onto the whole of \( X \) so as to remain \( \gamma \)-linear. Some sufficient conditions for extensibility are known.

**Theorem 13.** Let \( \langle X, \| \cdot \|, \| \cdot \|^* \rangle \) be a \( \gamma \)-reflective subspace of a \( \gamma \)-normal space \( \langle X, \| \cdot \| \rangle \). Then every \( \gamma \)-linear functional on \( X \) possesses a \( \gamma \)-linear extension on \( X \).

A universal space. Two two-norm spaces, \( \langle X, \| \cdot \|, \| \cdot \|^* \rangle \) and \( \langle Y, \| \cdot \|, \| \cdot \|^* \rangle \) are called \( \gamma \)-equivalent if there exists a distributive operation \( T \) from \( X \) onto \( Y \) which establishes an isometry of \( \langle X, \| \cdot \| \rangle \) and \( \langle Y, \| \cdot \| \rangle \) and, at the same time, \( T \) is a homomorphism between \( \langle X, \| \cdot \|^* \rangle \) and \( \langle Y, \| \cdot \|^* \rangle \).

Let us consider the following example: suppose we are given a linear space \( Z \) with a sequence \( \{ z_i \} \) of seminorms such that \( \| x \|_i = |x| \) for \( i = 1, 2, \ldots \) implies \( x = 0 \). Let \( Z_n = \{ x : \sup \{ |x|_i : 1 \leq i \leq n \} \} \). Then \( \langle Z_n, \| \cdot \|, \| \cdot \|^* \rangle \) is a \( \gamma \)-normal space.

In particular, let \( O \) denote the space of continuous functions \( x = x(t) \) on the half-line \( 0 \leq t < \infty \) with \( \| x \|_i = \sup \{ |x(t)| : 0 \leq t \leq i \} \). Then \( \gamma \)-convergence in the space \( \langle G_n, \| \cdot \|, \| \cdot \|^* \rangle \) means uniform boundedness plus uniform convergence on compact subsets of \( [0, \infty) \).

**Theorem 14.** Every \( \gamma \)-normal two-norm space is \( \gamma \)-equivalent to a subspace of a certain space \( \langle Z_n, \| \cdot \|, \| \cdot \|^* \rangle \).

The space \( \langle X, \| \cdot \|, \| \cdot \|^* \rangle \) is called \( \gamma \)-separable if there exists a countable set dense for the convergence \( \gamma \).

**Theorem 15.** Every \( \gamma \)-separable space is \( \gamma \)-equivalent to a subspace of the space \( \langle C_n, \| \cdot \|, \| \cdot \|^* \rangle \).

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The group of invertible elements of a commutative Banach algebra

by

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Let \( f \) be continuous, complex-valued on a compact subset \( D \) of the complex plane \( C \). Then \( f \) has the form \( f = \alpha^2 \) where \( \alpha \) is rational, and \( g \) continuous on \( D \). This classical theorem we generalize in a Banach algebra manner (see 1, below). Reformulated as in 4 (below) it represents another step along the path begun by Shilov [3] of finding algebraic invariants of a commutative Banach algebra \( A \) (over \( C \) with unit) depending only on the space \( A \) of complex-algebra homomorphisms. In a sense, Shilov shows that the cohomology homomorphism \( H^0(\Lambda, Z) \) is isomorphic to the subring of \( A \) generated by its idempotents; and we show that \( H^0(\Lambda, Z) \) is isomorphic to \( C / G \) (see 4).

Notations: \( C, \Lambda, A \) always the meaning as above. \( C^* = C - \{ 0 \} \).

**Theorem 16.** The space of continuous functions. If \( F \subset \mathcal{W}(X, C) \) then \( \mathcal{W}(X, C) \) is the space of continuous functions. If \( F \subset \mathcal{W}(X, C) \) then \( \mathcal{W}(X, C) \) is the space of continuous \( \mathcal{W}(X, C) \) valued functions on \( W, X = C \) or \( C^* \). \( f \neq 0 \) is the set where \( f \neq 0 \). For \( b \in \Lambda \) and \( \delta \in \Lambda, \delta \in \Lambda \) is \( \delta \in \Lambda, \delta \in \Lambda \) for some \( \delta \in \Lambda \).

We shall deduce this from the following mere combination of two theorems of H. Cartan's. For our notation we refer closely to [3].

**Proposition.** Let \( P_1, \ldots, P_n \) be polynomials in \( n \) complex variables, and form

\[
W = \{ (P_1, \ldots, P_n) : 1 \leq i \leq n \}.
\]

Then there is a natural isomorphism of the multiplicative groups.

We sketch the proof. For the Stein manifold \( W \) we have the exact sequence of sheaves (22), 27 (11)) \( 0 \to Z \to G \to C^* \to 0 \), and the exact
sequence (22), 35 (2.10.1), of chronology groups of \( W \),

\[ 0 \to H^i(Z) \to H^i(C_0) \to H^i(C_0') \to H^i(Z) \to H^i(C_0) \to \ldots \]

By Cartan's theorem ([2], 119), the group written last is 0, so that
\[ H^i(Z) = H^i(C_0')/\text{ex}H^i(C_0) \],
and this quotient-group is ([2], 24, 29 (2.62)) the one on the right side of 3.

But we also have ([2], 26 (9)) \( 0 \to Z \to C_0' \to C_0 \to 0 \), and so by analogous reasoning, using [2], 37 (2.11.1) (noting that \( C_0 \) is fine) we obtain Brucklin's theorem (generalized): \( H^i(Z) = \Psi(W, C_0')/\text{ex}\Psi(W, C_0) \).

A careful tracing of the isomorphism (3) shows that for \( \varphi \in \Psi(W, C_0) \) there is an \( \alpha \in \text{Hol}(W, C_0) \) and a \( \psi \in \Psi(W, C_0) \) such that \( \varphi = \alpha \psi \).

**Proof of 1.** By Stone-Weierstrass, \( f = f_1e^\varphi \) where \( f_1 = a_{1,1}g_{1,1} + \ldots + a_{1,n}g_{1,n} \), and \( g_{1,k} \varphi(A, C_0) \). Define \( \mu_k(c) = (c(a_1), \ldots, c(a_n)) \).

Then \( \mu_k(c) = (a_1, \ldots, a_n) \cap C_0 \) is the joint spectrum of these elements relative to \( A \).

Evidently \( a = a_{1,1}z_1 + \ldots + a_{n-1,n}z_n \) never vanishes on \( C_0 \). One can find \( a_{n-1} \neq a_{n,n} \) such that \( a \neq 0 \) on \( c = c(a_1, \ldots, a_n) \).

This \( a \) is the subalgebra generated by \( a_{1,1}, \ldots, a_{1,n} \), understanding \( a_{1,1}, \ldots, a_{n,n} \) as the first \( k \) coordinate-functions in \( C_0 \). From Shilov's observation ([1], 206), there exist polynomials \( P_1, \ldots, P_n \) such that \( c \cap W = \cap \{P_i < 1 \} \in C_0 \).

Thus \( 2.2 \) \( a = a \) where \( \alpha \in \text{Hol}(W, C_0) \), \( \psi \in \Psi(W, C_0) \). By the theorem of Oka-Weil [4] there is a polynomial \( P \) such that \( \max\{P(x) - a(x)\} > 0 \). Take \( \epsilon \) so small that \( P \neq 0 \) on \( \sigma \) and also \( a = a \) where \( P \neq 0 \) on \( \sigma \). Then \( a = a \) where \( a \neq \sigma \).

Thus \( a = P(a_{1,1}, \ldots, a_{1,n}) \).

Let \( a = P(a_{1,1}, \ldots, a_{1,n}) \). Then \( f_1 = f_1e^\varphi \) where \( g_{1,k} \varphi(A, C_0) \), yielding half of the lemma. Suppose now that \( a_{1,1}e^{\varphi} = b_{1,1}e^{\varphi} \) on \( A \), and, by [1], 9.1, \( ab^{-1} = e^{\varphi} \) for some \( a_{1,1}e^{\varphi} \).

**4. Theorem.** Let \( G \) be the group of invertible elements of \( A \). Then there is a subgroup \( \Gamma \) of \( G \) such that \( G = \hat{\Gamma} = \hat{\Gamma} \) where \( \hat{\Gamma} = (e^{\varphi} : a \in \Gamma) \) is the component of 1 in \( G \), and \( G/\hat{\Gamma} \to \hat{\Gamma} \to H^i(A, Z) \).

Sending \( a \) into \( a \) induces a homomorphism \( H \) of \( G/\hat{\Gamma} \) into \( \hat{\Gamma} \hat{\Gamma} \) (where \( \hat{\Gamma} = \varphi(A, C_0) \)). The lemma shows that \( H \) is “onto” and 1:1. \( G/\hat{\Gamma} \) has no elements of finite order, so \( \Gamma \) exists. Clearly \( \hat{\Gamma} = e^{\varphi} \).

Lemma 1 can be extended to commutative \( F \) algebras, by the use of ([5], 2.4).

I should like to add that when I announced my reduction of the problem to the set \( \sigma \), Professor H. L. Royden independently took up the matter and also arrived at [1]. Royden also used Cartan's Theorem B.

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**Bibliography**