convergence to zero implies coordinatewise convergence to zero, we have, for n sufficiently large, $\sum_{i\leqslant N} \|b_i^n\|^p < \varepsilon^p$. Then

$$\begin{aligned}\|b^1+b^n\| &\leqslant \|(b_1^1, \ldots, b_N^1, b_{N+1}^n, \ldots)\| + \|(b_1^n, \ldots, b_N^n, b_{N+1}^1, \ldots)\| \\ &= \Big(\sum_{i\leqslant N} \|b_i^1\|^p + \sum_{i>N} \|b_i^n\|^p\Big)^{1/p} + \Big(\sum_{i\leqslant N} \|b_i^n\|^p + \sum_{i>N} \|b_i^1\|^p\Big)^{1/p} \\ &\leqslant (1+1)^{1/p} + (\varepsilon^p+\varepsilon^p)^{1/p} = 2^{1/p}(1+\varepsilon).\end{aligned}$$

Thus we see that $(*)$ holds for $\theta \in (2^{\frac{1}{p}-1}, 1)$.

2. A summability method T is a real matrix (c_{mn}), $m = 1, 2, \ldots$, $n = 1, 2, \ldots$; the T-means of a sequence $\{x_n\}$ are $t_m = \sum_{n=1}^{\infty} c_{mn} x_n$. T is said to be *regular* if x_n real, $x_n \to x$ (finite), implies $t_m \to x$. The theorem of Toeplitz and Silverman gives the following necessary and sufficient conditions that T be regular:

1) $\sum_{n=1}^{\infty} |c_{mn}| < H$ for all m,

2) $c_{mn} \to 0$ as $m\to\infty$ for all n,

3) $\sum_{n=1}^{\infty} c_{mn} \to 1$ as $m\to\infty$.

We will be particularly interested in a class of methods which in addition to being regular have the property

4) $\sum_{n=1}^{\infty} |c_{mn}| \to 1$ as $m\to\infty$.

A matrix satisfying this requirement will be called *essentially positive*.

We obtain the following property from the theorem of Banach and Saks by replacing $(C,1)$-summability with general methods. A Banach space is said to have *property* $\mathscr{S}$ if for every bounded sequence there is a regular summability method T and a subsequence whose T-means converge strongly.

A Banach space is said to have *property* $w\mathscr{S}$ if for every bounded sequence there is a regular method T and a subsequence whose T-means converge weakly.

Since the insertion of columns of zeros into the matrix T does not affect the conditions 1) through 4) and we do not require that the same T be used for all sequences, property $\mathscr{S}$ ($w\mathscr{S}$) is equivalent to the existence of a regular method T for each bounded sequence such that the T means of the sequence converge strongly (weakly).

THEOREM 2. *For a Banach space B the following three statements are equivalent*:

(i) *B is reflexive.*

(ii) *B has property $\mathscr{S}$ with essentially positive T.*

(iii) *B has property $w\,\mathscr{S}$ with essentially positive T.*

Proof. If our sequence is bounded and the Banach space is reflexive then there is a weakly convergent subsequence $\{x_n\}$ with weak limit x. According to a well known theorem of Mazur ([6], p. 81), there exists a sequence of finite convex combinations of x_n which converge strongly to x. Thus there exist $c_{1n}\geqslant 0$, $n = 1, \ldots, n_1$, $\sum_{n=1}^{n_1} c_{1n} = 1$, such that $\big\|\sum_{n=1}^{n_1} c_{1n}x_n - x\big\| < 1$. Omitting $x_1, \ldots, x_{n_1}$ from consideration, we can find $c_{2n}\geqslant 0$, $n = n_1+1, \ldots, n_2$, such that

$$\Big\|\sum_{n=n_1+1}^{n_2} c_{2n}x_n - x\Big\| < 1/2.$$

Proceeding in this manner, we find a sequence of convex combinations with coefficients (c_{mn}), $n_{m-1} < n \leqslant n_m$, where $n_0 = 0$, such that

$$\Big\|\sum_{n=n_{m-1}+1}^{n_m} c_{mn}x_n - x\Big\| < 1/m.$$

If we define $c_{mn} = 0$ for $n \notin [n_{m-1}+1, n_m]$, then $T = (c_{mn})$ is a positive regular method which meets the requirements of the theorem. Thus (i) implies (ii).

Clearly, (ii) implies (iii).

Let us now assume that our space has property $w\mathscr{S}$, that $\{x_n\}$ is a sequence in U, the unit ball, that $T = (c_{mn})$ is an essentially positive regular method and x is the point of our space such that $t_m \to x$ weakly. If we demonstrate that $x \in U$ and, for every continuous linear functional f, $\underline{\lim} f(x_n) \leqslant f(x) \leqslant \overline{\lim} f(x_n)$, then U is weakly sequentially compact ([3], p. 48), which implies that the space is reflexive. Since $t_m \to x$ weakly,

$$\|x\| \leqslant \underline{\lim} \|t_m\| \quad \text{and} \quad \|t_m\| = \Big\| \sum_{n=1}^{\infty} c_{mn} x_n \Big\| \leqslant \sum_{n=1}^{\infty} |c_{mn}| \to 1,$$

we see that $\|x\| \leqslant 1$, or $x \in U$. We have for fixed N,

$$\inf_{n>N} \{f(x_n)\} = \lim_{m\to\infty} \Big\{ \sum_{n \leqslant N} c_{mn} f(x_n) + \sum_{n>N} c_{mn}^{-} f(x_n) + [\inf_{n>N} \{f(x_n)\}] \sum_{n>N} c_{mn}^{+} \Big\}$$

$$\leqslant \lim_{m\to\infty} f(t_m) = f(x).$$

Thus we have $\underline{\lim} f(x_n) \leqslant f(x)$. $f(x) \leqslant \overline{\lim} f(x_n)$ may be demonstrated in a similar fashion.

Since the $(C,1)$-method is a positive regular summability method, we have, at once, the

COROLLARY. *Banach spaces in which bounded sequences have subsequences whose $(C,1)$ means converge strongly are reflexive.*

3. In a Banach space, a Schauder base $\{\Phi_i\}$ is called *boundedly complete* if $\|\sum_{i=1}^{n} a_i \Phi_i\| < M$ for all n implies that $\sum_{i=1}^{\infty} a_i \Phi_i$ converges.

THEOREM 3. *A Schauder base in a Banach space with property $w\mathscr{S}$ is boundedly complete.*

Proof. Suppose $\|\sum_{i=1}^{n} a_i \Phi_i\| < M$ for all n. Then there is a regular method $T = (c_{mn})$ and a point $x = \sum_{i=1}^{\infty} b_i \Phi_i$ such that the T means of

$$x_n = \sum_{i=1}^{n} a_i \Phi_i$$

converge weakly to x. We will show that $a_i = b_i$ for all i.

We have formally

$$t_m = \sum_{n=1}^{\infty} c_{mn} x_n = \sum_{i=1}^{\infty} a_i \Big(\sum_{n=i}^{\infty} c_{mn} \Big) \Phi_i .$$

We verify this identity by demonstrating the equiconvergence of the two series;

$$\Big\| \sum_{i=1}^{N} a_i \Big(\sum_{n=i}^{\infty} c_{mn} \Big) \Phi_i - \sum_{n=1}^{N} c_{mn} \sum_{i=1}^{n} a_i \Phi_i \Big\|$$

$$= \Big\| \sum_{i=1}^{N} a_i \Big(\sum_{n=i}^{\infty} c_{mn} \Big) \Phi_i - \sum_{i=1}^{N} a_i \Big(\sum_{n=i}^{N} c_{mn} \Big) \Phi_i \Big\|$$

$$= \Big\| \sum_{i=1}^{N} a_i \Phi_i \sum_{n=N+1}^{\infty} c_{mn} \Big\| < M \sum_{n=N+1}^{\infty} |c_{mn}| \to 0 \quad \text{as} \quad N \to \infty .$$

Thus

$$\sum_{i=1}^{\infty} \Big(a_i \Big(\sum_{n=1}^{\infty} c_{mn} \Big) - b_i \Big) \Phi_i \to 0 \text{ weakly as } m \to \infty$$

which implies that for each i,

$$\lim_{m\to\infty} \Big[a_i \Big(\sum_{n=i}^{\infty} c_{mn} \Big) - b_i \Big] = a_i - b_i = 0 .$$

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