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Power series as orthogonal expansions

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In the theory of functions of a complex variable are considered expansions of an analytic function f in the series

$$f = \sum_{n=0}^{\infty} a_n u_n, \tag{1}$$

where the sequence $\{u_n\}$ of analytic functions is fixed and coefficients a_n depend on f. The case of extreme importance is that of u_n being polynomials.

If we consider functions f which are analytic in a domain G, then it is natural to investigate expansions of type (1) under the assumption that u_n are orthogonal with respect to the scalar product defined by the plane integral

$$(f, g) = \iint_G f(z)\cdot\overline{g(z)}\,dx\,dy.$$

By the application of the orthogonalization procedure to the system $\{z^n\}$ it is transformed into Carleman's polynomials for the domain G. It is known that the system of Carleman's polynomials is complete in the space of all square integrable analytic functions on G (this space is denoted by $H_2'(G)$) if G is a Carathéodory's domain.

In particular, if G is the interior D of the unit circle, the system $u_n(z) = z^n$, $n = 0, 1, \ldots$, is orthogonal and complete. Let us remark that this system admits the following properties:

$$u_n \cdot u_m = u_{n+m}, \tag{2}$$

$$u_n' = n u_{n-1}. \tag{3}$$

The purpose of the present note is to prove that the conditions of type (2) or (3) characterize the system $\{z^n\}$ among other complete orthogonal systems $\{u_n\}$ in $H_2'(D)$ (u_n are not supposed to be the polynomials).

1. We shall denote by D the interior of the unit circle. The set of all functions analytic in D and square integrable (i. e. $\iint\limits_D |f(z)|^2 dx\,dy < \infty$) will be denoted by H_2'. It is a Hilbert space with respect to the scalar product

$$(f, g) = \iint\limits_D f(z)\cdot\overline{g(z)}\,dx\,dy.$$

In the polar coordinates the above formula assumes the form

$$(f, g) = \int\limits_0^1\int\limits_0^{2\pi} f(re^{i\theta})\cdot\overline{g(re^{i\theta})}\,r\,d\theta\,dr.$$

If the functions $f, g \in H_2'$ are represented as power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

then we have formula

$$(f, g) = \pi\sum_{n=0}^{\infty}\frac{a_n\cdot\bar{b}_n}{n+1}, \tag{4}$$

and, in particular,

$$\|f\|^2 = \pi\sum_{n=0}^{\infty}\frac{|a_n|^2}{n+1}.$$

Then it is easy to see that $f' \in H_2'$ implies $f \in H_2'$.

If λ, μ are arbitrary complex numbers, then

$$\iint\limits_D e^{\lambda z}\cdot\overline{e^{\mu z}}\,dx\,dy = \pi F(\lambda\bar{\mu}),$$

where the function F is defined by the series

$$F(z) = \sum_{n=0}^{\infty}\frac{z^n}{n!(n+1)!}.$$

It is known (see [1]) that the convergence in H_2' is stronger than almost uniform convergence. Under some additional conditions both types of convergence are equivalent for partial sums of an orthogonal expansion. We have

LEMMA 1. *If $\{u_n\}$ is a system of orthogonal functions in H_2', any u_n, is analytic in the closed circle $|z| \leqslant 1$, $f \in H_2'$, and $f(z) = \sum_{n=0}^{\infty} a_n u_n(z)$ (the series being almost uniformly convergent), then $a_n\|u_n\|^2 = (f, u_n)$, and the series is convergent in the mean.*

Proof. Let us write $s_m(z) = \sum_{k=0}^{m} a_k u_k(z)$ and, for a fixed n, $u_n(z) = \sum_{k=0}^{\infty} \alpha_k z^k$; then by formula (4)

$$(s_m, u_n) = \pi\sum_{k=0}^{\infty}\frac{s_m^{(k)}(0)\bar{\alpha}_k}{(k+1)!} = \frac{1}{2i}\sum_{k=0}^{\infty}\frac{\bar{\alpha}_k}{k+1}\int\limits_K\frac{s_m(z)}{z^{k+1}}\,dz,$$

where K is such a circumference $|z| = \varrho < 1$ that the series

$$\sum_{k=0}^{\infty}\frac{|\alpha_k|}{k+1}\varrho^{-k}$$

is convergent. If $|s_m(z) - f(z)| < \varepsilon$ for $|z| = \varrho$, $m \geqslant M$, then

$$|(s_m, u_n) - (f, u_n)| \leqslant \varepsilon\pi\sum_{k=0}^{\infty}\frac{|\alpha_k|}{k+1}\varrho^{-k},$$

and consequently

$$(f, u_n) = \lim_{m\to\infty}(s_m, u_n).$$

Since $(s_m, u_n) = a_n\|u_n\|^2$ for $m \geqslant n$, $(f, u_n) = a_n\|u_n\|^2$. The function f belongs to H_2', so that

$$\sum_{n=0}^{\infty}|a_n|^2\|u_n\|^2 < \infty$$

and this implies the last part of the Lemma.

2. In this section we prove the following

THEOREM 1. *If $\{u_n\}$ is an orthogonal and complete* (1) *in H_2' system consisting of bounded analytic functions, and the system satisfies the condition*

(i) *for any pair of indices n, m there exist an index k and a complex number λ_{nm} such that $u_n\cdot u_m = \lambda_{nm}\cdot u_k$,*

then the system $\{u_n\}$ differs from $\{z^n\}$ at most by ordering and by numerical coefficients.

Proof. The functions u_n are linearly independent, so that the index k in (i) is uniquely determined by n, m: $k = \varrho(n, m)$, and, moreover, $\lambda_{nm} \neq 0$. If $\varrho(n, m) = \varrho(n_1, m)$, then we have $(\lambda_{n_1m}u_n - \lambda_{nm}u_{n_1})\cdot u_m = 0$, and $n = n_1$ in virtue of the linear independence of the analytic functions u_n.

(1) i. e. if $f \in H_2'$ and $(f, u_n) = 0$, $n = 0, 1, \ldots$, then $f = 0$.

Let an expansion of the function 1 be

$$1 = \sum_{n=0}^{\infty} a_n u_n;$$

this implies for an arbitrary m

$$u_m = \sum_{n=0}^{\infty} a_n u_n u_m = \sum_{n=0}^{\infty} a_n \lambda_{nm} u_{\varrho(n,m)},$$

and the series is convergent in the mean, because u_m is bounded. Consequently, for some n_0 we have $m = \varrho(n_0, m)$. Then $u_{n_0} u_m = \lambda_{n_0 m} u_m$ and $u_{n_0}(z) = \lambda_{n_0 m}$ for all $z \in D$. To simplify the notation we can assume that $n_0 = 0$ and $\lambda_{n_0 m} = 1$; then using (4) we have $u_n(0) = \pi^{-1}(u_n, 1) = \pi^{-1}(u_n, u_0) = 0$ for all $n > 0$.

Let us consider an expansion

$$z = \sum_{n=0}^{\infty} \beta_n u_n(z);$$

by the preceding formula we have $\beta_0 = 0$, and by differentiation we get

$$1 = \sum_{n=1}^{\infty} \beta_n u_n'(0).$$

Consequently $u_{n_1}'(0) \neq 0$ for at least one index $n_1 > 0$. We write $u = u_{n_1}$.

We shall prove that *for any positive integer p there exists in the sequence $\{u_n\}$ precisely one function which has 0 as its p-multiple value in $z = 0$.*

In fact, the function u^p differs from some u_k by a non-zero coefficient and has 0 as its p-multiple value in $z = 0$. On the other hand, if $u_r(z) = z^p v(z)$, $u_s(z) = z^p w(z)$ and $v(0) \neq 0 \neq w(0)$, then $v u_s = w u_r$ and we have expansions

$$v = \sum_{n=0}^{\infty} \gamma_n u_n, \quad w = \sum_{n=0}^{\infty} \delta_n u_n \quad \text{with} \quad \gamma_0 \neq 0 \neq \delta_0.$$

Consequently we get

$$\sum_{n=0}^{\infty} \gamma_n \lambda_{ns} u_{\varrho(n,s)} = \sum_{n=0}^{\infty} \delta_n \lambda_{nr} u_{\varrho(n,r)}$$

and the terms with $n = 0$ are $\gamma_0 u_s$, $\delta_0 u_r$, respectively. The functions $\varrho(n, r)$, $\varrho(n, s)$ are one-to-one as functions of n; consequently there

exist integers k, l such that $s = \varrho(k, r)$, $r = \varrho(l, s)$, and this implies $u_k u_r = \lambda_{kr} u_s$, $u_l u_s = \lambda_{ls} u_r$. Hence we get $\lambda_{kr} u_s = \lambda_{ls}^{-1} u_k u_l u_s$ or $(\lambda_{kr}\lambda_{ls} - u_k u_l) u_s = 0$, and consequently $u_k u_l = \lambda_{kr}\lambda_{ls}$. We have $\lambda_{kr}\lambda_{ls} \neq 0$ and $u_n(0) = 0$ for $n > 0$; so $k = l = 0$ and $s = r$.

We have proved above that $u'(0) \neq 0$ for some $u = u_{n_1}$; in virtue of the preceding considerations it follows that the sequences $\{u_n\}$ and $\{u^n\}$ differ by ordering and numerical coefficients. Consequently, to complete the proof of Theorem 1 it is sufficient to prove that $u(z) = cz$ for some complex number $c \neq 0$.

The function z may be represented as

$$z = \sum_{k=0}^{\infty} \varepsilon_n u^n(z),$$

and hence it follows that if $u(z_1) = u(z_2)$, then $z_1 = z_2$; thus u is a univalent function.

Let us write $E = u(D)$; the set E is a bounded domain and $0 \in E$. Let us consider a new variable $\zeta = u(z)$ with $z \in D$, $\zeta \in E$, and let v be the inverse function of u. Then we have

$$\iint_D u^n(z)\overline{u^m(z)}\,dx\,dy = \iint_E \zeta^n \overline{\zeta^m} |v'(\zeta)|^2 d\xi\, d\eta$$

with $\zeta = \xi + i\eta$. Using Fubini's theorem we get for sufficiently large R and $n \neq m$

$$\int_0^R r^{n+m+1}\Big[\int_0^{2\pi} e^{(n-m)i\theta} |v'(re^{i\theta})|^2 \chi(r, \theta)\, d\theta\Big] dr = 0$$

with χ being the characteristic function of the set E. Consequently the functions

$$\varphi_k(r) = \int_0^{2\pi} e^{ki\theta} |v'(re^{i\theta})|^2 \chi(r, \theta)\, d\theta$$

are defined a. e., integrable and for all $k \neq 0$

$$\int_0^R r^{2m+k+1} \varphi_k(r)\, dr = 0 \qquad (m = 0, 1, \ldots).$$

By the well-known theorem on moments it follows, that $\varphi_k(r) = 0$ a. e. for $k \neq 0$. This means that for almost all r the Fourier coefficients of $|v'(re)^{i\theta})|^2 \chi(r, \theta)$ (considered as a function of θ) are all 0 except the case with index 0. For sufficiently small r we have $\chi(r, \theta) = 1$, and then there exists a function ψ such that $|v'(re^{i\theta})|^2 = \psi(r)$ for almost all (sufficiently small) r, and if r is fixed, for almost all θ. The function v' is an analytic one, so that $|v'(\zeta)|^2 = \psi(|\zeta|)$ for all sufficiently small ζ, and thus $v'(\zeta) = c^{-1}\zeta^t$. Since $v(0) = 0$ and v is a univalent function, $u(z) = cz$, which completes the proof.

3. In this section we prove the following

THEOREM 2. *If $\{u_n\}$ is an orthogonal and complete in H_2' system consisting of functions analytic in the closed circle $|z| \leqslant 1$, and the system satisfies the condition*

(ii) *for any index n there exists an index k and a complex number λ_n such that $u_n' = \lambda_n u_k$,*

then the system $\{u_n\}$ differs from $\{z^n\}$ at most by ordering and by numerical coefficients.

Let us first prove the following

LEMMA 2. *If all assumptions of Theorem 2 are satisfied, then a constant (non-zero) function belongs to the system $\{u_n\}$.*

Proof. Let us suppose that all functions u_n are non-constant. We denote by $\varrho(n)$ the uniquely determined index such that $u_n' = \lambda_n u_{\varrho(n)}$. Then the function ϱ has the following properties:

(a) any non-negative integer k is of the form $k = \varrho(n)$,

(b) there exists precisely one pair of indices $n, m, m \neq n$, such that $\varrho(n) = \varrho(m)$,

(c) if $\varrho(n) = \varrho(m)$ and $n \neq m$, then $\varrho(n) \neq n$ and $\varrho(m) \neq m$.

To prove (a) let v be such an analytic function that $v' = u_k$; then $v = \sum_{n=0}^{\infty} a_n u_n$ and by almost uniform convergence of this series

$$u_k(z) = v'(z) = \sum_{n=0}^{\infty} a_n u_n'(z) = \sum_{n=0}^{\infty} a_n \lambda_n u_{\varrho(n)}(z).$$

By Lemma 1 it follows that $k = \varrho(n)$ for some n.

To prove (b) let

$$1 = \sum_{n=0}^{\infty} \beta_n u_n(z);$$

then by differentiation we get $0 = \sum_{n=0}^{\infty} \beta_n \lambda_n u_{\varrho(n)}(z)$ and since not all coefficients $\beta_n \lambda_n$ are zero, it follows by Lemma 1 that ϱ is not one-to-one. On the other hand, if $\varrho(n) = \varrho(m) = k$, $n \neq m$, then $u_n' = \lambda_n u_k$, $u_m' = \lambda_m u_k$ and $\lambda_m u_n - \lambda_n u_m$ is a constant non-zero function. Consequently n, m is the unique pair of indices such that 1 is a linear form of u_n and u_m.

To prove (c) let us suppose $\varrho(n) = \varrho(m) = m \neq n$. Since $u_n' = \lambda_n u_m$, $u_m' = \lambda_m u_m$, we get $u_m(z) = ae^{\lambda_m z}$, $u_n(z) = b(e^{\lambda_m z} + c)$ with a, b, c being non-zero. By (a) it follows that $n = \varrho(l)$ for some integer $l \neq n, m$, and thus $u_l' = \lambda_l u_n = \lambda_l b(e^{\lambda_m z} + c)$. The function u_l is orthogonal to both u_n, u_m, it is also orthogonal to any constant, and from (4) we get $u_l(0) = 0$. Hence u_l can be represented as $u_l(z) = d(e^{\lambda_m z} + c\lambda_m z - 1)$, $d \neq 0$. By the relation $(u_n, u_m) = (u_l, u_m) = 0$ we get

$$F(|\lambda_m|^2) + c = 0, \qquad F(|\lambda_m|^2) + \tfrac{1}{2}c|\lambda_m|^2 - 1 = 0$$

(F being the function defined in the section 1) and hence

$$F(|\lambda_m|^2) = \frac{2}{2 - |\lambda_m|^2}.$$

We have $|\lambda_m|^2 < 2$ because $F(|\lambda_m|^2) > 0$, and consequently

$$\sum_{n=0}^{\infty} \frac{|\lambda_m|^{2n}}{n!(n+1)!} = \sum_{n=0}^{\infty} \frac{|\lambda_m|^{2n}}{2^n},$$

which is impossible, because $\lambda_m \neq 0$ and $1/n!(n+1)! < 1/2^n$ for all $n > 0$. Then the proof of (c) is complete.

By (a), (b) we can assume that $\varrho(1) = \varrho(2) = 0$ and we shall consider two cases:

(α) for any positive integer q we have $\varrho^q(0) \neq 0$,

(β) there exists a positive integer q such that $\varrho^q(0) = 0$.

If (α) holds, then there exist two sequences of integers $\{n_k\}$, $\{m_k\}$, $k = 1, 2, \ldots$, such that $n_1 = 1$, $m_1 = 2$, $\varrho(n_{k+1}) = n_k$, $\varrho(m_{k+1}) = m_k$. It is easy to see that all terms of these sequences are different and positive.

By the relation $\varrho(1) = \varrho(2) = 0$ it follows that for some number $a \neq 0$ we have

$$\lambda_2 u_1 - \lambda_1 u_2 = a. \tag{5}$$

Since $n_k > 1$, $m_k > 1$ for $k > 1$, by the orthogonality of functions u_n it follows that $u_{n_k}(0) = u_{m_k}(0) = 0$, $k = 2, 3, \ldots$ Integrating both sides of (5) we get

$$\mu_k u_{n_k}(z) - \nu_k u_{m_k}(z) = \frac{1}{(k-1)!} a z^{k-1} \qquad (k = 1, 2, \ldots),$$

μ_k, ν_k being some complex numbers. By this formula it follows that all integrals

$$\iint_D u_0(z) \bar{z}^r \, dx\, dy \qquad (r = 0, 1, \ldots)$$

are zero and so $u_0 = 0$, which is impossible.

If (β) holds, then let q be the least positive integer such that $\varrho^q(0)=0$; we have $q>1$ in view of (b). By our assumption it follows that all the integers $r_0=0$, $r_1=\varrho(0),\ldots,r_{q-1}=\varrho^{q-1}(0)$, are different, and r_{q-1} is 1 or 2. We can assume, of course, that $r_{q-1}=1$; we have moreover $u_0^{(q)}=b^q u_0$ for some complex number $b\neq 0$ and from the orthogonality of the functions $u_{r_0}, u_{r_1},\ldots,u_{r_{q-2}}$ to $\lambda_2 u_1-\lambda_1 u_2=a$ it follows that $u_0(0)=0$, $u_0'(0)=0,\ldots,u_0^{(q-2)}(0)=0$, $u_0^{(q-1)}(0)=c$ with $c\neq 0$. It is easy to check that the function u_0 is given by the formula

$$u_0(z)=cb^{1-q}q^{-1}\sum_{j=0}^{q-1}\varepsilon_j e^{\varepsilon_j bz}$$

with $\varepsilon_j=\varepsilon_1^j$, $j=0,1,\ldots,(q-1)$, being q-th roots of 1. As the functions u_{r_k} differ only by numerical coefficients from the functions

$$v_k(z)=cb^{k-q+1}q^{-1}\sum_{j=0}^{q-1}\varepsilon_j^{k+1}e^{\varepsilon_j bz},\qquad k=0,1,\ldots,(q-1),$$

we have by the relation of orthogonality

$$(v_{q-1},u_2)=(v_{q-2},u_p)=0 \tag{6}$$

with p satisfying $\varrho(p)=2$.

Since $v'_{q-2}=v_{q-1}$, $v_{q-1}=au_1$, $u'_p=\lambda_p u_2$ and $u_2=\lambda_1^{-1}(\lambda_2 u_1-a)=\lambda_1^{-1}(\beta v_{q-1}-a)$ with $\beta=\lambda_2 a^{-1}$, we have $u_p(z)=\lambda_p\lambda_1^{-1}[\beta v_{q-2}(z)-az]$ because $u_p(0)=0$ and $v_{q-2}(0)=0$.

Using formulae (6) we get

$$\bar\beta(v_{q-1},v_{q-1})-\bar a c\pi=0,$$

$$\bar\beta(v_{q-2},v_{q-2})-\bar a c\frac{\pi}{2}=0,$$

because

$$\iint_D v_{q-2}(z)\bar z\,dx\,dy=c\frac{\pi}{2}.$$

Consequently we have $2(v_{q-2},v_{q-2})=(v_{q-1},v_{q-1})$ and by a simple computation we get

$$\sum_{n=0}^{\infty}\frac{2}{(nq+1)!(nq+2)!}|b|^{2nq}=\sum_{n=0}^{\infty}\frac{1}{(nq)!(nq+1)!}|b|^{2nq},$$

which is impossible, because $b\neq 0$. The proof of the Lemma 2 is complete.

Proof of the Theorem 2. By Lemma 2 we can assume that $u_0(z)=1$. Let us denote by $\varrho(n)$ $(n=1,2,\ldots)$ such an index that $u'_n=\lambda_n u_{\varrho(n)}$ and $\varrho(0)=0$. In the same way as in the proof of Lemma 2 we can show that

(d) any non-negative integer k is of the form $k=\varrho(n)$,

(e) if $\varrho(n)=\varrho(m)$ and $n\neq m$, then $n=0$ or $m=0$.

By the above properties of the function ϱ it follows that there exists the unique sequence of different integers $\{n_k\}$ such that $n_0=0$, $\varrho(n_{k+1})=n_k$, $k=1,2,\ldots$ In fact, by the orthogonality of u_n with u_0 it follows that $u_n(0)=0$ for $n\geqslant 1$. If

$$z=\sum_{n=0}^{\infty}\beta_n u_n(z),$$

then $\beta_0=0$ and

$$1=\sum_{n=0}^{\infty}\beta_n\lambda_n u_{\varrho(n)}(z)$$

(by Lemma 2 the series is convergent in the mean). Consequently $\varrho(n_1)=0$ for some $n_1\geqslant 1$.

In view of the properties of the sequence $\{n_k\}$ it follows that $u_{n_k}(z)=\gamma_k z^k$ for some complex numbers $\gamma_k\neq 0$. From the completeness of $\{z^k\}$ it follows that the sequence $\{n_k\}$ exhausts all non-negative integers, which completes the proof of Theorem 2.

Added in proof. As was pointed out to me by Dr S. Rolewicz, multiplicative systems of orthogonal functions of a real variable were extensively studied in [2].

References

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