but it follows from (A) of the lemma and $g(z_n) = 0$ that

$$G(z_n) = \sum_1^\infty \beta_i g_i(z_n) > \frac{79}{80} \sum_1^n \beta_i - \sum_{n+1}^\infty \beta_i \to \frac{79}{80} \sum_1^\infty \beta_i \quad \text{as} \quad n \to \infty.$$

Therefore

$$\frac{79}{80} \sum_1^\infty \beta_i \leqslant G(u) \leqslant \sum_1^\infty \beta_i - \frac{1}{4},$$

which implies that $\sum_1^\infty \beta_i \geqslant 20$. However, $\sum_1^\infty \beta_i < 10$.

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HARVEY MUDD COLLEGE, CLAREMONT, CALIFORNIA
THE INSTITUTE FOR ADVANCED STUDY, PRINCETON, N. J.

Reçu par la Rédaction le 24. 11. 1962

STUDIA MATHEMATICA, T. XXIII. (1964)

Generalized convolutions

by

K. URBANIK (Wrocław)

Introduction

Let $\mathfrak{P}$ be the class of all probability measures defined on Borel subsets of the positive half-line. By E_a $(a \geqslant 0)$ we shall denote the probability measure concentrated at the point a. For any positive number a we define a transformation T_a of $\mathfrak{P}$ onto itself by means of the formula $(T_a P)(\mathscr{A}) = P(a^{-1}\mathscr{A})$, where $P \in \mathfrak{P}$, $\mathscr{A}$ is a Borel set and $a^{-1}\mathscr{A} = \{a^{-1}x: x \in \mathscr{A}\}$. Of course, the family T_a $(a > 0)$ forms a group under composition and $T_a T_b = T_{ab}$ $(a, b > 0)$. Further, we define the transformation T_0 by assuming $T_0 P = E_0$ for all P from $\mathfrak{P}$. It is very easy to verify that for every bounded continuous function f the equation

$$(1) \qquad \int_0^\infty f(x)(T_a P)(dx) = \int_0^\infty f(ax) P(dx) \qquad (a \geqslant 0,\, P \in \mathfrak{P})$$

holds.

We say that a sequence $P_1, P_2, \ldots$ of probability measures is *weakly convergent* to a probability measure P, in symbols $P_n \to P$, if for every bounded continuous function f the equation

$$\lim_{n\to\infty} \int_0^\infty f(x) P_n(dx) = \int_0^\infty f(x) P(dx)$$

holds. From this definition of weak convergence and from (1) it follows that

(*) $T_{a_n} P_n \to T_a P$ whenever $a_n \to a$ and $P_n \to P$.

In particular,

(**) if $a_n \to 0$ and $P_n \to P$, then $T_{a_n} P_n \to E_0$.

A commutative and associative $\mathfrak{P}$-valued binary operation $\circ$ defined on $\mathfrak{P}$ is called a *generalized convolution* if it satisfies the following conditions:

(i) the measure E_0 is a unit element, i.e. $E_0 \circ P = P$ for all $P \in \mathfrak{P}$;

(ii) $(aP+bQ)\circ R = a(P\circ R)+b(Q\circ R)$, whenever $P, Q, R \in \mathfrak{P}$ and $a \geqslant 0$, $b \geqslant 0$, $a+b = 1$ (linearity);

(iii) $(T_aP)\circ(T_aQ) = T_a(P\circ Q)$ for any $P, Q \in \mathfrak{P}$ and $a > 0$ (homogeneity);

(iv) if $P_n \to P$, then $P_n\circ Q \to P\circ Q$ for all $Q \in \mathfrak{P}$ (continuity);

(v) there exists a sequence $c_1, c_2, \ldots$ of positive numbers such that the sequence $T_{c_n}E_1^{\circ n}$ weakly converges to a measure different from E_0 (the law of large numbers for measures concentrated at a single point).

The power $E_a^{\circ n}$ is taken in the sense of the operation $\circ$, i.e. $E_a^{\circ 1} = E_a$, $E_a^{\circ(n+1)} = E_a^{\circ n}\circ E_a$ $(n = 1, 2, \ldots)$.

Now we shall quote some simple examples of generalized convolutions. In all examples generalized convolutions $P\circ Q$ will be defined by means of the functional $\int_0^\infty f(x)(P\circ Q)(dx)$ on all bounded continuous functions f.

1. *a-convolution* $(0 < a < \infty)$:

$$\int_0^\infty f(x)(P\circ Q)(dx) = \int_0^\infty\int_0^\infty f\big((x^a+y^a)^{1/a}\big)P(dx)Q(dy). \tag{2}$$

For $a = 1$ we obtain the ordinary convolution. We note that the sequence $c_n = n^{-1/a}$ satisfies condition (v) and E_1 is the weak limit of the sequence $T_{c_n}E_1^{\circ n}$.

2. *∞-convolution*:

$$\int_0^\infty f(x)(P\circ Q)(dx) = \int_0^\infty\int_0^\infty f\big(\max(x, y)\big)P(dx)Q(dy). \tag{3}$$

Since $E_1^{\circ n} = E_1$ $(n = 1, 2, \ldots)$, the sequence $c_n = 1$ $(n = 1, 2, \ldots)$ satisfies (v).

3. *$(a, 1)$-convolution* $(0 < a < \infty)$:

$$\int_0^\infty f(x)(P\circ Q)(dx) = \frac{1}{2}\int_0^\infty\int_0^\infty \big[f\big((x^a+y^a)^{1/a}\big)+f\big(|x^a-y^a|^{1/a}\big)\big]P(dx)Q(dy). \tag{4}$$

4. *(a, β)-convolution* $(0 < a < \infty,\ 1 < \beta < \infty)$:

$$\int_0^\infty f(x)(P\circ Q)(dx) = \frac{\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta-1}{2}\right)\sqrt{\pi}}\int_0^\infty\int_0^\infty\int_{-1}^1 f\big((x^{2a}+y^{2a}+2x^ay^az)^{1/2a}\big)(1-z^2)^{(\beta-3)/2}\,dz\,P(dx)Q(dy). \tag{5}$$

Taking $c_n = n^{-1/2a}$ for (a, β)-convolution $(0 < a < \infty,\ 1 \leqslant \beta < \infty)$ we obtain the probability measure

$$P(A) = \frac{2a\left(\frac{\beta}{2}\right)^{\beta/2}}{\Gamma\left(\frac{\beta}{2}\right)}\int_A x^{a+\beta-2}\exp\left(-\frac{\beta}{2}x^{2a}\right)dx$$

as the weak limit of the sequence $T_{c_n}E_1^{\circ n}$.

The $(1, \beta)$-convolutions for $\beta \geqslant 1$ were considered by Kingman [3]. The aim of this paper is to extend some Kingman's results on generalized convolutions. We shall give a necessary and sufficient condition for the existence of an analogue of characteristic functions associated with a generalized convolution. Moreover, we shall discuss some problems concerning infinitely decomposable and stable probability measures.

2. Generalized convolution algebras

The class $\mathfrak{P}$ with a generalized convolution $\circ$ will be called a *generalized convolution algebra* and denoted by $(\mathfrak{P}, \circ)$. A continuous mapping h of $\mathfrak{P}$ into the real field is called a *homomorphism* of the algebra $(\mathfrak{P}, \circ)$ if $h(aP+bQ) = ah(P)+bh(Q)$, whenever $a \geqslant 0$, $b \geqslant 0$, $a+b = 1$, and $h(P\circ Q) = h(P)h(Q)$ for all $P, Q \in \mathfrak{P}$. Of course, each generalized convolution algebra admits two trivial homomorphisms $h(P) \equiv 0$ and $h(P) \equiv 1$. Algebras admitting a non-trivial homomorphism are called *regular*.

Consider an a-convolution algebra $(0 < a < \infty)$. The mapping

$$h(P) = \int_0^\infty \exp(-x^a)P(dx) \tag{6}$$

is linear and continuous. Moreover, by (2), it satisfies the equation

$$h(P\circ Q) = \int_0^\infty\int_0^\infty \exp(-x^a-y^a)P(dx)Q(dy) = h(P)h(Q)$$

and, consequently, is a non-trivial homomorphism of the a-convolution algebra. In other words, a-convolution algebras $(0 < a < \infty)$ are regular. However, ∞-convolution algebra is not regular. Indeed, by (3), we have $E_a\circ E_a = E_a$ $(a \geqslant 0)$. Hence for any homomorphism h we obtain the equation $h^2(E_a) = h(E_a\circ E_a) = h(E_a)$, which implies $h(E_a) = 0$ or 1. Thus, by continuity of the homomorphism h, we have either $h(E_a) = 0$ for all $a \geqslant 0$ or $h(E_a) = 1$ for all $a \geqslant 0$. Furthermore, either $h(P) = 0$ for all convex linear combinations $P = \sum_{k=1}^n b_kE_{a_k}$ $(b_j \geqslant 0,\ j = 1, \ldots, n;\ \sum_{k=1}^n b_k = 1)$

or $h(P)=1$ for all convex linear combinations P. Since these convex linear combinations form a dense (in the sense of weak convergence) subset of $\mathfrak{P}$, the continuity of h implies that the homomorphism h is trivial.

The (a,β)-convolution algebras are regular. A non-trivial homomorphism can be constructed as follows. Put

$$g_\beta(x)=\begin{cases}\cos x & \text{if}\quad \beta=1,\\ \dfrac{1}{\sqrt{\pi}}\displaystyle\int_{-1}^{1}(1-t^2)^{(\beta-3)/2}\cos xt\,dt & \text{if}\quad \beta>1.\end{cases}\tag{7}$$

The function g_β can be also written in the form

$$g_\beta(x)=\Gamma\left(\frac{\beta}{2}\right)\left(\frac{2}{x}\right)^{\beta/2-1}J_{\beta/2-1}(x)\qquad(\beta\geqslant 1),$$

where J_ν is the Bessel function. The non-trivial mapping

$$h(P)=\int_0^\infty g_\beta(x^a)P(dx)\tag{8}$$

is obviously linear and continuous. If $\beta=1$, then we have, by virtue of (4),

$$h(P\circ Q)=\frac{1}{2}\int_0^\infty\int_0^\infty[\cos(x^a+y^a)+\cos|x^a-y^a|]P(dx)Q(dy)|$$

$$=\int_0^\infty\int_0^\infty\cos x^a\cos y^a P(dx)Q(dy)=h(P)h(Q).$$

Thus h is a non-trivial homomorphism of $(a,1)$-convolution algebra.

Now consider the case $\beta>1$. From a well-known formula concerning Bessel functions (see [4], formula 8.19.3, p. 243) for any pair u,v of positive numbers we get

$$\int_{-1}^{1}g_\beta\big((u^2+v^2+2uvz)^{1/2}\big)(1-z^2)^{(\beta-3)/2}dz=\frac{\Gamma\big(\frac12(\beta-1)\big)\sqrt{\pi}}{\Gamma(\beta/2)}g_\beta(u)g_\beta(v).$$

Hence and from (5) it follows that

$$h(P\circ Q)$$

$$=\frac{\Gamma(\beta/2)}{\sqrt{\pi}\Gamma\big(\frac12(\beta-1)\big)}\int_0^\infty\int_0^\infty\int_{-1}^{1}g_\beta\big((x^{2a}+y^{2a}+2x^ay^az)^{1/2}\big)(1-z^2)^{(\beta-3)/2}dz P(dx)Q(dy)$$

$$=\int_0^\infty\int_0^\infty g_\beta(x)g_\beta(y)P(dx)Q(dy)=h(P)h(Q).$$

Thus h is a non-trivial homomorphism.

We say that an algebra $(\mathfrak{P},\circ)$ admits a *characteristic function* if there exists one-to-one correspondence $P\leftrightarrow\Phi_P$ between probability measures P from $\mathfrak{P}$ and real-valued functions Φ_P defined on the positive half-line such that $\Phi_{aP+bQ}=a\Phi_P+b\Phi_Q$ $(a\geqslant 0,\ b\geqslant 0,\ a+b=1)$, $\Phi_{P\circ Q}=\Phi_P\Phi_Q$, $\Phi_{T_aP}(t)=\Phi_P(at)$ $(a\geqslant 0,\ t\geqslant 0)$ and the uniform convergence in every finite interval of Φ_{P_n} is equivalent to the weak convergence of P_n. The function Φ_P will be called the *characteristic function* of the probability measure P in the algebra $(\mathfrak{P},\circ)$. The characteristic function in generalized convolution algebras plays the same fundamental role as in ordinary convolution algebra, i. e. in classical problems concerning the addition of independent random variables.

Suppose that Φ_P is a characteristic function in a generalized convolution algebra. It is very easy to see that $\Phi_{E_0}(t)\equiv 1$. Let P_0 be a probability measure different from E_0. Since the correspondence between characteristic functions and probability measures is one-to-one, we infer that there exists a number t_0 such that $\Phi_{P_0}(t_0)\neq 1$. Setting $h(P)=\Phi_P(t_0)$ for any $P\in\mathfrak{P}$, we obtain a non-trivial homomorphism of the algebra in question. Thus each generalized convolution algebra admitting a characteristic function is regular. We shall prove in section 4 that the converse theorem is also true. The proof of this theorem will be based on some fundamental properties of homomorphisms in generalized convolution algebras, which will be proved in the next section.

3. Properties of homomorphisms

THEOREM 1. *For any homomorphism h of an algebra $(\mathfrak{P},\circ)$ we have $|h(P)|\leqslant 1$ $(P\in\mathfrak{P})$.*

Proof. Contrary to this, let us suppose that there exists a probability measure Q such that $c=|h(Q)|>1$. Put

$$P_n=c^{-2n}Q^{\circ 2n}+(1-c^{-2n})E_0\qquad(n=1,2,\ldots).\tag{9}$$

Of course, $P_n\in\mathfrak{P}$ and $P_n\to E_0$ as $n\to\infty$. Consequently,

$$\lim_{n\to\infty}h(P_n)=h(E_0).$$

Since $h(Q)\neq 0$ and $h(Q)=h(Q\circ E_0)=h(Q)h(E_0)$, we have $h(E_0)=1$. Thus

$$\lim_{n\to\infty}h(P_n)=1.\tag{10}$$

However, by (9), we have

$$h(P_n)=c^{-2n}h(Q^{\circ 2n})+(1-c^{-2n})h(E_0)=c^{-2n}\big(h(Q)\big)^{2n}+1-c^{-2n}=2-c^{-2n},$$

which implies $\lim_{n\to\infty} h(P_n) = 2$. But this contradicts (10). The theorem is thus proved.

THEOREM 2. *For any homomorphism h of an algebra $(\mathfrak{P}, \circ)$ we have*

$$h(P) = \int_0^\infty h(E_x)P(dx) \qquad (P \in \mathfrak{P}). \tag{11}$$

Proof. By continuity of the homomorphism h and Theorem 1, the function $h(E_x)$ of the variable x is bounded and continuous on the positive half-line. By linearity of the homomorphism h, formula (11) holds for every convex linear combination of measures E_a $(a \geqslant 0)$, i. e. for every measure P of the form $P = \sum_{k=1}^{n} b_k E_{a_k}$, where $b_j \geqslant 0$ $(j = 1, 2, \ldots)$ and $\sum_{k=1}^{n} b_k = 1$. Since the set of all such measures is dense in the sense of the weak convergence in $\mathfrak{P}$, we obtain (11) for all probability measures from $\mathfrak{P}$.

THEOREM 3. *Let h be a non-trivial homomorphism of an algebra $(\mathfrak{P}, \circ)$ and let $P \in \mathfrak{P}$. If $h(T_a P) = 1$ for all $a \geqslant 0$, then $P = E_0$.*

Proof. Since the homomorphism h is non-trivial, we infer, by Theorem 2, that there exists a non-negative number x_0 such that

$$h(E_{x_0}) \neq 1. \tag{12}$$

Further, by Theorem 2, the equation $h(T_a P) = 1$ can be written in the form

$$\int_0^\infty h(E_x)(T_a P)(dx) = \int_0^\infty h(E_{ax})P(dx) = 1.$$

Hence for all $a \geqslant 0$ we get

$$\int_0^\infty \big(1 - h(E_{ax})\big)P(dx) = 0.$$

Since, by Theorem 1, the integrand is non-negative, the last equation for each $a \geqslant 0$ implies $h(E_{ax}) = 1$ P-almost everywhere. Hence it follows that for any denumerable dense subset $a_1, a_2, \ldots$ of the positive half-line the equation $h(E_{a_n x}) = 1$ holds for P-almost all x and for all n. If P is not concentrated at the origin, then there exists a positive number x_1 such that the equation $h(E_{a_n x_1}) = 1$ holds for all n. Hence, by continuity of h, we have $h(E_{a x_1}) = 1$ for all $a \geqslant 0$, which, of course, contradicts (12). Thus the measure P is concentrated at the origin and, consequently, $P = E_0$.

THEOREM 4. *Let h be a non-trivial homomorphism of an algebra $(\mathfrak{P}, \circ)$, P a probability measure and $c_1, c_2, \ldots$ a sequence of positive numbers. If the sequence $T_{c_n} P^{\circ n}$ weakly converges to a probability measure Q satisfying the inequality $Q \neq E_0$, then*

$$\lim_{n\to\infty} c_n = 0, \tag{13}$$

$$\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = 1 \tag{14}$$

and there exist positive numbers c and λ such that

$$h(T_a Q) = \exp(-c a^\lambda) \qquad (a \geqslant 0). \tag{15}$$

Proof. Contrary to (13) let us suppose that there exists a subsequence $c_{k_1}, c_{k_2}, \ldots$ of the sequence $c_1, c_2, \ldots$ such that

$$\lim_{n\to\infty} c_{k_n}^{-1} = b < \infty.$$

Thus for any non-negative number a we have, by (*), the relation

$$(T_a P)^{\circ k_n} = T_{a c_{k_n}^{-1}} (T_{c_{k_n}} P)^{\circ k_n} \to T_{ab} Q.$$

Hence it follows that for any $a \geqslant 0$ the limit $\lim_{n\to\infty} h(T_a P)^{k_n}$ exists. Thus $h(T_a P)$ is one of the numbers $1, -1, 0$. By continuity of the homomorphism h and equation $h(E_0) = h(T_0 P) = 1$, valid for non-trivial homomorphisms, we obtain $h(T_a P) = 1$ for all $a \geqslant 0$. Hence, by Theorem 3, $P = E_0$ and, consequently, $T_{c_n} P^{\circ n} = E_0$. Thus $Q = E_0$, which contradicts the hypothesis. Formula (13) is thus proved.

Let us turn next to (14). Suppose that we could find a subsequence $c_{n_1}, c_{n_2}, \ldots$ satisfying the following condition:

$$d = \lim_{k\to\infty} \frac{c_{n_k}}{c_{n_k+1}} \neq 1.$$

By (**) and (13) we have the convergence

$$T_{a c_n} P \to E_0 \qquad (a \geqslant 0). \tag{16}$$

First we consider the case $d < \infty$. From (16) it follows that

$$h(T_{a c_n} P^{\circ(n+1)}) = h(T_a T_{c_n} P^{\circ n})\, h(T_{a c_n} P) \to h(T_a Q) \tag{17}$$

as $n \to \infty$. On the other hand, setting $d_k = c_{n_k}/c_{n_k+1}$, we have, by (*),

$$T_{a c_{n_k}} P^{\circ(n_k+1)} = T_{a d_k} T_{c_{n_k+1}} P^{\circ(n_k+1)} \to T_{ad} Q.$$

Hence and from (17) we get

$$h(T_a Q) = h(T_{ad} Q) \qquad (a \geqslant 0).$$

Thus, by successive iteration,

$$h(T_aQ) = h(T_{aq^n}Q) \quad (a \geqslant 0,\ n = 1, 2, \ldots), \tag{18}$$

where $q = \min(d, d^{-1})$. Since $0 \leqslant q < 1$, we have, by (**), $T_{aq^n}Q \to E_0$. Consequently, by (18), $h(T_aQ) = h(E_0) = 1$ for all $a \geqslant 0$. Now, applying Theorem 3, we obtain $Q = E_0$, which contradicts the hypothesis. Formula (14) is thus proved in the case $d < \infty$.

Now consider the case $d = \infty$. Setting $q_k = c_{n_k+1}/c_{n_k}$, we have $\lim\limits_{k\to\infty} q_k = 0$. Hence, according to (**), we get the convergence

$$T_{ac_{n_k+1}}P^{\circ n_k} = T_{aq_k}(T_{c_{n_k}}P^{\circ n_k}) \to E_0 \quad (a \geqslant 0). \tag{19}$$

On the other hand, by (16), we have the relation

$$h(T_{ac_{n+1}}P^{\circ n}) = \frac{h(T_{ac_{n+1}}P^{\circ(n+1)})}{h(T_{ac_{n+1}}P)} \to \frac{h(T_aQ)}{h(E_0)} = h(T_aQ).$$

Comparying it with (19) we get $h(T_aQ) = h(E_0) = 1$ $(a \geqslant 0)$. Now, applying Theorem 3, we get $Q = E_0$, which contradicts the hypothesis. Formula (14) is thus proved.

Now we proceed to the proof of (15). From (13) and (14) it follows that for any pair x, y of positive numbers there exist subsequences $c_{n_1}, c_{n_2}, \ldots$ and $c_{m_1}, c_{m_2}, \ldots$ of the sequence $c_1, c_2, \ldots$ such that

$$\lim_{k\to\infty} \frac{c_{n_k}}{c_{m_k}} = \frac{y}{x}. \tag{20}$$

Moreover, we may assume without loss of generality that the limit

$$s = \lim_{k\to\infty} \frac{c_{n_k}}{c_{n_k+m_k}},$$

perhaps infinite, exists. First of all we shall prove that the limit s is finite Contrary to this let as suppose that

$$\lim_{k\to\infty} v_k = 0, \quad \text{where} \quad v_k = \frac{c_{n_k+m_k}}{c_{n_k}}.$$

Setting $w_k = c_{n_k}/c_{m_k}$ we have

$$T_{ac_{n_k+m_k}}P^{\circ(n_k+m_k)} = T_{av_k}(T_{c_{n_k}}P^{\circ n_k}) \circ T_{av_kw_k})T_{c_{m_k}}P^{\circ m_k}).$$

Hence, by (**),

$$h(T_{ac_{n_k+m_k}}P^{\circ(n_k+m_k)}) \to h(E_0)h(E_0) = 1.$$

But, by the hypothesis,

$$T_{ac_{n_k+m_k}}P^{\circ(n_k+m_k)} \to T_aQ.$$

Thus $h(T_aQ) = 1$ for all $a \geqslant 0$. Applying Theorem 3 we infer that $Q = E_0$, which contradicts the hypothesis. The finiteness of the limit s is thus proved.

Using the notation $s_k = c_{n_k}/c_{n_k+m_k}$, $w_k = c_{n_k}/c_{m_k}$ we obtain the following equations:

$$T_{axc_{n_k}}P^{\circ(n_k+m_k)} = T_{axc_{n_k}}P^{\circ n_k} \circ T_{axw_k}(T_{c_{m_k}}P^{\circ m_k}), \tag{21}$$

$$T_{axc_{n_k}}P^{\circ(n_k+m_k)} = T_{axs_k}(T_{c_{n_k+m_k}}P^{\circ(n_k+m_k)}). \tag{22}$$

When $k \to \infty$ we get from (21), by virtue of (*) and (20),

$$h(T_{axc_{n_k}}P^{\circ(n_k+m_k)}) \to h(T_{ax}Q)h(T_{ay}Q).$$

Further, from (22) we obtain the convergence

$$h(T_{axc_{n_k}}P^{\circ(n_k+m_k)}) \to h(T_{axs}Q).$$

Thus for any non-negative number a we have the equation

$$h(T_{ax}Q)h(T_{ay}Q) = h(T_{axs}Q).$$

We define an auxiliary function $g(x, y)$ by means of the formulas $g(x, 0) = x$, $g(0, y) = y$ and $g(x, y) = sx$ for $x > 0$, $y > 0$. The function g satisfies the equation

$$h(T_{ax}Q)h(T_{ay}Q) = h(T_{ag(x,y)}Q) \quad (x \geqslant 0,\ y \geqslant 0) \tag{23}$$

for all $a \geqslant 0$.

We shall prove that the function g is the only function satisfying (23) for all $a \geqslant 0$. Suppose that there exist two functions g_1 and g_2 satisfying (23) for all $a \geqslant 0$ and $g_1(x_0, y_0) < g_2(x_0, y_0)$ for a pair of non-negative numbers x_0, y_0. Setting

$$u = \frac{g_1(x_0, y_0)}{g_2(x_0, y_0)}$$

we get, according to (23), the equation

$$h(T_aQ) = h(T_{au}Q) \quad (a \geqslant 0).$$

Hence it follows that $h(T_aQ) = h(T_{au^n}Q)$ for all positive integers n. Since $0 \leqslant u < 1$, we have, by (**), $h(T_{au^n}Q) \to h(E_0) = 1$, as $n \to \infty$. Thus $h(T_aQ) = 1$ for all $a \geqslant 0$. Applying Theorem 3 we infer that $Q = E_0$ which contradicts the hypothesis. The uniqueness of the function g is thus proved.

As a direct consequence of equation (23) and the uniqueness of its solution we obtain

$$g(x,y) = g(y,x), \tag{24}$$

$$g\big(g(x,y),z\big) = g\big(x,g(y,z)\big), \tag{25}$$

$$g(zx,zy) = zg(x,y) \tag{26}$$

for all non-negative numbers x, y and z. Now we shall prove that the function g is continuous in the quadrant $x \geqslant 0$, $y \geqslant 0$. Let $x_n \to x$ and $y_n \to y$. Moreover, suppose that $g(x_n, y_n) \to z$, where $0 \leqslant z \leqslant \infty$. The equation $z = \infty$ is impossible. Indeed, it would imply, by (23) and (**),

$$h(T_a Q) = h(T_{ap_n}Q)h(T_{aq_n}Q) \to h(E_0)h(E_0) = 1,$$

where $p_n = x_n/g(x_n, y_n)$ and $q_n = y_n/g(x_n, y_n)$. But the equation $h(T_a Q) = 1$ for all $a \geqslant 0$ implies, by Theorem 3, the equation $Q = E_0$ which contradicts the hypothesis. Thus the limit z is finite. By continuity of the homomorphism h and by equation (23) we obtain $h(T_{az}Q) = h(T_{ax}Q)h(T_{ay})$ for all $a \geqslant 0$. Hence, by the uniqueness of solution of (23), we get $z = g(x,y)$. Thus the function g is continuous.

From (23) for any $a \geqslant 0$ we obtain

$$h^2(T_a Q) = h(T_{ag(1,1)}Q). \tag{27}$$

If $g(1,1) = 1$, then from (27) and from the equation $h(T_0 Q) = h(E_0) = 1$ we get, by virtue of the continuity of $h(T_a Q)$ with respect to a, $h(T_a Q) = 1$ $(a \geqslant 0)$. But this, according to Theorem 3, contradicts the inequality $Q \neq E_0$. Thus $g(1,1) \neq 1$. Further, if $g(1,1) < 1$, then, by induction, from (27) we get

$$h^{2^n}(T_a Q) = h(T_{ag^n_{\cdot}(1,1)}Q) \quad (a \geqslant 0, \quad n = 1, 2, \ldots).$$

Hence, by (**), $h^{2^n}(T_a Q) \to 1$ as $n \to \infty$. Thus $|h(T_a Q)| = 1$ which, by continuity of $h(T_a Q)$ with respect to a and the condition $h(T_0 Q) = 1$, implies $h(T_a Q) = 1$ for all $a \geqslant 0$. Applying Theorem 3 we get a contradiction. Thus we have proved that the inequality

$$g(1,1) > 1 \tag{28}$$

is valid.

By (26), to prove the inequality

$$g(x,y) > x \quad (x \geqslant 0,\ y > 0) \tag{29}$$

it suffices to prove it for $y = 1$. Let us suppose that there exists a number x_1 such that $g(x_1, 1) < x_1$. Since $g(0,1) = 1$ and the function g is continuous, we infer that there exists a number x_0 lying between 0 and x_1 for which the equation $g(x_0, 1) = x_0$ holds. From this equation, using (25) and (26), we obtain by induction with respect to n the formula

$$g\big(x_0, g^n(1,1)\big) = x_0 \quad (n = 1, 2, \ldots).$$

Hence, setting $z_n = x_0/g^n(1,1)$, we get, by (26),

$$g(z_n, 1) = z_n \quad (n = 1, 2, \ldots).$$

From inequality (28) it follows that $\lim_{n\to\infty} z = 0$. Thus, by the continuity of g, the last equation implies $g(0,1) = 0$ which contradicts the definition of $g(0,1) = 1$. This completes the proof of (29).

Now we shall prove that for all $x \geqslant 0$

$$g(x, y_1) > g(x, y_2) \quad \text{whenever} \quad y_1 > y_2. \tag{30}$$

If $y_2 = 0$, then (30) is a direct consequence of (29) and the definition of $g(x,0) = x$. Suppose that $y_2 > 0$. Since $g(0, y_2) = y_2$ and, by (29), $g(y_1, y_2) > y_1$, we infer, by virtue of continuity of g, that there exists a number y satisfying the inequality $0 < y < y_1$ for which the equation $g(y, y_2) = y_1$ holds. Hence, taking into account (24), (25) and (29), we obtain

$$g(x, y_1) = g\big(x, g(y, y_2)\big) = g\big(g(x,y), y_2\big) = g\big(g(y,x), y_2\big)$$
$$= g\big(y, g(x, y_2)\big) = g\big(g(x, y_2), y\big) > g(x, y_2)$$

which completes the proof of (30).

F. Bohnenblust proved in [1] (p. 630-632) that the functions g satisfying conditions (24), (25), (26), (30) and the boundary condition $g(0,x) = x$ are of the form $g(x,y) = (x^\lambda + y^\lambda)^{1/\lambda}$, where λ is a positive constant. Thus, setting $H(x^\lambda) = h(T_x Q)$, we obtain from (23) a functional equation $H(x)H(y) = H(x+y)$ $(x \geqslant 0, y \geqslant 0)$. By Theorem 3 the function H is not identically equal to 1. Since $H(0) = h(E_0) = 1$, it is not a constant function. Moreover, the function H is continuous. It is well-known that all continuous non-constant solutions of the considered functional equation are of the form $H(x) = \exp(-cx)$, where c is a constant different from 0. Thus $h(T_a Q) = \exp(-ca^\lambda)$ $(a \geqslant 0)$. From Theorem 1 it follows that the constant c cannot be negative and, consequently, $c > 0$. Theorem 4 is thus proved.

As a consequence of Theorem 4 we obtain the following theorem:

THEOREM 5. *For any non-trivial homomorphism h of a generalized convolution algebra there exists a positive number a_0 such that $h(E_a) < 1$ whenever $0 < a \leqslant a_0$.*

Proof. We have, by Theorem 1, the inequality $h(E_a) \leqslant 1$ for all $a \geqslant 0$. Suppose that there exists a sequence $b_1, b_2, \ldots$ of positive numbers such that $b_n \to 0$ and

$$h(E_{b_n}) = 1 \quad (n = 1, 2, \ldots). \tag{31}$$

By condition (v) of the definition of generalized covolutions there exist a sequence $c_1, c_2, \ldots$ of positive numbers and a measure Q different from E_0 such that

$$T_{c_n} E_1^{\circ n} \to Q. \tag{32}$$

Since, by Theorem 4, $c_n \to 0$, we can find a subsequence $c_{k_1}, c_{k_2}, \ldots$ satisfying the condition

$$c_{k_{n+1}} < b_n \leqslant c_{k_n} \quad (n = 1, 2, \ldots).$$

Setting $d_n = b_n / c_{k_n}$, we obtain, by virtue of (14), $\lim_{n\to\infty} d_n = 1$. Since

$$E_{b_n}^{\circ k_n} = T_{b_n} E_1^{\circ k_n} = T_{d_n c_{k_n}} E_1^{\circ k_n} = T_{d_n}(T_{c_{k_n}} E_1^{\circ k_n}),$$

we have, by (*) and (32), $h^{k_n}(E_{b_n}) \to h(Q)$. Hence and from (31) it follows that $h(Q) = 1$. Since $h(Q) = h(T_1 Q)$, this contradicts (15). The Theorem is thus proved.

4. Characteristic functions

We already know that generalized convolution algebras admitting characteristic functions are regular. Now we shall prove the converse theorem:

THEOREM 6. *Every regular generalized convolution algebra admits characteristic functions. Each non-trivial homomorphism h induces a characteristic function Φ_P by means of the formula*

$$\Phi_P(t) = h(T_t P) \quad (t \geqslant 0,\ P \in \mathfrak{P}). \tag{33}$$

Conversely, each characteristic function is of the form (33). *Moreover, Φ_P is an integral transform*

$$\Phi_P(t) = \int_0^\infty \Omega(tx) P(dx), \tag{34}$$

where the kernel Ω is defined by formula $\Omega(x) = h(E_x)$.

Proof. First of all we shall prove that every characteristic function Φ_P is of the form (33), where h is a non-trivial homomorphism of the algebra in question. Put $h(P) = \Phi_P(1)$. It is clear that h is a homomorphism. From the formula $\Phi_{T_a P}(t) = \Phi_P(at)$ we obtain the equation $\Phi_P(t) = \Phi_{T_t P}(1) = h(T_t P)$. Hence, in particular, it follows that the homomorphism h is non-trivial.

Now let us suppose that h is a non-trivial homomorphism. We have to prove that function (33) is a characteristic function. We note that the integral representation (34) is a direct consequence of (1), (11) and (33). Further, the conditions $\Phi_{aP+bQ} = a\Phi_P + b\Phi_Q$ $(a \geqslant 0,\ b \geqslant 0,\ a+b = 1)$ and $\Phi_{P\circ Q} = \Phi_P \Phi_Q$ are a consequence of (33) and the corresponding conditions for the homomorphism h. From definition (33) we get also $\Phi_{T_a P}(t) = \Phi_P(at)$. Now we shall prove that the correspondence between probability measures P and the functions Φ_P is one-to-one.

Suppose that $\Phi_{P_1} = \Phi_{P_2}$. From condition (v) for generalized convolutions and from Theorem 4 (formula (15)) it follows that there exists a probability measure Q such that

$$\Phi_Q(t) = \exp(-ct^\lambda), \tag{35}$$

where c and λ are positive constants. Since, by (34),

$$\int_0^\infty \Phi_{P_j}(ty) Q(dy) = \int_0^\infty \int_0^\infty \Omega(txy) P_j(dx) Q(dy) = \int_0^\infty \int_0^\infty \Omega(txy) Q(dy) P_j(dx)$$
$$= \int_0^\infty \exp(-ct^\lambda x^\lambda) P_j(dx) \quad (j = 1, 2),$$

we have

$$\int_0^\infty \exp(-ct^\lambda x^\lambda) P_1(dx) = \int_0^\infty \exp(-ct^\lambda x^\lambda) P_2(dx).$$

By the change of variable $z = x^\lambda$, $u = ct^\lambda$ in the last equation we obtain the equality for Laplace-Stieltjes transforms

$$\int_0^\infty \exp(-uz) P_1^*(dz) = \int_0^\infty \exp(-uz) P_2^*(dz),$$

where $P_j^*(dz) = P_j(dx)$ $(j = 1, 2)$. Since the measures P_j^* are uniquely determined by their Laplace-Stieltjes transforms (see [5], p. 290), we have $P_1^* = P_2^*$ and, consequently, $P_1 = P_2$. Thus the measure P is uniquely determined by the function Φ_P.

If $P_n \to P$ and $t_n \to t$, then, by (*), $h(T_{t_n} P_n) \to h(T_t P)$. In other words, if $P_n \to P$, then the functions $\Phi_{P_n}(t)$ tend to the function $\Phi_P(t)$ uniformly in every finite interval. Now suppose that a sequence $\Phi_{P_1}(t)$, $\Phi_{P_2}(t), \ldots$ converges to a function $\Phi(t)$ uniformly in every finite interval. To prove that P_n weakly converges to a probability measure P and $\Phi = \Phi_P$ it suffices to show that the sequence $P_1, P_2, \ldots$ is compact, i. e. each subsequence of $P_1, P_2, \ldots$ contains a convergent subsequence.

Indeed, if P' and P'' are weak limits of subsequences P_{n_k} and P_{m_k} respectively, then, by the previous part of the proof,

$$\Phi_{P_{n_k}}(t) \to \Phi_{P'}(t) \quad \text{and} \quad \Phi_{P_{m_k}}(t) \to \Phi_{P''}(t)$$

uniformly in every finite interval. Thus $\Phi_{P'}(t) = \Phi(t) = \Phi_{P''}(t)$ which implies, by the uniqueness of the correspondence $P \leftrightarrow \Phi_P$, the equation $P' = P''$. Hence it follows that the sequence $P_1, P_2, \ldots$ is weakly convergent.

The compactness of the sequence $P_1, P_2, \ldots$ and consequently the Theorem 6 will follow from the following lemma:

LEMMA. *Let $P_1, P_2, \ldots$ be a sequence of probability measures and let h be a non-trivial homomorphism of a generalized convolution algebra. If there exists a positive number t_0 such that the sequence $h(T_tP_1), h(T_tP_2), \ldots$ is uniformly convergent in the interval $0 \leqslant t \leqslant t_0$, then the sequence $P_1, P_2, \ldots$ is compact.*

Proof. Since the functions $h(T_tP_n)$ are continuous and $h(T_0P_n) = h(E_0) = 1$, the uniform convergence in $0 \leqslant t \leqslant t_0$ implies for any $\varepsilon > 0$ the existence of an integer n_0 and a positive number u_0 $(u_0 \leqslant t_0)$ such that

$$h(T_tP_n) > 1-\varepsilon \quad \text{if} \quad n \geqslant n_0 \text{ and } 0 \leqslant t \leqslant u_0. \tag{36}$$

Let Q be the probability measure defined by (35). Since, by (**), $T_aQ \to E_0$ ad $a \to 0$, there exists a positive number a_0 such that

$$\int_{u_0}^{\infty} Q_0(dt) < \varepsilon, \tag{37}$$

where $Q_0 = T_{a_0}Q$. Moreover, by (35),

$$h(T_xQ_0) = \exp(-c_0x^{\lambda}),$$

where c_0 is a positive constant. Thus we can find a positive number x_ε such that

$$h(T_xQ_0) < \varepsilon \quad \text{if} \quad x \geqslant x_\varepsilon. \tag{38}$$

From (36) and (37) we get

$$\int_0^{\infty} h(T_tP_n)Q_0(dt) = \int_0^{u_0} h(T_tP_n)Q_0(dt) + \int_{u_0}^{\infty} h(T_tP_n)Q_0(dt) \tag{39}$$
$$> (1-\varepsilon)\int_0^{u_0} Q_0(dt) - \int_{u_0}^{\infty} Q_0(dt) > (1-\varepsilon)^2 - \varepsilon$$

for all $n \geqslant n_0$ and $0 < \varepsilon \leqslant 1$. Further, for all $n \geqslant n_0$ we have, by (33), (34) and (38),

$$\int_0^{\infty} h(T_tP_n)Q_0(dt) = \int_0^{\infty}\int_0^{\infty} \Omega(tx)P_n(dx)Q_0(dt) = \int_0^{\infty}\int_0^{\infty} \Omega(tx)Q_0(dt)P_n(dx)$$
$$= \int_0^{\infty} h(T_xQ_0)P_n(dx) \leqslant \int_0^{x_\varepsilon} P_n(dx) + \int_{x_\varepsilon}^{\infty} h(T_xQ_0)P_n(dx) \leqslant \int_0^{x_\varepsilon} P_n(dx) + \varepsilon.$$

Hence and from (39) it follows that for every number ε $(0 < \varepsilon \leqslant 1)$ there exist an integer n_0 and a positive number x_ε such that

$$\int_0^{x_\varepsilon} P_n(dx) > (1-\varepsilon)^2 - 2\varepsilon \quad \text{if} \quad n \geqslant n_0.$$

It is well-known that this condition implies the compactness of the sequence $P_1, P_2, \ldots$ (see [2], Chapter 2, Theorem 3). The Lemma is thus proved.

We note that from (6), (8) and Theorem 6 it follows that the functions

$$\Phi_P(t) = \int_0^{\infty} \exp(-t^{\alpha}x^{\alpha})P(dx) \quad \text{and} \quad \Phi_P(t) = \int_0^{\infty} g_\beta(t^{\alpha}x^{\alpha})P(dx)$$

are characteristic functions in the α-convolution algebra and the (α, β)-convolution algebra respectively.

THEOREM 7. *Let $(\mathfrak{P}, \circ)$ be a regular generalized convolution algebra. There exist a probability measure M and a positive number $\varkappa$ such that for every characteristic function Φ_P in $(\mathfrak{P}, \circ)$ we have*

$$\Phi_M(t) = \exp(-c_\Phi t^{\varkappa}), \tag{40}$$

where c_Φ is a positive number depending on Φ_P. Moreover, the kernel Ω of Φ_P satisfies the condition

$$\lim_{x\to 0} \frac{1-\Omega(tx)}{1-\Omega(x)} = t^{\varkappa} \tag{41}$$

uniformly in every finite interval.

Proof. From condition (v) for generalized convolutions it follows that there exist a sequence $c_1, c_2, \ldots$ of positive numbers and a probability measure M different from E_0 such that

$$T_{c_n}E_1^{\circ n} \to M. \tag{42}$$

Furthermore, from (15) and (33) it follows that for any characteristic function Φ_P there exist positive numbers c_Φ and $\varkappa_\Phi$ such that

$$\Phi_M(t) = \exp(-c_\Phi t^{\varkappa_\Phi}).$$

To prove (40) it suffices to show that $\varkappa_\Phi$ does not depend upon the choice of Φ_P. Contrary to this let us suppose that $\varkappa_\Phi < \varkappa_\Psi$ for a pair Φ_P, Ψ_P of characteristic functions. From the equations

$$\Phi_{M\circ M}(t) = \Phi_M(t)\Phi_M(t) = \exp(-2c_\Phi t^{\varkappa_\Phi}) = \Phi_{T_{b_\Phi}M}(t),$$
$$\Psi_{M\circ M}(t) = \Psi_M(t)\Psi_M(t) = \exp(-2c_\Psi t^{\varkappa_\Psi}) = \Psi_{T_{b_\Psi}M}(t),$$

where $b_\Phi = 2^{1/\varkappa_\Phi}$ and $b_\Psi = 2^{1/\varkappa_\Psi}$ it follows that

$$M\circ M = T_{b_\Phi}M = T_{b_\Psi}M. \tag{43}$$

Put $d = b_\Psi/b_\Phi$. Since $\varkappa_\Phi < \varkappa_\Psi$ and, consequently, $b_\Phi > b_\Psi$, we have $d<1$. Moreover, from (43) we obtain $M = T_dM$ and, consequently, $M = T_{d^n}M$ $(n = 1, 2, \ldots)$. Since, by (**), $T_{d^n}M \to E_0$, we get $M = E_0$ which contradicts the hypothesis $M \neq E_0$. Formula (40) is thus proved.

Now we proceed to the proof of (41). Let Ω be the kernel of the characteristic function Φ_P. Since the characteristic function of the measure $T_{c_n}E_1^{\circ n}$ is equal to $\Omega^n(c_nt)$, we infer, by (40) and (42), that

$$\Omega^n(c_nt) \to \exp(-c_\Phi t^\varkappa)$$

uniformly in every finite interval. Hence it follows that

$$n\big(1-\Omega(c_nt)\big) \to c_\Phi t^\varkappa \tag{44}$$

uniformly in every finite interval. Given an arbitrary sequence $x_1, x_2, \ldots$ of positive numbers tending to 0. Without loss of generality we may assume that $1-\Omega(x_n) > 0$ $(n = 1, 2, \ldots)$ (see Theorem 5). Since, by Theorem 4, $c_n \to 0$, we can find a subsequence $c_{k_1}, c_{k_2}, \ldots$ for which the inequalities

$$c_{k_n+1} < x_n \leqslant c_{k_n} \quad (n = 1, 2, \ldots)$$

hold. Setting $s_n = x_n/c_{k_n}$, we have, by (14), $s_n \to 1$ and, consequently, by (44),

$$\frac{1-\Omega(tx_n)}{1-\Omega(x_n)} = \frac{1-\Omega(c_{k_n}s_nt)}{1-\Omega(c_{k_n}s_n)} \to t^\varkappa$$

uniformly in every finite interval. This completes the proof of the Theorem.

The exponent $\varkappa$ is uniquely determined by formula (41). We shall call it a *characteristic exponent* of the algebra $(\mathfrak{P}, \circ)$. Moreover, each measure M satisfying (40) with a constant c_Φ will be called a *characteristic measure* of the algebra $(\mathfrak{P}, \circ)$. It is very easy to verify that the characteristic exponent of the a-convolution algebra is equal to a and the characteristic exponent of the (a, β)-convolution algebra is equal to $2a$.

Every probability measure P is uniquely determined by its characteristic function Φ_P. Now we shall give an inversion formula analogous to the classical result of Lévy.

THEOREM 8. *Let M and Φ_P be a characteristic measure and a characteristic function of a generalized convolution algebra, respectively. Put*

$$V_P(t) = \int_0^\infty \Phi_P(t^{1/\varkappa}vx)M(dx),$$

where

$$v = \left(\frac{-1}{\log\Phi_M(1)}\right)^{1/\varkappa},$$

and $\varkappa$ is the characteristic exponent of the algebra. If $[a, b]$ is an interval with endpoints of P-measure zero, then

$$P([a, b]) = \lim_{n\to\infty}(-1)^n \int_{a^\varkappa}^{b^\varkappa} \left(\frac{n}{t}\right)^{n+1} V_P^{(n)}\left(\frac{n}{t}\right)dt.$$

Proof. By (34) and (40) we have

$$V_P(t) = \int_0^\infty\int_0^\infty \Omega(t^{1/\varkappa}vxy)P(dy)M(dx) = \int_0^\infty\int_0^\infty \Omega(t^{1/\varkappa}vxy)M(dx)P(dy)$$
$$= \int_0^\infty \exp(-ty^\varkappa)P(dy).$$

Making the change of variable $z = y^\varkappa$ we obtain

$$V_P(t) = \int_0^\infty \exp(-tz)P^*(dz),$$

where $P^*(dz) = P(dy)$. Thus $V_P(t)$ is the Laplace-Stieltjes transform of the measure P^*. Applying the inversion formula for Laplace-Stieltjes transform (see [5], p. 290) we get

$$P^*([a^\varkappa, b^\varkappa]) = \lim_{n\to\infty}(-1)^n \int_{a^\varkappa}^{b^\varkappa} \left(\frac{n}{t}\right)^{n+1} V_P^{(n)}\left(\frac{n}{t}\right)dt$$

provided the endpoints of the interval $[a^\varkappa, b^\varkappa]$ are of P^*-measure zero. Hence, taking into account the equation $P([a, b]) = P^*([a^\varkappa, b^\varkappa])$, we obtain required inversion formula.

5. Infinitely decomposable measures

This section is devoted to the study of certain limit distributions. We assume that the algebra $(\mathfrak{P}, \circ)$ is regular and Φ_P is a fixed characteristic function in $(\mathfrak{P}, \circ)$.

A measure $P \in \mathfrak{P}$ is said to be *infinitely decomposable* if for every positive integer n there exists a measure $P_n \in \mathfrak{P}$ such that $P = P_n^{\circ n}$.

THEOREM 9. *The characteristic function of an infinitely decomposable measure is positive.*

Proof. Suppose that P is infinitely decomposable and Φ_P is not positive. Taking into account the continuity of Φ_P and formula $\Phi_P(0) = 1$, we can find a positive number t_0 such that $\Phi_P(t_0) = 0$ and $\Phi_P(t) > 0$ in the interval $0 \leqslant t < t_0$. Let P_n be a probability measure satisfying the condition $P_n^{\circ n} = P$. Since $\Phi_{P_n}^n(t) = \Phi_P(t)$, we have $\Phi_{P_n}(t) = \Phi_P^{1/n}(t)$ in the interval $0 \leqslant t \leqslant t_0$. Thus

$$\Phi_{P_n}(t_0) = 0 \qquad (n = 1, 2, \ldots) \tag{45}$$

and

$$\Phi_{P_n}(t) \to 1 \tag{46}$$

uniformly in every compact contained in the interval $0 \leqslant t < t_0$. Hence, by Lemma in section 4, the sequence of probability measures $P_1, P_2, \ldots$ is compact. Let Q be its limit point. From (45) and (46) it follows that $\Phi_Q(t) = 1$ in the interval $0 \leqslant t < t_0$ and $\Phi_Q(t_0) = 0$, which contradicts the continuity of the characteristic function. The theorem is thus proved.

From this theorem it follows that the equation $\Phi_{P_n}(t) = \Phi_P^{1/n}(t)$ is equivalent to the equation $\Phi_{P_n}^n(t) = \Phi_P(t)$. Since the last equation is equivalent to the formula $P_n^{\circ n} = P$, we obtain the following condition for infinite decomposability:

THEOREM 10. *A probability measure P is infinitely decomposable if and only if, for every positive integer n, $\Phi_P^{1/n}(t)$ is a characteristic function.*

As a direct consequence of Theorem 10 we obtain the following Theorem:

THEOREM 11. *The family of infinitely decomposable measures is closed under generalized convolution, transformations T_a $(a \geqslant 0)$ and passages to the limit.*

Let c be a non-negative number and Q a probability measure. The probability measure

$$P = \sum_{s=0}^{\infty} \frac{c^s}{s!} Q^{\circ s} \exp(-c)$$

is said to be of *Poisson type*. It is easy to verify that $\Phi_P(t) = \exp\big(c(\Phi_Q(t) - 1)\big)$. Hence it follows that $P = P_n^{\circ n}$, where P_n is a measure of Poisson type associated with the same measure Q and the constant c/n. Thus measures of Poisson type are infinitely decomposable.

Probability measures P_{nk} $(k = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots)$ are said to be *uniformly asymptotically neglegible* if for any positive number ε

$$\lim_{n\to\infty} \max_{1\leqslant k\leqslant k_n} \int_{\varepsilon}^{\infty} P_{nk}(dx) = 0. \tag{47}$$

THEOREM 12. *A probability measure is a weak limit of a sequence $P_{n1}\circ P_{n2}\circ \ldots \circ P_{nk_n}$, where P_{nk} $(k = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots)$ are uniformly asymptotically neglegible if and only if it is infinitely decomposable.*

Proof. Consider a sequence P_{nk} $(k = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots)$ of uniformly asymptotically neglegible probability measures such that

$$P_{n1}\circ P_{n2}\circ \ldots \circ P_{nk_n} \to P. \tag{48}$$

We define an auxiliary sequence of measures of Poisson type

$$P_n = \sum_{s=0}^{\infty} \frac{k_n^s}{s!} Q_n^{\circ s} \exp(-k_n), \qquad \text{where} \qquad Q_n = k_n^{-1} \sum_{k=1}^{k_n} P_{nk}.$$

Of course

$$\Phi_{P_n}(t) = \exp \sum_{k=1}^{k_n} \big(\Phi_{P_{nk}}(t) - 1\big). \tag{49}$$

By continuity of $\Phi_P(t)$ and equation $\Phi_P(0) = 1$ we conclude that $\Phi_P(t)$ is positive in a neighborhood of the origin. Consequently, there exists an interval $[0, t_1)$ such that $\Phi_P(t) > 0$ if $0 \leqslant t < t_1$ and either t_1 is a finite number and $\Phi_P(t_1) = 0$ or $t_1 = \infty$. From (48) it follows that

$$\prod_{k=1}^{k_n} \Phi_{P_{nk}}(t) \to \Phi_P(t)$$

uniformly in every finite interval. Thus

$$\sum_{k=1}^{k_n} \log \Phi_{P_{nk}}(t) \to \log \Phi_P(t) \tag{50}$$

uniformly in every compact contained in $[0, t_1)$. Given a positive number ε and a positive number t_0, there exists a positive number δ such that $1 - \Omega(tx) < \varepsilon$ whenever $0 \leqslant x \leqslant \delta$ and $0 \leqslant t \leqslant t_0$. Hence for any number t satisfying the inequality $0 \leqslant t \leqslant t_0$ and for any integer k satisfying the inequality $1 \leqslant k \leqslant k_n$ we get the formula

$$0 \leqslant 1 - \Phi_{P_{nk}}(t) = \int_0^{\infty} \big(1 - \Omega(tx)\big) P_{nk}(dx)$$

$$\leqslant \int_0^{\delta} \big(1 - \Omega(tx)\big) P_{nk}(dx) + \int_{\delta}^{\infty} \big(1 - \Omega(tx)\big) P_{nk}(dx) \leqslant \varepsilon + 2 \max_{1\leqslant k\leqslant k_n} \int_{\delta}^{\infty} P_{nk}(dx),$$

which, by (47), implies $\max\limits_{1\leqslant k\leqslant k_n} \big(1 - \Phi_{nk}(t)\big) \to 0$ uniformly in every finite interval. Hence and from (50) it follows that

$$\sum_{k=1}^{k_n} \big(\Phi_{P_{nk}}(t) - 1\big) \to \log \Phi_P(t)$$

uniformly in every compact contained in $[0, t_1)$. Consequently, by (49),

$$\Phi_{P_n}(t) \to \Phi_P(t) \tag{51}$$

uniformly in every compact contained in $[0, t_1)$. Hence, by Lemma in section 4, we infer that the sequence $P_1, P_2, \ldots$ is compact. Let P_* be its limit point. Since the measures $P_1, P_2, \ldots$ are of Poisson type and, consequently, infinitely decomposable, the measure P_* is, according to Theorem 11, also infinitely decomposable. Moreover, from (51) it follows that $\Phi_{P_*}(t) = \Phi_P(t)$ whenever $0 \leqslant t < t_1$. Hence it follows that t_1 cannot be a finite number. Indeed, by continuity of the characteristic function the last equation would imply $\Phi_{P_*}(t_1) = \Phi_P(t_1) = 0$, which contradicts Theorem 9. Thus $t_1 = \infty$ and, consequently, $\Phi_{P_*} = \Phi_P$, which implies $P = P_*$. The limit measure P is thus infinitely decomposable.

Conversely, let us suppose that P is an infinitely decomposable measure and $P = P_n^{\circ n}$ $(n = 1, 2, \ldots)$. Put $P_{nk} = P_n$ $(k = 1, 2, \ldots, n;$ $n = 1, 2, \ldots)$. Since $P_{n1} \circ P_{n2} \circ \ldots \circ P_{nn} = P$, to prove the Theorem it suffices to prove that P_{nk} are uniformly asymptotically neglegible.

Let M be a characteristic measure of the algebra in question. Since, by Theorem 9, Φ_P is positive and $\Phi_{P_{nk}}(t) = \Phi_{P_n}(t) = \Phi_P^{1/n}(t)$, we have the formula

$$\lim_{n\to\infty} \max_{1 \leqslant k \leqslant n} \int_0^\infty \big(1 - \Phi_{P_{nk}}(t)\big) M(dt) = 0. \tag{52}$$

Given a positive number ε, we have, by (34), and (40),

$$\begin{aligned}\int_0^\infty \big(1 - \Phi_{P_{nk}}(t)\big) M(dt) &= \int_0^\infty \int_0^\infty \big(1 - \Omega(tx)\big) P_{nk}(dx) M(dt) \\ &= \int_0^\infty \int_0^\infty \big(1 - \Omega(tx)\big) M(dt) P_{nk}(dx) \\ &= \int_0^\infty \big(1 - \exp(-c_\Phi x^\varkappa)\big) P_{nk}(dx) \\ &\geqslant \int_\varepsilon^\infty \big(1 - \exp(-c_\Phi x^\varkappa)\big) P_{nk}(x) \\ &\geqslant \big(1 - \exp(-c_\Phi \varepsilon^\varkappa)\big) \int_\varepsilon^\infty P_{nk}(dx).\end{aligned}$$

Hence and from (52) it follows that the measures P_{nk} are uniformly asymptotically neglegible, which completes the proof of the Theorem.

By Theorem 5 there exists a positive number x_0 such that $\Omega(x) < 1$ whenever $0 < x \leqslant x_0$. Put

$$\omega(x) = \begin{cases} 1 - \Omega(x) & \text{if} \quad 0 \leqslant x \leqslant x_0, \\ 1 - \Omega(x_0) & \text{if} \quad x > x_0. \end{cases} \tag{53}$$

Of course, the function ω is positive except the origin. From (41) it follows that

$$\lim_{x\to 0} \frac{1 - \Omega(tx)}{\omega(x)} = t^\varkappa \tag{54}$$

uniformly in every finite interval, where $\varkappa$ is the characteristic exponent of the algebra. Thus the function $\big(1 - \Omega(tx)\big)/\omega(x)$ defined, according to (54), to be $t^\varkappa$ for $x = 0$, becomes continuous in the quadrant $0 \leqslant t < \infty$, $0 \leqslant x < \infty$.

Now we shall prove an analogue of the Lévy-Khintchine representation for the characteristic functions of infinitely decomposable measures in a generalized convolution algebra.

THEOREM 13. *A function Φ is a characteristic function of an infinitely decomposable measure if and only if it is of the form*

$$\Phi(t) = \exp \int_0^\infty \frac{\Omega(tx) - 1}{\omega(x)} m(dx), \tag{55}$$

where m is a finite Borel measure on the positive half-line.

Proof. First we shall prove the necessity of the condition. Let P be an infinitely decomposable measure. Suppose that $P = P_n^{\circ n}$ and, consequently, $\Phi_{P_n}(t) = \Phi_P^{1/n}(t)$ $(n = 1, 2, \ldots)$. Hence, by Theorem 9, we get

$$n\big(\Phi_{P_n}(t) - 1\big) \to \log \Phi_P(t) \tag{56}$$

uniformly in every finite interval. Setting $m_n(\mathscr{A}) = n \int_{\mathscr{A}} \omega(x) P_n(dx)$ and taking into account (34), we obtain

$$n\big(\Phi_{P_n}(t) - 1\big) = \int_0^\infty \frac{\Omega(tx) - 1}{\omega(x)} m_n(dx). \tag{57}$$

Let ε be a positive number satisfying the inequality $\varepsilon < \frac{1}{3}$. Since the function $\log \Phi_P(t)$ is continuous and $\log \Phi_P(0) = 0$, we can find, by virtue of (56), an integer n_0 and a positive number t_0 such that $n\big(1 - \Phi_{P_n}(t)\big) < \varepsilon$ whenever $0 \leqslant t \leqslant t_0$ and $n \geqslant n_0$. Hence and from (57) it follows that

$$\int_0^\infty \frac{1 - \Omega(tx)}{\omega(x)} m_n(dx) < \varepsilon \quad (0 \leqslant t \leqslant t_0;\ n \geqslant n_0). \tag{58}$$

Let M be a characteristic measure of the algebra in question. Since, by (**), $T_a M \to E_0$ as $a \to 0$, there exists a positive number a_0 such that

(59) $$\int_{t_0}^{\infty} M_0(dx) < \varepsilon,$$

where the measure M_0, being also a characteristic measure, is defined by the formula $M_0 = T_{a_0} M$. By (40) the characteristic function of the measure M_0 is given by the formula

(60) $$\Phi_{M_0}(t) = \exp(-ct^{\varkappa}),$$

where $\varkappa$ is the characteristic exponent of the algebra and c is a positive constant. Put

(61) $$x_\varepsilon = \left(\frac{\log\varepsilon}{-c}\right)^{1/\varkappa}.$$

Since the integrand in (58) is non-negative, we conclude that the inequality

$$\int_0^{t_0}\int_{x_\varepsilon}^{\infty} \frac{1-\Omega(tx)}{\omega(x)}\, m_n(dx)\big) M_0(dt) \leqslant \varepsilon \qquad (n \geqslant n_0)$$

holds. Since the integrand is bounded in the strip $0 \leqslant t \leqslant t_0$, $x_\varepsilon \leqslant x < \infty$, we can change the order of integrations:

(62) $$\int_{x_\varepsilon}^{\infty} \frac{1}{\omega(x)} \int_0^{t_0} \big(1-\Omega(tx)\big) M_0(dt)\, m_n(dx) \leqslant \varepsilon \qquad (n \geqslant n_0).$$

Further, by a simple computation, from (59), (60) and (61) we get

$$\int_0^{t_0} \big(1-\Omega(tx)\big) M_0(dt) = 1-\Phi_{M_0}(x) - \int_{t_0}^{\infty} \big(1-\Omega(tx)\big) M_0(dt)$$
$$\geqslant 1-\Phi_{M_0}(x_\varepsilon) - 2\int_{t_0}^{\infty} M_0(dt) \geqslant 1-3\varepsilon$$

whenever $x \geqslant x_\varepsilon$. Thus, setting $b = \max_{0\leqslant x<\infty} \omega(x)$, we obtain, by (62) the inequality

(63) $$\frac{1-3\varepsilon}{b} \int_{x_\varepsilon}^{\infty} m_n(dx) \leqslant \varepsilon \qquad (n \geqslant n_0).$$

Consequently, to prove that the sequence $m_1, m_2, \ldots$ is compact, it is sufficient to prove that the measures m_n are bounded in common.

From the definition of the function ω and from (54) it follows that there exists a positive number t_* such that $t_* \leqslant t_0$ and

$$d = \min_{0\leqslant x\leqslant x_\varepsilon} \frac{1-\Omega(t_* x)}{\omega(x)} > 0.$$

Since the integrand in (58) is non-negative, we have

$$d\int_0^{x_\varepsilon} m_n(dx) \leqslant \int_0^{x_\varepsilon} \frac{1-\Omega(t_* x)}{\omega(x)}\, m_n(dx) \leqslant \varepsilon \qquad (n \geqslant n_0)$$

which together with (63) shows that m_n are bounded in common. Consequently, the sequence $m_1, m_2, \ldots$ is compact. Let m be its limit point. Since for any t the function $\big(\Omega(tx)-1\big)/\omega(x)$ is bounded and continuous on the half-line $0 \leqslant x < \infty$, from (56) and (57) we get

$$\log \Phi_P(t) = \int_0^{\infty} \frac{\Omega(tx)-1}{\omega(x)}\, m(dx),$$

which completes the proof of the necessity of our condition.

Conversely, suppose that the function Φ is given by formula (55). Consider the function $\exp\big(-m(\{0\})t^{\varkappa}\big)$. If $m(\{0\}) = 0$, then it is the characteristic function of the measure E_0 which is obviously infinitely decomposable. If $m(\{0\}) > 0$, then, by (40), it is the characteristic function of a characteristic measure of the algebra. Since, by (40) and Theorem 10, characteristic measures are also infinitely decomposable, we infer that there exists an infinitely decomposable measure P_0 such that $\Phi_{P_0}(t) = \exp\big(-m(\{0\}t^{\varkappa})\big)$. Now we define a sequence of measures of Poisson type by means of the formula

$$P_n = \sum_{s=0}^{\infty} \frac{c_n^s}{s!} Q_n^{\circ s} \exp(-c_n),$$

where

$$c_n = \int_{1/n}^{\infty} \frac{1}{\omega(x)}\, m(dx), \qquad Q_n = E_0 \quad \text{if} \quad c_n = 0$$

and

$$Q_n(\mathscr{A}) = c_n^{-1} \int_{\mathscr{A}\cap[1/n,\infty)} \frac{1}{\omega(x)}\, m(dx) \quad \text{if} \quad c_n > 0.$$

Since measures of Poisson type are infinitely decomposable, the measure $P_0 \circ P_n$ is, by Theorem 11, also infinitely decomposable. From the equation

$$\Phi_{P_0\circ P_n}(t) = \exp\big(-m(\{0\})t^{\varkappa} - c_n!(\Phi_{Q_n}(t)-1)\big)$$
$$= \exp \int_{\{0\}\cup[1/n,\infty)} \frac{\Omega(tx)-1}{\omega(x)}\, m(dx) \qquad (n = 1, 2, \ldots)$$

it follows that $\Phi_{P_0\circ P_n}(t) \to \Phi(t)$ uniformly in every finite interval. Thus, by Theorem 11, the sequence $P_0\circ P_n$ converges to an infinitely decomposable measure. It is clear that the function $\Phi(t)$ is the characteristic function of this limit measure, so that the Theorem is completely established.

6. Stable measures

A probability measure P is said to be *stable* in a generalized convolution algebra if for any pair a, b of positive numbers there exists a positive number c such that $T_aP\circ T_bP = T_cP$.

THEOREM 14. *The family of stable measures is closed under the transformations T_a $(a \geqslant 0)$ and passages to the limit.*

Proof. Let P be a stable measure. The stability of T_aP $(a\geqslant 0)$ is a direct consequence of condition (iii) for generalized convolutions. Now suppose that a sequence $P_1, P_2, \ldots$ of stable probability measures is weakly convergent to a probability measure Q. If $Q = E_0$, then it is obviously stable. Therefore we may assume that $Q \neq E_0$. Given positive numbers a and b, there exists a positive number c_n such that $T_aP_n\circ T_bP_n = T_{c_n}P_n$. Hence it follows that

$$\Phi_{P_n}(at)\Phi_{P_n}(bt) = \Phi_{P_n}(c_nt) \quad (n = 1, 2, \ldots). \tag{64}$$

The sequence $c_1, c_2, \ldots$ cannot tend to ∞. Indeed, by (64), this would imply $\Phi_{P_n}(t) = \Phi_{P_n}(at/c_n)\,\Phi_{P_n}(bt/c_n) \to \Phi_Q^2(0) = 1$, which contradicts the hypothesis $Q\neq E_0$. Thus $c_1, c_2, \ldots$ contains a convergent subsequence. Let c be its limit. From (64) we obtain the equation $\Phi_Q(at)\Phi_Q(bt) = \Phi_Q(ct)$. Since the left-hand side of this equation is not identically equal to 1, we have $c > 0$. Furthermore, the last equation implies $T_aQ\circ T_bQ = T_cQ$ which completes the proof.

THEOREM 15. *A probability measure is a weak limit of a sequence $T_{c_n}P^{\circ n}$, where $c_n > 0$ $(n = 1, 2, \ldots)$ and $P\in\mathfrak{P}$, if and only if it is stable.*

Proof. Let Q be the limit of a sequence $T_{c_n}P^{\circ n}$. Since E_0 is stable, we may assume that $Q\neq E_0$. By Theorem 4, there are positive numbers c_0 and λ such that $\Phi_Q(t) = \exp(-c_0t^\lambda)$. Setting, for any pair a, b of positive numbers, $c = (a^\lambda+b^\lambda)^{1/\lambda}$, we have $\Phi_{T_aQ\circ T_bQ}(t) = \Phi_{T_aQ}(t)\Phi_{T_bQ}(t) = \Phi_{T_cQ}(t)$ and, consequently, $T_aQ\circ T_bQ = T_cQ$ which shows that Q is a stable measure.

Conversely, let Q be a stable measure. For any positive integer n there exists a positive number a_n such that $Q^{\circ n} = (T_1Q)^{\circ n} = T_{a_n}Q$. Setting $c_n = a_n^{-1}$, we have $T_{c_n}Q^{\circ n} = Q$ and, consequently, Q is the limit of the sequence $T_{c_n}Q^{\circ n}$.

THEOREM 16. *A function Φ is a characteristic function of a stable measure if and only if it is of the form*

$$\Phi(t) = \exp(-ct^\lambda), \tag{65}$$

where $c\geqslant 0$ and λ is either the characteristic exponent $\varkappa$ of the algebra or $\lambda > 0$ and

$$\int_0^1 \frac{\omega(x)}{x^{1+\lambda}}\,dx < \infty. \tag{66}$$

Proof. Let Q be a stable measure. Since the characteristic function of E_0 is of the form (65) with $c = 0$, we may assume that $Q\neq E_0$. From Theorems 4 and 15 it follows that $\Phi_Q(t) = \exp(-ct^\lambda)$, where c and λ are positive numbers. Suppose that $\lambda\neq\varkappa$. We have to prove condition (66). First we shall prove the formula

$$\lim_{x\to 0}\frac{\omega(x)}{x^\lambda} = 0. \tag{67}$$

Contrary to (67) let us suppose that there exists a sequence $x_1, x_2, \ldots$ of positive numbers tending to 0 such that

$$\lim_{n\to\infty}\frac{\omega(x_n)}{x_n^\lambda} = v \quad (0 < v\leqslant\infty).$$

From (41) and (53) in the case $v < \infty$ we get

$$\lim_{n\to\infty}\frac{1-\Omega(tx_n)}{x_n^\lambda} = \lim_{n\to\infty}\frac{1-\Omega(tx_n)}{1-\Omega(x_n)}\lim_{n\to\infty}\frac{\omega(x_n)}{x_n^\lambda} = vt^\varkappa \tag{68}$$

uniformly in every finite interval. If $v = \infty$, then

$$\lim_{n\to\infty}\frac{1-\Omega(tx_n)}{x_n^\lambda} = \begin{cases}\infty & \text{if } t > 0,\\ 0 & \text{if } t = 0.\end{cases} \tag{69}$$

Using the formula

$$\int_0^\infty \big(1-\Omega(txx_n)\big)Q(dx) = 1-\exp(-cx_n^\lambda t^\lambda)$$

and Fatou Lemma we obtain

$$ct^\lambda = \lim_{n\to\infty}\int_0^\infty\frac{1-\Omega(txx_n)}{x_n^\lambda}Q(dx) \geqslant \int_0^\infty \lim_{n\to\infty}\frac{1-\Omega(txx_n)}{x_n^\lambda}Q(dx). \tag{70}$$

Hence and from (69) it follows that in the case $v = \infty$ the measure Q is concentrated at the origin, i. e. $Q = E_0$, which contradicts the hypothesis. If $v < \infty$, then from (68) and (70) we get $ct^\lambda \geqslant vt^\varkappa \int_0^\infty x^\varkappa Q(dx)$ for all $t \geqslant 0$. Since $Q \neq E_0$, the integral $\int_0^\infty x^\varkappa Q(dx)$ is positive. Thus the function $t^{\varkappa-\lambda}$ is bounded on the positive half-line. But this implies $\lambda = \varkappa$ which contradicts the hypothesis. Formula (67) is thus proxed.

Now we proceed to the proof of (66). Contrary to (66) let us suppose that

$$\int_0^1 \frac{\omega(x)}{x^{1+\lambda}} dx = \infty. \tag{71}$$

From (53) and (67) it follows that both integrals

$$\int_0^\infty \frac{\omega(x)}{x^{1+\nu}} dx \quad \text{and} \quad \int_0^\infty \frac{1-\Omega(x)}{x^{1+\nu}} dx$$

are finite for all positive $\nu < \lambda$. Since the kernel Ω is not identically equal to 1, we have also the inequality

$$\int_0^\infty \frac{1-\Omega(x)}{x^{1+\nu}} dx > 0.$$

Hence it follows that the formula

$$m_\nu(\mathscr{A}) = b_\nu \int_{\mathscr{A}} \frac{\omega(x)}{x^{1+\nu}} dx, \tag{72}$$

where

$$b_\nu = \left(\int_0^\infty \frac{1-\Omega(x)}{x^{1+\nu}} dx\right)^{-1}, \tag{73}$$

defines a finite measure on the positive half-line provided $0 < \nu < \lambda$. Taking into account (53), (71) and (73), we conclude that $\lim_{\nu\to\lambda} b_\nu = 0$. Hence and from (72) it follows that

$$\lim_{\nu\to\lambda} \int_a^\infty m_\nu(dx) = 0 \quad \text{for} \quad a > 0. \tag{74}$$

The measures m_ν $(\nu < \lambda)$ are bounded in common. Indeed, from (72) by simple computations we get the formula

$$\int_0^\infty \frac{1-\Omega(tx)}{\omega(x)} m_\nu(dx) = t^\nu \quad (\nu < \lambda). \tag{75}$$

Since the integrand is non-negative, we have

$$\int_0^{x_0} m_\nu(dx) = \int_0^{x_0} \frac{1-\Omega(x)}{\omega(x)} m_\nu(dx) \leqslant 1 \quad (\nu < \lambda),$$

where x_0 is defined by (53). Comparying this inequality with (74) we infer that m_ν $(\nu < \lambda)$ are bounded in common. Moreover, condition (74) implies the compactness of the family m_ν $(\nu < \lambda)$. Let m_* be a limit point of this family as $\nu \to \lambda$. By (74) the measure m_* is concentrated at the origin, i. e. $m_* = a_* E_0$, where a_* is a non-negative number. Since for any number t the function $\big(1-\Omega(tx)\big)/\omega(x)$ is continuous and bounded on the positive half-line, we have for a sequence of indices ν tending to λ the relation

$$\int_0^\infty \frac{1-\Omega(tx)}{\omega(x)} m_\nu(dx) \to \int_0^\infty \frac{1-\Omega(tx)}{\omega(x)} m_*(dx).$$

By (75) the left-hand side of this formula tends to t^λ and, by (54), the right-hand side is equal to $a_* t^\varkappa$. Thus $\lambda = \varkappa$ which contradicts the hypothesis $\lambda \neq \varkappa$. Condition (66) is thus proved.

Now we shall prove that any function of the form (65) is a characteristic function of a stable probability measure. If $c = 0$, then (65) is the characteristic function of the measure E_0. Suppose that $c > 0$. If $\lambda = \varkappa$, then (65) is the characteristic function of a characteristic measure of the algebra. It remains the case $\lambda \neq \varkappa$ for which condition (66) is satisfied. Hence it follows that both integrals

$$\int_0^\infty \frac{\omega(x)}{x^{1+\lambda}} dx \quad \text{and} \quad \int_0^\infty \frac{1-\Omega(x)}{x^{1+\lambda}} dx$$

are finite. Moreover, since the kernel Ω is not identically equal to 1, the last integral is positive. Thus the measure m defined by the formula

$$m(\mathscr{A}) = bc \int_{\mathscr{A}} \frac{\omega(x)}{x^{1+\lambda}} dx, \quad \text{where} \quad b = \left(\int_0^\infty \frac{1-\Omega(x)}{x^{1+\lambda}} dx\right)^{-1},$$

is finite on the positive half-line. Thus, by Theorem 13, there exists an infinitely decomposable measure Q such that

$$\Phi_Q(t) = \exp \int_0^\infty \frac{\Omega(tx)-1}{\omega(x)} m(dx).$$

Hence, by simple computations, we get the formula

$$\Phi_Q(t) = \exp bc \int_0^\infty \frac{\Omega(tx)-1}{x^{1+\lambda}} dx = \exp(-ct^\lambda).$$

It is very easy to verify that the measure Q is stable. This completes the proof.

We conclude this paper with a characterization theorem for α-convolution algebras.

THEOREM 17. *Let* $(\mathfrak{P}, \circ)$ *be a regular generalized convolution algebra. If there exists in* $\mathfrak{P}$ *a stable purely atomic measure* Q *different from* E_0, *then* $(\mathfrak{P}, \circ)$ *is an* α*-convolution algebra and* $Q = E_a$ *for a positive number* a.

Proof. Let Q be a stable purely atomic measure different from E_0. By Theorem 16 its characteristic function is of the form $\Phi_Q(t) = \exp(-ct^\alpha)$, where c and α are positive constants. This formula implies the equation

$$\Phi_{T_x Q}(t)\Phi_{T_y Q}(t) = \Phi_{T_{(x^\alpha+y^\alpha)^{1/\alpha}}Q}(t).$$

Hence it follows that

$$T_x Q \circ T_y Q = T_{(x^\alpha+y^\alpha)^{1/\alpha}} Q. \tag{76}$$

Let $\mathscr{A}$ be the set of all atoms of the measure Q. Of course, the set $\mathscr{A}$ is at most denumerable and the measure Q can be written in the form $Q = \sum_{a\in\mathscr{A}} c_a E_a$, where $\sum_{b\in\mathscr{A}} c_a = 1$ and $c_a > 0$ for all $a \in \mathscr{A}$. Hence and from (76) we get the equation

$$\sum_{a\in\mathscr{A}} \sum_{b\in\mathscr{A}} c_a c_b E_{ax} \circ E_{by} = \sum_{\mathscr{A}} c_a E_{(x^\alpha+y^\alpha)^{1/\alpha} a},$$

which shows that for any pair $a, b \in \mathscr{A}$ the measure $E_{ax} \circ E_{by}$ is purely atomic and all its atoms belong to the set $\{(x^\alpha+y^\alpha)^{1/\alpha} a \colon a \in \mathscr{A}\}$. Thus, denoting the last set by $\mathscr{A}(x, y)$ and the set of all atoms of $E_x \circ E_y$ by $\mathscr{B}(x, y)$, we have the inclusion

$$\mathscr{B}(ax, by) \subset \mathscr{A}(x, y) \qquad (x, y \geqslant 0;\ a, b \in \mathscr{A}). \tag{77}$$

Let x be a positive number. Since $Q \neq E_0$, the set $\mathscr{A}$ contains a positive number, say a_0. Suppose that $\mathscr{A}$ contains a number b_0 different from a_0. It is very easy to verify that for any pair a_1, a_2 $(a_2 > 0)$ of elements of $\mathscr{A}$ both expressions $a_0^\alpha a_1^\alpha - b_0^\alpha a_2^\alpha$ and $a_0^\alpha a_2^\alpha - a_0^\alpha a_1^\alpha$ cannot vanish simultaneously. Since the set $\mathscr{A}$ is at most denumerable, we can find a positive number z satisfying for all a_1 and a_2 $(a_2 > 0)$ from $\mathscr{A}$ the inequality

$$z^\alpha(a_0^\alpha a_1^\alpha - b_0^\alpha a_2^\alpha) \neq x^\alpha a_0^\alpha (a_2^\alpha - a_1^\alpha). \tag{78}$$

Moreover, we may suppose that the number z satisfies the inequality

$$z^\alpha a_1^\alpha \neq x^\alpha (a_0^\alpha - 1) \tag{79}$$

for all $a_1 \in \mathscr{A}$.

Suppose that $q \in \mathscr{A}(x, z) \cap \mathscr{A}(x, a_0^{-1} b_0 z)$. There exist then elements b_1 and b_2 belonging to $\mathscr{A}$ such that

$$q^\alpha = (x^\alpha + z^\alpha) b_1^\alpha = \big(x^\alpha + (a_2^{-1} b_0 z)^\alpha\big) b_2^\alpha.$$

Hence, by simple computation, we get the equation

$$z^\alpha (a_0^\alpha b_1^\alpha - b_0^\alpha b_2^\alpha) = x^\alpha a_0^\alpha (b_2^\alpha - b_1^\alpha),$$

which, according to (78), implies $b_2 = 0$ and, consequently, $q = 0$. Thus,

$$\mathscr{A}(x, z) \cap \mathscr{A}(x, a_0^{-1} b_0 z) \subset \{0\}. \tag{80}$$

Since $\mathscr{B}(b_0 x, b_0 z) = \mathscr{B}\big(b_0 x, a_0^{-1} b_0 z\big)$, we infer, by (77), that the intersection (80) is non-void. Thus it contains the number 0. Hence it follows that $0 \in \mathscr{A}$. Substituting in (77) $a = a_0$, $b = 0$, $y = z$ and taking into account the obvious equation $\mathscr{B}(a_0 x, 0) = \{a_0 x\}$, we obtain the relation $a_0 x \in \mathscr{A}(x, z)$. Consequently, there exists an element c_1 in $\mathscr{A}$ such that $a_0 x = (x^\alpha + z^\alpha)^{1/\alpha} c_1$. But this equation contradicts (79). Thus $\mathscr{A}$ is a one-point set $\{a_0\}$ and, consequently, $Q = E_{a_0}$. Moreover, from (77) it follows that $\mathscr{B}(a_0 x, a_0 y) = \{(x^\alpha + y^\alpha)^{1/\alpha} a_0\}$. Hence we get $E_x \circ E_y = E_{(x^\alpha+y^\alpha)^{1/\alpha}}$ $(x \geqslant 0,\ y \geqslant 0)$. Now it is very easy to verify that for convex linear combinations of the measures E_a $(a \geqslant 0)$ formula (2) holds. Since they form a dense subset of $\mathfrak{P}$ in the sense of weak convergence, formula (2) holds for all measures P, Q from $\mathfrak{P}$. In other words, the algebra in question is an α-convolution algebra. The Theorem is thus proved.

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MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES
INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS, WROCŁAW UNIVERSITY
INSTYTUT MATEMATYCZNY, UNIWERSYTET WROCŁAWSKI

Reçu par la Rédaction le 4. 3. 1963