Interpolation in Hilbert spaces of analytic functions

by

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Let \( H \) be a separable Hilbert space and let \( x_n \) be a sequence of unit vectors that span \( H \). We wish to know when the sequence \( x_n \) will have the following two properties.

(i) \( \sum |a_n|^2 \leq M \|a\|^2 \) for some constant \( M \) and all \( a \in H \).

(ii) For each sequence \( (a_n) \) in \( l^2 \), there is an \( x \) in \( H \) with \( (x, x_n) = a_n \) and \( \|x\| \leq M \sum |a_n| \), where \( M \) is a constant.

Let \( a_n = (x_n, x_n) \) and let \( A \) be the infinite matrix \( (a_n) \). Then it is not difficult to prove the following lemma:

**Lemma.** Property (i) is equivalent to each of the following two statements.

1. Given any orthonormal basis \( e_n \), there is a bounded operator \( T: H \rightarrow H \) with \( T(e_n) = x_n \) (all \( n \)).

2. The matrix \( A \) is a bounded transformation from \( l^2 \) to itself.

Property (ii) is equivalent to each of the following two statements.

1. Given any orthonormal basis \( e_n \), there is a bounded operator \( T: H \rightarrow H \) with \( T(e_n) = e_n \) (all \( n \)).

2. The matrix \( A \) is bounded below on \( l^2 \).

Suppose now that \( H \) is a Hilbert space of analytic functions in some domain \( D \) of the complex plane. For example:

I. \( H \) is the set of \( f = 2a_n z^n \) with \( \sum |a_n| < \infty \) (this is the Hardy space \( H_2 \)).

II. \( H \) is the set of \( f \) with \( \sum |a_n|^2 |n+1| < \infty \) (this is the Bergman space of \( f \) analytic in the unit circle for which \( \int f \overline{f} \, dA < \infty \)).

III. \( H \) is the set of \( f \) with \( \sum |a_n|^2 |n+1| < \infty \).

We assume that \( H \) has a reproducing kernel, that is, a function \( K(z) \) \( \in D \) such that \( (f, K) = f(z) \) for all \( f \) in \( H \) and \( z \in D \). In the three examples we have:

I. \( K = 1/(1-\overline{z}) \).

II. \( K = 1/(1-\overline{z})^2 \).

III. \( K = \exp(\overline{z}) \).

Let \( z_n \) be a sequence of points in \( D \), and form the unit vectors \( x_n = K_n \) \( n \) where \( K_n = K(z_n) \). We now seek necessary and sufficient
Determinants in Banach spaces

by

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§ 1. Terminology and notation. \(X\) is a Banach space, \(\mathbb{E}\) is the dual of \(X\), \(x, y, z\) denote elements of \(X\), and \(\xi, \eta, \zeta\) — elements of \(\mathbb{E}\). \(\xi \hat{\otimes} \eta\) is the value of \(\xi \otimes \eta\) at \(x, E\) is the Banach algebra of all bounded endomorphisms in \(X\) (called also operators), with the unit \(I\). If \(y = \xi \hat{\otimes} \eta\) is a bounded endomorphism in \(X\), then \(\eta = \xi \otimes \zeta\) denotes the adjoint endomorphism in \(\mathbb{E}\). Endomorphisms (operators) will be often interpreted as bilinear functionals \(\xi \otimes y = \xi(y)\) in \(\mathbb{E}\). For fixed \(x_0, \xi, \eta\) the symbol \(x_0 \hat{\otimes} \xi\eta\) denotes the one-dimensional operator \(\xi \otimes \eta\) at \(x_0\).

\(T \otimes E = \xi \hat{\otimes} \eta\) is said to be quasinuclear if there exists a bounded linear functional \(F\) on \(E\) such that \(\xi \hat{\otimes} \eta = F((\xi \hat{\otimes} \eta))\). Then \(T\) is denoted by \(\text{Tr} \), \(F\), and the symbol \(\text{quasinuclear}\) of \(T\). E.g., if \(x_0, \xi, \eta\) are fixed, then \(F(A) = \xi(A \hat{\otimes} \eta)\) is a functional on \(E\), denoted by \(\text{Tr} \), which is a quasinuclear of \(x_0 \hat{\otimes} \xi\eta\). All functionals in the closure of the set of all finite sums \(\sum_{i=1}^{n} \xi_i \otimes x_i\) are called nucleus. If \(E\) is a nucleus, then \(T = \text{Tr} \) is called nuclear and \(F\) is the nucleus of \(T\). For any quasinuclear \(F, F(I)\) is called trace of \(F\) and denoted by \(\text{Tr} F\). The space \(Q\) of all quasinuclear is a Banach algebra with multiplication: \(F_1 \circ F_2(A) = F_1(F_2(A \hat{\otimes} E))\). The canonical mapping \(F \rightarrow \text{Tr} F\) is a ring homomorphism. We write sometimes \(F_n((\xi \hat{\otimes} \eta))\) instead of \(F((\xi \hat{\otimes} \eta))\).

§ 2. The determinant system for an \(A \otimes E\) is an infinite sequence

\(D_0, D_1(x_0), D_2(x_1, x_2), \ldots, D_n(x_1, \ldots, x_n), \ldots\)

such that: (1) \(D_0\) is a scalar; (2) for \(n > 0, D_n(x_1, \ldots, x_n)\) is a \(2n\)-linear functional on \(\mathbb{E}^n \times X^n\), skew symmetric in variables \(\xi_1, \ldots, \xi_n\) and skew symmetric in variables \(x_1, \ldots, x_n\); (3) \(D_n(x_1, \ldots, x_n)\) interpreted as a function of \(\xi_1\) and \(x_0\) only is a bilinear functional on \(\mathbb{E} \times X\) of the form \(\xi_1 \hat{\otimes} x_0\), where \(C \in \mathbb{E}\); (4) for an integer \(r, D_r\) does not vanish identically (the smallest \(r\)