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Metric properties of normed algebras

by

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An algebra A over the real field R is a vector space over R which is closed with respect to a product xy which is linear in both x and y and satisfies the condition $\lambda(xy) = (\lambda x)y = x(\lambda y)$ for any $\lambda \in R$ and $x, y \in A$. The product is not necessarily associative. In the present note we assume that the algebra A contains a unit element e, i. e., an element satisfying the equation $ex = xe = x$ for any $x \in A$. Given any subset B of A, $\dim B$ will denote the linear dimension of B, i. e., the power of a maximal set of linearly independent elements of B. Further, $[B]$ will denote the linear set spanned by the elements of B. For arbitrary elements $x_1, x_2, \ldots, x_n$ by $A(x_1, x_2, \ldots, x_n)$ we shall denote the subalgebra generated by $x_1, x_2, \ldots, x_n$. An algebra A is called *power associative* if $A(x)$ is associative for every x in A. An algebra A is said to be *alternative* if for every pair x, y from A the equalities $x^2y = x(xy)$, $yx^2 = (yx)x$ hold. If only one of the above conditions is satisfied, then A is said to be *one-sided alternative*. A. A. Albert has proved ([1], p. 318-328), that every one-sided alternative algebra is power associative. An algebra is called *algebraic* if $A(x)$ is finite dimensional for every x in A. An algebra is called *normed* if it is a normed space over R under a submultiplicative norm $\| \ \|$, i. e., a norm satisfying in addition to the usual requirements the condition $\|xy\| \leqslant \|x\| \cdot \|y\|$ for any x and y in A. Moreover, in this paper we assume that $\|e\| = 1$.

The aim of the present note is to discuss the relation between metric and algebraic properties of normed algebras.

In the sequel by K we shall denote the unit ball of the normed space in question, i. e., the set $\{x: \|x\| \leqslant 1\}$ and by S the unit sphere, i. e., the boundary of K. By the well-known Hahn-Banach extension theorem for every $a \in S$ there exists a linear functional f such that $f(a) = 1$ and $|f(x)| \leqslant \|x\|$ for any $x \in A$. The functional f induces a hyperplane P consisting of all elements x satisfying the equality $f(x) = 1$. This hyperplane supports the unit ball at the point a. An element $a \in S$ is said to be *regular* if there exists exactly one hyperplane P supporting the unit ball at the point a. There exist algebras whose all elements of the unit sphere are regular; e.g. classical algebras: the real field, the complex field, the quaternion algebra and the Cayley algebra under the Euclidean norm.

LEMMA 1. *An element $a \in S$ is regular in A if and only if it is regular in the subspace $[a, b]$ for each element $b \in A$ linearly independent of a.*

Lemma 1 is a direct consequence of the Hahn-Banach extension theorem.

LEMMA 2. *Let $a \in S$ and b be linearly independent elements in A, P hyperplane supporting the ball K at the point a and $f(x)$ such a functional that $P = \{x: f(x) = 1\}$. Then the intersection $P \frown [a, b]$ consists of all elements $a + \gamma c$ ($\gamma \in R$) where c is a fixed element of the space $[a, b]$.*

Proof. Let U be the subspace of all elements u satisfying the equation $f(u) = 0$. Each element belonging to the intersection $P \frown [a, b]$ can be written in the form $a + u$, where $u \in U \frown [a, b]$. To prove our Lemma it is sufficient to show that $\dim\{U \frown [a, b]\} = 1$. Since a and b are linearly independent, the element $-f(b)a + b$ is different from 0. Moreover, $-f(b)a + b \in U \frown [a, b]$. Hence $\dim\{U \frown [a, b]\} \geqslant 1$. If $\dim\{U \frown [a, b]\} = 2$, then, of course, $U \frown [a, b] = [a, b]$. Hence it follows that $a \in U$ and, consequently, $f(a) = 0$, which is impossible. Lemma 2 is thus proved.

The set of elements $a + \gamma c$ ($\gamma \in R$) will be called a *line* and denoted by $p(a, c)$. Let P be a hyperplane supporting the ball K at the point a. If $p(a, c) = P \frown [a, b]$, then we shall call $p(a, c)$ a line supporting the unit ball $K \cap [a, b]$ at the point a. If $p(a, c)$ is a line supporting the ball $K \frown [a, b]$ at the point a, then there exists a linear functional $f(x)$ on $[a, b]$ such that $f(a + \gamma c) = 1$ for each $\gamma \in R$ and $f(x) \leqslant \|x\|$ for each $x \in [a, b]$. Consequently, for any $\gamma \in R$ we have the inequality

$$\|a + \gamma c\| \geqslant f(a + \gamma c) = 1. \tag{1}$$

Let a, b ($a \in S$) be two linearly independent elements. The element b is called *quasi-orthogonal* to a, if $p(a, b)$ is a line supporting the ball $K \frown [a, b]$ at the point a. We note that if an element b is quasi-orthogonal to a, then inequality (1) is true, i.e., $\|a + \gamma b\| \geqslant 1$ for each $\gamma \in R$. The converse implication is also true. Namely, if $\|a\| = 1$, $b \neq 0$ and for each $\gamma \in R$

$$\|a + \gamma b\| \geqslant 1, \tag{2}$$

then the element b is quasi-orthogonal to a.

In fact, since $b \neq 0$, from (2) we obtain the linear independence of the elements a and b. We define the linear functional f on $[a, b]$ by the formula $f(\alpha a + \beta b) = \alpha$ ($\alpha, \beta \in R$). For $\alpha \neq 0$ from (2) we get

$$|f(\alpha a + \beta b)| = |\alpha| \leqslant |\alpha| \cdot \left\| a + \frac{\beta}{\alpha} b \right\| = \|\alpha a + \beta b\|. \tag{3}$$

Since $f(b) = 0$, inequality (3) is also satisfied for $\alpha = 0$. The functional $f(x)$ can be extended to the whole space A without increasing its norm. Hence it follows that $p(a, b)$ is a line supporting the ball $K \frown [a, b]$ at the point a, i.e., that the element b is quasi-orthogonal to a. Consequently, we have proved the following

LEMMA 3. *An element $b \neq 0$ is quasi-orthogonal to an element $a \in S$ if and only if for each $\gamma \in R$ the inequality $\|a + \gamma b\| \geqslant 1$ holds.*

LEMMA 4. *If a is regular and b is quasi-orthogonal to a, then $\|a + \beta b\| = 1 + o(\beta)$ (i.e., $\lim_{\beta \to 0} \frac{1}{\beta}\{\|a + \beta b\| - 1\} = 0$) and $o(\beta) \geqslant 0$.*

Proof. Since a is regular, the line $p(a, b) = a + \gamma b$ is the only line supporting the ball $K \cap [a, b]$ at the point a. Consequently, taking into account the linear independence of elements b and $b - \alpha a$ ($\alpha \neq 0$), we infer that the line $p(a, b - \alpha a)$ does not support the ball $K \cap [a, b]$ at the point a. Therefore, there exists $\gamma_0 \in R$ such that $\|a + \gamma_0(b - \alpha a)\| < 1$. Since for $\gamma\alpha \leqslant 0$ we have the inequality

$$\|a + \gamma(b - \alpha a)\| = (1 - \alpha\gamma) \left\| a + \frac{\gamma}{1 - \alpha\gamma} b \right\| \geqslant 1 - \alpha\gamma \geqslant 1,$$

we infer that $\gamma_0 \alpha > 0$. Now let us suppose that $\alpha > 0$ and $\gamma_0 > 0$. The remaining case $\alpha < 0$ and $\gamma_0 < 0$ can be dealt with analogously.

Since $a \in K \cap [a, b]$ and $a + \gamma_0(b - \alpha a) \in K \frown [a, b]$, for each number γ satisfying the condition $0 < \gamma < \gamma_0$, we have $\|a + \gamma(b - \alpha a)\| < 1$. Furthermore, we can choose such a number γ_1 that the inequalities $0 < \gamma_1 < \gamma_0$, $1 - \alpha\gamma_1 > 0$ are satisfied. Thus, for $0 < \gamma < \gamma_1$ we have the inequality

$$(1 - \alpha\gamma) \cdot \left\| a + \frac{\gamma}{1 - \alpha\gamma} b \right\| = \|a + \gamma(b - \alpha a)\| \leqslant 1,$$

which implies

$$0 \leqslant \left\| a + \frac{\gamma}{1 - \alpha\gamma} b \right\| - 1 \leqslant \frac{\alpha\gamma}{1 - \alpha\gamma}$$

and, consequently,

$$0 \leqslant \frac{\left\| a + \frac{\gamma}{1 - \alpha\gamma} b \right\| - 1}{\frac{\gamma}{1 - \alpha\gamma}} \leqslant \alpha. \tag{4}$$

Since γ is an arbitrary positive number, we have, according to (4), the relation $\|a + \beta b\| = 1 + o(\beta)$. Finally, since the element b is quasi-orthogonal to a, we have the inequality $\|a + \beta b\| \geqslant 1$, which implies $o(\beta) \geqslant 0$.

Define $x^1 = x$, $x^{k+1} = x^k x$ ($k = 1, 2, \ldots$).

LEMMA 5. *If an element $j \in A$ $(j \neq 0)$ satisfies the equation $j^2 = 0$, then for each $\gamma \in R$ $(\gamma \neq 0)$ the inequality $\|e+\gamma j\| > 1$ holds.*

Proof. Of course, to prove our statement it is sufficient to show that for each non-zero $\gamma \in R$ the formula $\lim_{n\to\infty} \|e+\gamma j\|^n = \infty$ holds. But this formula is a direct consequence of the inequality

$$\|e+\gamma j\|^n \geqslant \|(e+\gamma j)^n\| = \|e+n\gamma j\| \geqslant n|\gamma|\cdot\|j\| - \|e\|.$$

From Lemma 5 we get the following

COROLLARY. *Each non-zero element $j \in A$ satisfying the equation $j^2 = 0$ is quasi-orthogonal to the unit element e.*

LEMMA 6. *If the normed algebra A contains an element $j \neq 0$, with $j^2 = 0$, then the unit element e is not regular.*

Proof. Contrary to this, let us suppose that e is a regular element. From Lemma 4 and Corollary to Lemma 5 it follows that for each $\beta \neq 0$ the equation $\|e+\beta j\| = 1+|\beta|\cdot\eta(\beta)$ holds, where $\eta(\beta) > 0$ and $\lim_{\beta\to 0} \eta(\beta) = 0$. Thus,

$$1+2|\beta|\cdot\eta(2\beta) = \|e+2\beta j\| = \|(e+\beta j)^2\| \leqslant \|e+\beta j\|^2$$
$$= [1+|\beta|\cdot\eta(\beta)]^2 = 1+2|\beta|\cdot\eta(\beta)+\beta^2\cdot\eta^2(\beta),$$
$$2|\beta|\cdot\eta(2\beta) \leqslant 2|\beta|\cdot\eta(\beta)+\beta^2\cdot\eta^2(\beta),$$

and

$$\eta(\beta) \geqslant \frac{\eta(2\beta)}{1+\dfrac{|\beta|}{2}\cdot\eta(\beta)}.$$

Moreover, there exist positive numbers M and δ such that $0 < \eta(\beta) < M$ whenever $0 < \beta < \delta$. Consequently, for each β satisfying the condition $0 < \beta < \delta$ we have the inequality

$$\eta(\beta) \geqslant \frac{\eta(2\beta)}{1+\dfrac{|\beta|}{2}M},$$

which implies the following ones:

$$(5) \qquad \eta\left(\frac{\beta}{2^n}\right) \geqslant \frac{\eta(2\beta)}{\prod_{k=1}^{n+1}\left(1+\dfrac{|\beta|M}{2^k}\right)} \geqslant \frac{\eta(2\beta)}{\exp(|\beta|M)} \qquad (n = 1, 2, \ldots).$$

Hence we obtain the relation $\varliminf_{n\to\infty} \eta(\beta/2^n) > 0$, which contradicts Lemma 4. Lemma 5 is thus proved.

An element a of the algebra A is called an *idempotent* if $a^2 = a$.

LEMMA 7. *If an element e_1 in A is an idempotent different from zero and from the unit element, then e_1 is quasi-orthogonal to e.*

Proof. By Lemma 3, it is sufficient to show that for each $\gamma \in R$ $\varliminf_{n\to\infty} \|(e+\gamma e_1)^n\| \neq 0$. Since e commutes with e_1 and $e_1^2 = e_1$, we have the equation

$$(e+\gamma e_1)^n = e+[(1+\gamma)^n-1]e_1.$$

Thus

$$\lim_{n\to\infty} \|(e+\gamma e_1)^n\| = \begin{cases} \infty & \text{if} \quad \gamma < -2 \text{ or } \gamma > 0, \\ \|e-e_1\| & \text{if} \quad -2 < \gamma < 0, \\ \|e\| & \text{if} \quad \gamma = 0 \end{cases}$$

and

$$\|(e-2e_1)^n\| \geqslant \min(\|e\|, \|e-2e_1\|),$$

which completes the proof.

LEMMA 8. *If the normed algebra A contains an idempotent e_1 different from zero and from the unit element e, then e is not regular.*

Proof. First of all we note that the elements e and e_1 are linearly independent and, consequently, the elements e_1 and $e_2 = e-e_1$ are also linearly independent. Since $e_2 \neq 0$, $e_2 \neq e$ and $e_2^2 = (e-e_1)^2 = e-e_1 = e_2$, by Lemma 7, the elements e_1 and e_2 are quasi-orthogonal to the unit element e. Consequently, the element e is not regular in $[e, e_1]$. Applying Lemma 1 we infer that e is not regular in A either.

Now we shall quote some elementary concepts of the theory of finite dimensional associative algebras. Let A be suh an algebra. If two non-zero elements $a, b \in A$ satisfy the equation $ab = 0$, then each of them is called a *divisor of zero*. The algebra is said to be a *division algebra* if for every a, b in A, with $a \neq 0$, the equations $ax = b$ and $ya = b$ are solvable in A. It is well-known that any finite dimensional associative algebra without divisors of zero is a division algebra; see e. g. [9, XVI, § 114]. An element a belonging to the algebra A is called a *nilpotent* if there exists such an integer n that $a^n = 0$. An element $a \in A$ is said to be a *proper nilpotent* if the elements xa and ax are nilpotents for every x in A. The set of all proper nilpotents is called a *radical*. An algebra is said to be *semisimple* if its radical contains only zero element. It can be proved that each finite dimensional associative semisimple algebra has a unit element [8, § 9, Theorem 12]. It is clear that an algebra which has no nilpotents different from zero is a semisimple algebra.

THEOREM 1. *A finite dimensional associative normed algebra with a regular unit element is algebraically isomorphic with one of the following: the real field, the complex field, the quaternion algebra.*

We note that the unit element is regular in all classical algebras: the real field, the complex field and the quaternion algebra considered under the Euclidean norm.

Proof. We shall prove first that the algebra A in question is semisimple. Contrary to this let us suppose that there exist an element $x \in A$ and an integer n such that $x^n \neq 0$ and $x^{n+1} = 0$. Setting $j = x^n$, we have, by the associative law, $j^2 = 0$. Thus, according to Lemma 6, the unit element of A is not regular, which contradicts the assumption of our Theorem.

Now we shall prove that the algebra A contains no divisors of zero. Contrary to this let us assume that there are non-zero elements $a, b \in A$ such that $ab = 0$. The subalgebra $A(a)$ generated by the element a is finite-dimensional, associative and semisimple. Thus, $A(a)$ contains a unit element e_1 which is, of course, a non-zero idempotent. Furthermore, e_1 is a divisor of zero, because for any $x \in A(a)$ the equation $xb = 0$ holds. Since the unit element e of algebra A is not a divisor of zero, we infer that $e_1 \neq e$ and, consequently, e_1 is an idempotent different from zero and from e. Hence, by Lemma 8, the unit element e is not regular, which is impossible. Thus, we have proved that the algebra A contains no divisors of zero. Consequently, the algebra A is a division algebra. Now the assertion of our Theorem is a direct consequence of the well-known Frobenius Theorem; see e. g. [6, X, § 52].

LEMMA 9. *If the normed algebra A contains such an element i that $i^2 = -e$, then the element i is quasi-orthogonal to e.*

Proof. Obviously, the subalgebra $A(i)$ is isomorphic with the complex field. The absolute value $|\alpha e + \beta i| = \sqrt{\alpha^2 + \beta^2}$ of the complex number $\alpha e + \beta i$ is a multiplicative norm in $A(i)$. Evidently, for $\gamma \neq 0$ we have the equation

$$\lim_{n\to\infty} |(e+\gamma i)^n| = \lim_{n\to\infty} (1+\gamma^2)^{n/2} = \infty. \tag{6}$$

Since the algebra $A(i)$ is finite-dimensional, the norms $|\ |$ and $\|\ \|$ are equivalent in $A(i)$; i. e., there exist two positive constants m and M such that $m|x| \leqslant \|x\| \leqslant M|x|$ for any $x \in A(i)$. Thus, by (6), we have $\lim_{n\to\infty} \|(e+\gamma i)^n\| = \infty$ for each $\gamma \neq 0$. Consequently, for each $\gamma \neq 0$ the relation $\lim_{n\to\infty} \|e+\gamma i\|^n = \infty$ holds. Therefore for each $\gamma \in R$ the inequality $\|e+\gamma i\| \geqslant 1$ is true, which completes the proof of the Lemma.

LEMMA 10. *If the normed algebra A contains an element i such that $i^2 = -e$ and the unit element e is a regular one, then for each pair $\alpha, \beta \in R$*

$$\|\alpha e + \beta i\| = \sqrt{\alpha^2 + \beta^2}.$$

Proof. To prove our Lemma it is sufficient to show that $\|\alpha e + \beta i\| = 1$, whenever $\alpha^2 + \beta^2 = 1$. Consider the elements $x = e\cdot\cos\varphi + i\cdot\sin\varphi$. We know that both norms $|\ |$ and $\|\ \|$ are equivalent in $A(i)$. Consequently, there exists a positive number m for which the inequalities

$$\|x^n\| > m \quad (n = 1, 2, \ldots) \tag{7}$$

hold. Since the norm $\|\ \|$ is submultiplicative, inequality (7) implies the inequality $\|e\cdot\cos\alpha + i\cdot\sin\varphi\| \geqslant 1$. Suppose that there exists a number φ_0 for which the inequality

$$\|e\cos\varphi_0 + i\sin\varphi_0\| = \varrho > 1$$

holds. Without loss of generality we may suppose that $\varphi_0 > 0$. Let $\{y_n\}$ denote the sequence of elements

$$y_n = e\cos\frac{\varphi_0}{n} + i\sin\frac{\varphi_0}{n} \quad (n = 1, 2, \ldots).$$

We choose such an index N that for each $n > N$ the condition $0 < \varphi_0/n < \pi/2$ is fulfilled. Thus for each $n > N$ we get the equation

$$\|y_n\| = \cos\frac{\varphi_0}{n}\, e + i\,\mathrm{tg}\,\frac{\varphi_0}{n}.$$

Furthermore, according to Lemmas 4 and 9, we have the formula

$$\|y_n\| = \cos\frac{\varphi_0}{n}\left[1 + o\left(\mathrm{tg}\,\frac{\varphi_0}{n}\right)\right]. \tag{8}$$

Hence, taking into account the submultiplicativity of the norm $\|\ \|$, we obtain the inequality

$$\|y_n\| \geqslant \sqrt[n]{e\cdot\cos\varphi_0 + i\cdot\sin\varphi_0} = \sqrt[n]{\varrho}. \tag{9}$$

For each $n > N$ from (8) and (9) we get the inequality

$$\frac{o\left(\mathrm{tg}\,\frac{\varphi_0}{n}\right)}{\mathrm{tg}\,\frac{\varphi_0}{n}} \geqslant \frac{\sqrt[n]{\varrho} - \cos\frac{\varphi_0}{n}}{\sin\frac{\varphi_0}{n}}.$$

But the right-hand side of this inequality tends to $\log\varrho/\varphi_0$, when $n \to \infty$, which gives a contradiction. Thus $\|e\cdot\cos\varphi + i\cdot\sin\varphi\| = 1$ for any φ, which completes the proof.

Now we shall prove the following generalization of Theorem 1.

THEOREM 2. *Each algebraic one-sided alternative-normed algebra with a regular unit element is isometrically isomorphic with one of the following: the real field, the complex field, the quaternion algebra or the Cayley algebra.*

Proof. We suppose that the algebra A is left-alternative. The case of right-alternative algebras can be discussed analogously. If $\dim A = 1$, then, of course, the algebra A is isometrically isomorphic with the real field. Now let A contain an element a linearly independent of the unit element e. Then the subalgebra $A(a)$ is finite-dimensional, associative and commutative. Consequently, the subalgebra $A(e, a)$ is also finite-dimensional, associative and commutative ([1], p. 319). Moreover, by Lemma 1 the unit element e is regular in $[e, a]$. Taking into account the inequality $\dim A(e, a) \geqslant 2$, we infer, by Theorem 1, that the subalgebra $A(e, a)$ is algebraically isomorphic with the complex field. Thus there exists an element $i_1 \in A(e, a)$ such that $i_1^2 = -e$ and every element $x \in A(e, a)$ can be written in the form $x = \alpha e + \beta i_1$ $(\alpha, \beta \in R)$. Hence, by Lemma 10, we obtain the isometric isomorphism of $A(e, a)$ and the complex field. Thus, our Theorem is proved in the case $\dim A = 2$. Now we consider the case $\dim A \geqslant 3$. Let e, a, b be three linearly independent elements of A. From the first part of the proof it follows that there exist elements i_1 and i_2 such that $i_1^2 = i_2^2 = -e$, $a \in A(i_1)$, $b \in A(i_2)$ and $\|\alpha e + \beta i_1\| = \|\alpha e + \beta i_2\| = \sqrt{\alpha^2 + \beta^2}$ for any α, β in R.

In the sequel by z_n we shall denote every element of $A(i_1)$ satisfying the equation $(z_n)^{2^n} = e$. We shall prove that

$$\|z_n b\| = \|b\|. \tag{10}$$

Since $A(i_1)$ is isometrically isomorphic with the complex field and $(z_n)^{2^n} = e$, we have the equation

$$\|z_n\| = 1 \quad (n = 0, 1, 2, \ldots) \tag{11}$$

We shall prove formula (10) by induction with respect to n. For $n = 0$ formula (10) is true in virtue of the equality $z_0 = e$. Now we suppose that (10) holds for every $n \leqslant k$. Since

$$[(z_{k+1})^2]^{2^k} = (z_{k+1})^{2^{k+1}} = e,$$

the element $z_k = (z_{k+1})^2$ satisfies (10). Taking into account (11), the left-alternative law and the submultiplicativity of the norm, we have the relation

$$\|b\| = \|z_{k+1}^2 b\| = \|z_{k+1}(z_{k+1} b)\| \leqslant \|z_{k+1}\| \cdot \|z_{k+1} b\|$$
$$= \|z_{k+1} b\| \leqslant \|z_{k+1}\| \cdot \|b\| = \|b\|,$$

which implies the equation $\|z_{k+1} b\| = \|b\|$. Equation (10) is thus proved From (10) and (11) it follows the equation

$$\|z_n b\| = \|z_n\| \cdot \|b\|. \tag{12}$$

Since $A(i_1)$ is isomorphic with the complex field and $(z_n)^{2^n} = e$, each element z_n is of the form

$$z_n = e \cos \frac{m}{2^n} 2\pi + i_1 \sin \frac{m}{2^n} 2\pi,$$

where m is an integer. Hence it follows that the elements z_n $(n = 0, 1, 2, \ldots)$ form a dense set in $S \frown [e, i_1]$. Thus, by the continuity of the multiplication and (12), we have the equality

$$\|zb\| = \|z\| \cdot \|b\| \tag{13}$$

for every $z \in S \frown [e, i_1]$. Since the norm $\| \; \|$ is homogeneous, i. e., $\|\alpha x\| = |\alpha| \cdot \|x\|$ for each $\alpha \in R$ and $x \in A$, equation (13) holds for any $z \in A(i_1)$. In particular, we have the equation

$$\|ab\| = \|a\| \cdot \|b\|.$$

In other words, A is an absolute-valued algebra ([2], p. 495). Therefore Theorem 2 is a direct consequence of Albert's Theorem ([3], p. 768).

Now we shall give two examples of algebras not isomorphic with the four classical algebras; they show that some assumptions of Theorem 2 are essential.

Example 1. We consider an n-dimensional $(n \geqslant 2)$, associative, commutative and normed algebra $A(e_1, e_2, \ldots, e_n)$ in which the product is defined by the formulas

$$e_r e_s = e_s e_r = \begin{cases} 0 & \text{if} \quad r \neq s, \\ e_r & \text{if} \quad r = s. \end{cases}$$

The norm of the element $x = \sum_{r=1}^{n} \alpha_r e_r$ is defined by means of the formula

$$\|x\| = \max_r |\alpha_r| \quad (r = 1, 2, \ldots, n).$$

Since the algebra A contains the idempotents e_r $(r = 1, 2, \ldots, n)$ different from the unit element $e = \sum_{r=1}^{n} e_r$, the element e is not regular (see Lemma 8). All the other assumptions of Theorem 2 are satisfied. Therefore the assumption concerning the regularity of the unit element is essential.

Example 2. Let $A(e, e_1, e_2, \ldots, e_{n-1})$ be an n-dimensional normed algebra $(n \geqslant 3)$ with ordinary Euclidean norm, where $e, e_1, \ldots, e_{n-1}$

form an orthonormal basis. The multiplication of elements is defined by means of the formulas

$$ee_r = e_r e = e_r \quad (r = 1, 2, \ldots, n-1),$$
$$e_r^2 = -e \quad (r = 1, 2, \ldots, n-1),$$
$$e_r e_s = e_s e_r = 0 \quad \text{if} \quad r \neq s.$$

It is very easy to verify that the norm is submultiplicative. But the algebra A is not one-sided alternative. Indeed,

$$e_1^2 e_2 = -e_2, \quad e_1(e_1 e_2) = 0,$$
$$e_1 e_2^2 = -e_1, \quad (e_1 e_2) e_2 = 0.$$

Consequently, one-sided alternativity of algebras is an essential condition in Theorem 2.

A normed space A is said to be *metrically homogeneous* if for any pair $x, y \in S$ there exists an isometry T of A preserving S such that $T(x) = y$. As a consequence of Theorem 2 we get the following result, which is an answer to a problem raised by K. Urbanik:

THEOREM 3. *Every metrically homogeneous finite dimensional one-sided alternative algebra is isometrically isomorphic with one of the following: the real field, the complex field, the quaternion algebra and the Cayley algebra.*

Proof. Since the unit ball K is convex, the sphere S contains at least one regular element ([5], p. 228). By Mazur and Ulam Theorem [7] each isometry T of finite dimensional linear normed space, with $T(0) = 0$, is a linear transformation. Since A is metrically homogeneous, all elements of the unit sphere are regular. In particular, the unit element is regular. Now our statement is a direct consequence of Theorem 2.

THEOREM 4. *For finite dimensional one-sided alternative normed algebras the following conditions are equivalent*:

(i) *the unit element is regular*,

(ii) *the algebra is metrically homogeneous*,

(iii) *the norm is induced by an inner product.*

Proof. The implication (iii) → (ii) is obvious, because the unit sphere is then simply an Euclidean sphere. Furthermore, in the proof of Theorem 3 we have shown that (ii) implies (i). Finally, the implication (i) → (iii) follows from Theorem 2.

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