Reflexivity and summability

by

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A theorem of Banach and Saks [1] asserts that any bounded sequence in $L_p(0,1)$ or $L_p (p > 1)$ has a subsequence whose $(O, 1)$ means converge strongly. Kakutani [4] later showed that for weakly convergent sequences in a uniformly convex Banach space the same conclusion holds. Since uniformly convex spaces are now known to be reflexive [3, 5, 8], "weakly convergent" here may be replaced by "bounded". Schreier [9] has shown that the theorem of Banach and Saks cannot be extended to $C[0,1]$. Thus we observe that for a subclass of the reflexive Banach spaces the conclusion of the Banach-Saks theorem holds and for some non-reflexive Banach spaces the theorem does not hold. We shall show (Corollary, Theorem 2) that Banach spaces for which the Banach-Saks theorem holds and for which the strong limit is reflexive. Two natural questions which arise are:

1) Are there reflexive Banach spaces which are not uniformly convex and for which the Banach-Saks theorem holds?

2) Is reflexivity equivalent to a summability property?

In § 1 we give a class of spaces for which the Banach-Saks theorem holds which includes the isomorphs of uniformly convex spaces. By exhibiting examples we show the inclusion to be proper. In § 2 we give an affirmative answer to the second question. In § 3 we give a summability property which, in a Banach space with a Schauder base, implies that it is boundedly complete.

1. Consider the class of Banach spaces satisfying the property:

(*) There is a $\theta \neq (0,1)$ such that in every sequence $(\{a_n\})$, $\|a_n\| \leq 1$, which converges weakly to zero, there is a pair $a_{n_1}, a_{n_2}$ such that $\|a_{n_1} + a_{n_2}\| < \theta$.

Kakutani has shown ([4], p. 191), that every uniformly convex space satisfies another formulation of this property. The equivalence of these formulations is easily seen. It is clear from the proof of Kakutani that

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this property implies that a weakly convergent sequence has a subsequence whose \((0, 1)\) means converge strongly. Thus, in the class of reflexive Banach spaces satisfying \((*)\), which includes the uniformly convex spaces, the Banach-Saks theorem holds.

Let \(B_1\) be the \(i\)-dimensional space of points \(b_1 = (b_1, \ldots, b_i)\) and let \(\|b\| = \sup_{i < \infty} |b_i|\). For \(p > 1\), let \(P^p(B_1)\) be the class of sequences \(b = (b_0, b_1, \ldots, b_n, \ldots)\) and \(\|b\| = (\sum_{i=0}^{\infty} \|b_i\|^p)^{1/p} < \infty\). Day [2] has shown that the Banach space \(P^p(B_1)\) is reflexive, but not isomorphic to any uniformly convex space.

**Theorem 1.** The spaces \(P^p(B_1)\) have property \((*)\).

**Proof.** Suppose \(\{b^n\}\) is a sequence in the unit ball of \(P^p(B_1)\) which converges weakly to zero.

Given \(\varepsilon > 0\), there exists an \(N\) such that \(\sum_{i=N}^{\infty} \|b^n_i\|^p < \varepsilon^p/2^n\). Then weak convergence to zero implies coordinatewise convergence to zero, we have, for \(n\) sufficiently large, \(\sum_{i=0}^{N-1} \|b^n_i\|^p < \varepsilon^p/2^n\). Then

\[
\|b^1 + b^n\|^p \leq \langle\|b_1\|, \ldots, \|b_N\|, b_{N+1}, \ldots\| \rangle + \langle\|b_1\|, \ldots, b_{N-1}, b_N, b_{N+1}, \ldots\| \rangle
\]

\[
= \left( \sum_{i=0}^{N-1} \|b^n_i\|^p + \sum_{i=N+1}^{\infty} \|b^n_i\|^p \right)^{1/p} + \left( \sum_{i=0}^{N-1} \|b_i\|^p + \sum_{i=N+1}^{\infty} \|b_i\|^p \right)^{1/p}
\]

\[
\leq (1 + 1)^{1/p} + (\varepsilon^p + \varepsilon^p)^{1/p} = 2(1 + \varepsilon).
\]

Thus we see that \((*)\) holds for \(\theta \leq (2^{p-1} + 1)\).

2. A summability method \(T\) is a real matrix \((\epsilon_{mn})\), \(m = 1, 2, \ldots, n = 1, 2, \ldots\); the \(T\)-means of a sequence \((x_n)\) are \(\epsilon_{mn} = \sum_{m=1}^{n} \epsilon_{mn} x_n\). \(T\) is said to be regular if \(x_n\) real, \(x_n \rightarrow x\) (finite), implies \(\epsilon_{mn} x_n \rightarrow x\). The theorem of Toeplitz and Silverman gives the following necessary and sufficient conditions that \(T\) be regular:

1. \(\sum_{m=1}^{\infty} |\epsilon_{mn}| < M\) for all \(m\),
2. \(\epsilon_{mn} \rightarrow 0\) as \(m \rightarrow \infty\) for all \(n\),
3. \(\sum_{m=1}^{\infty} \epsilon_{mn} \rightarrow 1\) as \(m \rightarrow \infty\).

We will be particularly interested in a class of methods which in addition to being regular have the property

4. \(\sum_{n=1}^{\infty} |\epsilon_{mn}| \rightarrow 1\) as \(m \rightarrow \infty\).

A matrix satisfying this requirement will be called essentially positive.

We obtain the following property from the theorem of Banach and Saks by replacing \((0, 1)\)-summability with general methods. A Banach space is said to have property \(\mathcal{S}\) if for every bounded sequence there is a regular summability method \(T\) and a subsequence whose \(T\)-means converge strongly.

A Banach space is said to have property \(w\mathcal{S}\) if for every bounded sequence there is a regular method \(T\) and a subsequence whose \(T\)-means converge weakly.

Since the insertion of columns of zeros into the matrix \(T\) does not affect the conditions 1) through 4) and we do not require that the same \(T\) be used for all sequences, property \(\mathcal{S}\) \((w\mathcal{S})\) is equivalent to the existence of a regular method \(T\) for each bounded sequence such that the \(T\)-means of the sequence converge strongly (weakly).

**Theorem 2.** For a Banach space \(B\) the following three statements are equivalent:

1. \(B\) is reflexive.
2. \(B\) has property \(\mathcal{S}\) with essentially positive \(T\).
3. \(B\) has property \(w\mathcal{S}\) with essentially positive \(T\).

**Proof.** If our sequence is bounded and the Banach space is reflexive then there is a weakly convergent subsequence \((x_{n_k})\) with weak limit \(x\). According to a well known theorem of Mazur ([6], p. 81), there exists a sequence of finite convex combinations of \(x_n\) which converge strongly to \(x\). Thus there exist \(c_{n_k} \geq 0, n = 1, 2, \ldots, n_{k+1}, n_{k+1} = 1\), such that \(\sum_{n_{k+1}}^{n_{k+1}} c_{n_k} x_n - x < 1/2\). Omitting \(x_{n_{k+1}}, \ldots, x_{n_{k+1}}\) from consideration, we can find \(c_{n_{k+1}} \geq 0, n = n_{k+1} + 1, \ldots, n_{k+1}\), such that \(\left\| \sum_{n=n_{k+1}+1}^{n_{k+1}} c_{n_{k+1}} x_n - x \right\| < 1/2\).

Proceeding in this manner, we find a sequence of convex combinations with coefficients \((c_{n_m})\), \(n_{m+1} = n_m + 1\), \(n_m \geq 0\), where \(n_{m+1} = 0\), such that

\[
\left\| \sum_{n=n_{m+1}+1}^{n_{m+1}} c_{n_m} x_n - x \right\| < 1/m.
\]

If we define \(c_{n_m} = 0\) for \(n \notin [n_{m+1} + 1, n_{m+1}]\), then \(T = (c_{n_m})\) is a positive regular method which meets the requirements of the theorem. Thus (i) implies (ii).

Clearly, (ii) implies (iii).

Let us now assume that our space has property \(w\mathcal{S}\), that \((x_n)\) is a sequence in \(U\), the unit ball, that \(T = (c_{n_m})\) is an essentially positive
regular method and \( x \) is the point of our space such that \( t_n \to x \) weakly. If we demonstrate that \( x \in U \) and, for every continuous linear functional \( f, \lim f(x_n) = f(x) \leq \lim f(\sigma_n) \), then \( U \) is weakly sequentially compact ([3], p. 48), which implies that the space is reflexive. Since \( t_n \to x \) weakly,

\[
||x|| \leq \lim \|x_n\| \quad \text{and} \quad ||x_n|| = \left\| \sum_{n=1}^{\infty} a_{m_n} \phi_n \right\| \leq \lim \left\| \sum_{n=1}^{\infty} a_{m_n} \right\| \to 1,
\]

we see that \( ||x|| \leq 1 \), or \( x \in U \). We have for fixed \( N \),

\[
\inf_{n > N} (f(x_n)) = \lim_{n \to \infty} \left\{ \sum_{n=1}^{\infty} a_{m_n} f(x_n) + \sum_{n > N} a_{m_n} f(x_n) + \left[ \inf_{n > N} (f(x_n)) \right] \sum_{n > N} a_{m_n} \right\}
\]

\[
\leq \lim f(t_n) = f(x).
\]

Thus we have \( \lim f(x_n) \leq f(x) \). \( f(x) \leq \lim f(x_n) \) may be demonstrated in a similar fashion.

Since the \((C,1)\)-method is a positive regular summability method, we have, at once, the

**Corollary.** Banach spaces in which bounded sequences have subsequences whose \((C,1)\) means converge strongly are reflexive.

3. In a Banach space, a Schauder base \( \{\phi_n\} \) is called boundedly complete if \( \| \sum \alpha_n \phi_n \| < M \) for all \( \alpha \) implies that \( \sum \alpha_n \phi_n \) converges.

**Theorem 3.** A Schauder base in a Banach space with property \( w^\sigma \) is boundedly complete.

**Proof.** Suppose \( \| \sum \alpha_n \phi_n \| < M \) for all \( \alpha \). Then there is a regular method \( T = (c_{m_n}) \) and a point \( x = \sum \beta_n \phi_n \) such that the \( T \) means of

\[
\sigma_n = \sum_{k=1}^{n} a_k \phi_k
\]

converge weakly to \( x \). We will show that \( a_i = b_i \) for all \( i \).

We have formally

\[
t_n = \sum_{n=1}^{\infty} c_{m_n} a_n = \sum_{n=1}^{\infty} a_n \left( \sum_{n=1}^{\infty} c_{m_n} \right) \phi_i.
\]

We verify this identity by demonstrating the equiconvergence of the two series:

\[
\left\| \sum_{i=1}^{N} a_i \left( \sum_{n=1}^{\infty} c_{m_n} \right) \phi_i - \sum_{i=1}^{N} a_i \left( \sum_{n=1}^{\infty} c_{m_n} \right) \phi_i \right\|
\]

\[
= \left\| \sum_{i=1}^{N} a_i \left( \sum_{n=1}^{\infty} c_{m_n} \right) \phi_i - \sum_{i=1}^{N} a_i \left( \sum_{n=1}^{\infty} c_{m_n} \right) \phi_i \right\|
\]

\[
= \left\| \sum_{i=1}^{N} a_i \left( \sum_{n=1}^{\infty} c_{m_n} \right) \phi_i - \sum_{i=1}^{\infty} \left( \sum_{n=1}^{\infty} c_{m_n} \right) \phi_i \right\|
\]

Thus

\[
\sum_{n=1}^{\infty} \left( a_i \left( \sum_{n=1}^{\infty} c_{m_n} \right) - b_i \phi_i \right) \to 0 \text{ weakly as } m \to \infty
\]

which implies that for each \( i \),

\[
\lim_{m \to \infty} \left( a_i \left( \sum_{n=1}^{\infty} c_{m_n} \right) - b_i \phi_i \right) = a_i - b_i = 0.
\]

References


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