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The Hilbert transform and rearrangement of functions

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1. Introduction. Let f be a measurable function on a measure space M with measure m. We define the *distribution function of* f, $\lambda_{|f|} = \lambda$, by letting $\lambda(y) = m\{x \in M;\ |f(x)| > y\}$ for each $y > 0$. It is easy to see that λ is non-increasing and continuous from the right. Two measurable functions, not necessarily defined on the same measure space, are said to be *equimeasurable* if they have the same distribution function. Given any measurable function f on M we can always find a non-increasing extended-real valued function defined on $(0, \infty)$ that is equimeasurable with f, namely $f^*(t) = \inf\{y > 0;\ \lambda_{|f|}(y) \leqslant t\}$. This function will be called the *non-increasing rearrangement of f onto* $(0, \infty)$ (note that if λ is one-to-one and onto $(0, \infty)$, then f^* is simply λ^{-1}).

We now state some elementary properties of the functions λ and f^*:

(1) *If* $f \in L^p(M)$, $p \geqslant 1$, *then*

$$\int_M |f|^p\, dm = p\int_0^\infty y^{p-1}\lambda(y)\, dy.$$

Consequently

(2) $\|f\|_p = \Big(\int_M |f|^p\, dm\Big)^{1/p} = \Big(\int_0^\infty [f^*(t)]^p\, dt\Big)^{1/p} = \|f^*\|_p$.

(3) *If f and g are two measurable functions on M, then*

$$\int_M fg\, dm \leqslant \int_0^\infty f^*(t) g^*(t)\, dt.$$

We shall be particularly interested in the integral mean of f^*,

$$f^{**}(s) = \frac{1}{s}\int_0^s f^*(t)\, dt.$$

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It is not hard to see that $f^{**}(s)$ is the Hardy-Littlewood maximal function (see [4], p. 32) of f^* evaluated at s. It can be shown easily, with the aid of (3), that

(4) *If f is a measurable function on an atom-free M and $s>0$, then*

$$sf^{**}(s) = \sup_{E\subset M,\, m(E)=s}\Big(\int_E |f|\,dm\Big).$$

This result, in turn, can be used to derive the following basic property of the function f^{**}:

(5) *If $s>0$, let V be the space of all measurable functions f on M such that $sf^{**}(s)<\infty$. Then V is a linear space and the mapping $f\to sf^{**}(s)=\|f\|^{(s)}$ is a norm making V a Banach space.*

Since, for each $s>0$,

$$\frac{1}{s}\|f\|^{(s)} = f^{**}(s)$$

is a norm, so is the mapping

$$f\to N_p(f) = \Big(\int_0^\infty [f^{**}(s)]^p\,ds\Big)^{1/p}$$

when $p\geqslant 1$. From property (2) and the obvious inequality $f^*(s)\leqslant f^{**}(s)$ we obtain the relation $\|f\|_p\leqslant N_p(f)$. On the other hand, a special case of Hardy's inequality (see [4], p. 20) gives us

(6) *If $p>1$,*

$$N_p(f)\leqslant \frac{p}{p-1}\|f\|_p;$$

thus N_p and $\|\ \|_p$ are equivalent norms.

We shall apply these notions to the Hilbert transform

$$\tilde f(x) = \lim_{\varepsilon\to 0+}\frac{1}{\pi}\int_{|t-x|>\varepsilon}\frac{f(t)}{t-x}\,dt.$$

It is well known that $\tilde f(x)$ exists a. e. if $f\in L^p(-\infty,\infty)$, $1\leqslant p<\infty$ (see [3]). We shall assume this fact and the following result (see [2])(1):

(1) We shall really use the fact that if f is locally integrable then

$$\lim_{\varepsilon\to 0+}\int_{a>|t-x|>\varepsilon}\frac{f(t)}{x-t}\,dt$$

exists a. e., where $a>0$ is any positive real number (see [4], pp. 57 and 131). We could limit ourselves to a dense class of functions, say finite linear combinations of characteristic functions of intervals, on which the Hilbert transform is obviously defined and, with the help of our results, then pass to all of L^p by extending the definition of $\tilde f$ as a limit in the norm of Hilbert transforms of functions in this class. This would make this paper self-contained, but we choose not to do this in order to avoid unnecessary technical difficulties.

(7) *If χ_E is the characteristic function of a measurable set $E\subset(-\infty,\infty)$ of finite Lebesgue measure $|E|$, then the distribution function of $\tilde\chi_E$ is $\lambda_{|\tilde\chi_E|}(y) = 2|E|/\sinh\pi y$. Consequently,*

$$\tilde\chi_E^*(t) = \frac{1}{\pi}\sinh^{-1}(2|E|/t).$$

The principal result of this paper is the following theorem:

THEOREM 1. *If*

$$\int_0^\infty f^*(t)\sinh^{-1}\Big(\frac{1}{t}\Big)dt<\infty,$$

then $\tilde f(x)$ exists a. e. and for each $s>0$

$$s\tilde f^{**}(s)\leqslant \frac{2}{\pi}\int_0^\infty f^*(t)\sinh^{-1}\Big(\frac{s}{t}\Big)dt = \frac{2s}{\pi}\int_0^\infty \frac{f^{**}(t)}{\sqrt{s^2+t^2}}\,dt.$$

The second section of this paper will be devoted to the proof of this theorem and an application of it deriving the well-known M. Riesz inequality for the Hilbert transform of functions in L^p, $1<p<\infty$. In the third section we shall extend this result to certain singular integrals defined in Euclidean n-dimensional space E^n. In the fourth section we prove the analog of Theorem 1 for the conjugate functions of functions defined on $[0,2\pi)$. Of special interest is the fact that our methods enable us to obtain a best possible result (Theorem 5) when $|f|\log^+|f|$ is integrable.

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2. The Hilbert transform. We have

$$\lim_{\varepsilon\to 0+}\frac{1}{\pi}\int_{|t-x|>\varepsilon}\frac{f(t)}{t-x}\,dt = \lim_{\varepsilon\to 0+}\frac{1}{\pi}\int_{a>|t-x|>\varepsilon}\frac{f(t)}{t-x}\,dt + \frac{1}{\pi}\int_{|t-x|\geqslant a}\frac{f(t)}{t-x}\,dt.$$

As was observed before the principal-value integral

$$\lim_{\varepsilon\to 0+}\frac{1}{\pi}\int_{a>|t-x|>\varepsilon}\frac{f(t)}{t-x}\,dt$$

exists a. e. while the second integral is absolutely convergent, since, by (3),

$$\int_{|t-x|\geqslant a}\Big|\frac{f(t)}{t-x}\Big|\,dt\leqslant 2\int_0^\infty\frac{f^*(t)}{t+a}\,dt$$

and the last integral converges whenever

$$\int_0^\infty f^*(t)\sinh^{-1}\Big(\frac{1}{t}\Big)dt<\infty.$$

Because of property (4) Theorem 1 will be established if we can show that

$$\int_E |\tilde{f}(x)|\,dx \leqslant \frac{2}{\pi}\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)dt = \frac{2s}{\pi}\int_0^\infty \frac{f^{**}(t)}{\sqrt{s^2+t^2}}\,dt$$

for each set E of measure s. Given such a set, let $E_1 \subset E$ be the subset on which $\tilde{f} \geqslant 0$ and $E_2 = E - E_1$. Then, using property (3),

$$\int_E |\tilde{f}(x)|\,dx = \int_{E_1} \tilde{f}(x)\,dx - \int_{E_2} \tilde{f}(x)\,dx = \int_{-\infty}^{\infty} \tilde{f}(x)\chi_{E_1}(x)\,dx - \int_{-\infty}^{\infty} \tilde{f}(x)\chi_{E_2}(x)\,dx$$

$$= -\int_{-\infty}^{\infty} f(x)\tilde{\chi}_{E_1}(x)\,dx + \int_{-\infty}^{\infty} f(x)\tilde{\chi}_{E_2}(x)\,dx$$

$$\leqslant \int_0^\infty f^*(t)\tilde{\chi}^*_{E_1}(t)\,dt + \int_0^\infty f^*(t)\tilde{\chi}^*_{E_2}(t)\,dt$$

$$= \frac{1}{\pi}\int_0^\infty f^*(t)\left\{\sinh^{-1}\frac{2|E_1|}{t} + \sinh^{-1}\frac{2|E_2|}{t}\right\}dt$$

$$\leqslant \frac{2}{\pi}\int_0^\infty f^*(t)\sinh^{-1}\left\{\frac{|E_1|}{t} + \frac{|E_2|}{t}\right\}dt = \frac{2}{\pi}\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)dt$$

and the desired inequality is proved. The equality

$$\frac{2}{\pi}\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)dt = \frac{2s}{\pi}\int_0^\infty \frac{f^{**}(t)}{\sqrt{s^2+t^2}}$$

follows by integrating by parts and observing that the integrated term vanishes.

Many classical results follow from this theorem. For example, the following theorem of M. Riesz (see [1]) is an immediate consequence:

THEOREM 2. *If $1 < p < \infty$, there exists a constant A_p, independent of $f \in L^p$, such that $\|\tilde{f}\|_p \leqslant A_p\|f\|_p$.*

Proof. We have

$$\|\tilde{f}\|_p \leqslant N_p(\tilde{f}) = \left\{\int_0^\infty [\tilde{f}^{**}(s)]^p ds\right\}^{1/p} \leqslant \frac{2}{\pi}\left\{\int_0^\infty \left[\frac{1}{s}\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)dt\right]^p ds\right\}^{1/p}$$

$$= \frac{2}{\pi}\left\{\int_0^\infty \left[\frac{1}{s}\int_0^\infty f^*\left(\frac{s}{u}\right)\sinh^{-1}(u)\,\frac{s}{u^2}\,du\right]^p ds\right\}^{1/p}$$

$$\leqslant \frac{2}{\pi}\int_0^\infty \frac{\sinh^{-1}(u)}{u^2}\left\{\int_0^\infty \left[f^*\left(\frac{s}{u}\right)\right]^p ds\right\}^{1/p} du = \frac{2}{\pi}\int_0^\infty \frac{\sinh^{-1}(u)}{u^2}\,u^{1/p}\|f^*\|_p\,du$$

$$= \frac{2}{\pi}\|f^*\|_p\int_0^\infty \frac{\sinh^{-1}u}{u^{1+1/p'}}\,du,$$

where $(1/p)+(1/p') = 1$.

Thus we obtain Theorem 2 with

$$A_p = \frac{2}{\pi}\int_0^\infty \frac{\sinh^{-1}u}{u^{1+1/p'}}\,du.$$

3. Singular integrals with odd kernels. We shall use capital letters $X, Y, \ldots$ to denote points $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n), \ldots$ of Euclidean n-dimensional space E^n and primed capital letters $X', Y', \ldots$ to denote points on the surface of the unit sphere Σ of E^n. For example, we shall consistently write $Y = rY'$, where $r = |Y| = \sqrt{y_1^2+y_2^2+\ldots+y_n^2}$. The element of volume in E^n will be denoted by dY while dY' will represent the element of area of the surface Σ. The operator

$$\hat{f}(X) = \lim_{\varepsilon\to 0+}\int_{|Y|>\varepsilon} \frac{\Omega(Y)}{|Y|^n} f(X-Y)\,dY,$$

where Ω is a homogeneous function of degree zero (that is, $\Omega(Y) = \Omega(Y')$) satisfying (a) $\Omega(Y') = -\Omega(-Y')$ for all $Y' \in \Sigma$ and (b) $\|\Omega\| = \int_\Sigma |\Omega(Y')|\,dY' < \infty$, is called a *singular integral operator with odd kernel*. It is well known (see [5]) that $\hat{f}(X)$ exists a. e. when $f \in L^p(E^n)$, $1 < p < \infty$, and that the mapping $f \to \hat{f}$ is a bounded linear operator on $L^p(E^n)$.

W shall prove the following extension of Theorem 1:

THEOREM 3. *If $1 < p < \infty$, then*

$$s\hat{f}^{**}(s) \leqslant \|\Omega\|\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)dt$$

for all $s > 0$.

The proof will depend on the following lemma, an analog of property (7):

LEMMA. *If E is a measurable subset of E^n of finite n-dimensional Lebesgue measure $m(E)$ and χ_E is the characteristic function of E, then*

$$s\hat{\chi}_E^{**}(s) \leqslant \frac{\|\Omega\|}{2}\int_0^s \sinh^{-1}\left(\frac{2m(E)}{t}\right)dt,$$

for all $s > 0$.

Proof. Let $\Sigma_1 = \{Y' \epsilon \Sigma;\ \Omega(Y') \geqslant 0\}$, $r = |Y|$ and $Y' = Y/r$ whenever $r \neq 0$. We may suppose that E is a finite union of disjoint cubes. (By approximating a general set of finite measure by such sets we then obtain the lemma.)(2). Then, if X is not on the boundary of E,

$$\begin{aligned}\hat{\chi}_E(X) &= \lim_{\varepsilon\to 0+} \int\limits_{|Y|>\varepsilon} \frac{\Omega(Y')}{|Y|^n} \chi_E(X-Y)\,dY \\ &= \lim_{\varepsilon\to 0+} \int\limits_{\Sigma_1} \Omega(Y') \left\{ \int\limits_{\varepsilon}^{\infty} \frac{\chi_E(X-rY')-\chi_E(X+rY')}{r}\,dr \right\} dY' \\ &= \int\limits_{\Sigma_1} \Omega(Y') \left\{ \lim_{\varepsilon\to 0+} \int\limits_{\varepsilon}^{\infty} \frac{\chi_E(X-rY')-\chi_E(X+rY')}{r}\,dr \right\} dY',\end{aligned}$$

the passage of the limit under the integral sign is justified since the convergence is uniform in Y' as long as X is not on the boundary of E.

For each fixed $Y' \epsilon \Sigma_1$ let $\Xi = \Xi_{Y'}$ be the hyperplane through the origin that is perpendicular to Y'. For $H \epsilon \Xi$ let $L = L_H$ be the line through H parallel to Y'. Each X in L has the form $X = H + xY'$ for some real number x. Let $E_L = E \cap L$ and ξ_L be the characteristic function of the set of all x such that $X = H + xY'$ belongs to E_L. Thus, $\xi_L(x) = \chi_E(X)$ whenever $X = H + xY'$ belongs to L. We then have

$$-\pi\tilde{\xi}_L(x) = \lim_{\varepsilon\to 0+} \int\limits_{|t|>\varepsilon} \frac{\xi_L(x-t)}{t}\,dt = \lim_{\varepsilon\to 0+} \int\limits_{\varepsilon}^{\infty} \frac{\xi_L(x-t)-\xi_L(x+t)}{t}\,dt.$$

Thus, by property (7), the measure of the set $\{x \epsilon (-\infty, \infty);\ |\tilde{\xi}_L(x)| > y > 0\}$ is $2m_1(E_L)/\sinh \pi y$, where $m_1(E_L)$ denotes the 1-dimensional Lebesgue measure of $E_L \subset L$.

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(2) More specifically, using the boundedness of our operator in L^p, $p > 1$, we have

$$\|\hat{\chi}_E\|^{(s)} = s\hat{\chi}_E^{**}(s) = \sup_{m(F)=s} \int\limits_F |\hat{\chi}_E|\,dm \leqslant \sup_{m(F)=s} [m(F)]^{1/p'} \Big[\int\limits_F |\hat{\chi}_E|^p\,dm\Big]^{1/p} \leqslant s^{1/p'} A_p \|\chi_E\|_p.$$

Thus, if E' is a finite disjoint union of cubes such that $m(E'-E)$ and $m(E-E')$ are small, then

$$\begin{aligned}|s\hat{\chi}_E^{**}(s) - s\hat{\chi}_{E'}^{**}(s)| &= \big|\|\chi_E\|^{(s)} - \|\hat{\chi}_{E'}\|^{(s)}\big| \leqslant \|\hat{\chi}_E - \hat{\chi}_{E'}\|^{(s)} \\ &\leqslant \|\hat{\chi}_{E-E'}\|^{(s)} + \|\hat{\chi}_{E'-E}\|^{(s)} \leqslant s^{1/p} A_p (\|\chi_{E-E'}\|_p + \|\chi_{E'-E}\|_p),\end{aligned}$$

which is small. Moreover,

$$\int\limits_0^s \sinh^{-1} \frac{2m(E)}{t}\,dt - \int\limits_0^s \sinh^{-1} \frac{2m(E')}{t}\,dt$$

is also small with $m(E)-m(E')$.

Now, to each $X \epsilon E^n$ there corresponds a unique $H \epsilon \Xi$ and real number x such that $X = H + xY'$. With x and X so related let $\hat{\zeta}_{E,Y'}(X) = \tilde{\xi}_L(x)$, where $L = L_H$. Also, we put

$$F = F(y, E, Y') = \{X \epsilon E^n;\ |\hat{\zeta}_{E,Y'}(X)| > y > 0\}$$

and

$$F_L = F(y, E, H, Y') = \{X \epsilon L_H;\ X = H + xY' \text{ and } |\tilde{\xi}_L(x)| > y > 0\} = F \cap L.$$

Then F_L has the same 1-dimensional Lebesgue measure along L as the set $\{x \epsilon (-\infty, \infty);\ |\tilde{\xi}_L(x)| > y > 0\}$ and, moreover, $F = \bigcup_{H\epsilon\Xi} F_L$. Thus, letting dx denote the element of measure along L and dH the element of measure in Ξ, we have, by Fubini's theorem,

$$\begin{aligned}m(F) &= \int\limits_{\Xi} \Big(\int\limits_{F_L} 1\,dx\Big)\,dH = \int\limits_{\Xi} \frac{2m_1(E_L)}{\sinh \pi y}\,dH \\ &= \frac{2}{\sinh \pi y} \int\limits_{\Xi} \Big(\int\limits_{E_L} dx\Big)\,dH = \frac{2}{\sinh \pi y} \int\limits_E dX = \frac{2m(E)}{\sinh \pi y}.\end{aligned}$$

In particular, we see that $m(F)$ is independent of Y'. Moreover, since $m(F)$ is the distribution function of $\hat{\zeta}_{E,Y'}$ evaluated at y, we obtain

$$s\hat{\zeta}_{E,Y'}^{**}(s) = \frac{1}{\pi} \int\limits_0^s \sinh^{-1}\Big(\frac{2m(E)}{t}\Big)\,dt. \tag{8}$$

Since

$$\hat{\chi}_E(X) = -\pi \int\limits_{\Sigma_1} \Omega(Y') \hat{\zeta}_{E,Y'}(X)\,dY',$$

(8) and the fact that the mapping $f \to sf^{**}(s)$ is a norm (property (5)) give us the inequality

$$\begin{aligned}s\hat{\chi}_E^{**}(s) &\leqslant \pi \int\limits_{\Sigma_1} \Omega(Y') s\hat{\zeta}_{E,Y'}^{**}(s)\,dY' = \int\limits_{\Sigma_1} \Omega(Y') \int\limits_0^s \sinh^{-1}\Big(\frac{2m(E)}{t}\Big)\,dt\,dY' \\ &= \frac{\|\Omega\|}{2} \int\limits_0^s \sinh^{-1}\Big(\frac{2m(E)}{t}\Big)\,dt\end{aligned}$$

and the lemma is proved.

Proof of Theorem 3. The argument is similar to that used in establishing Theorem 1. For $s > 0$ fixed we want to estimate $\int\limits_E |\hat{f}(X)|\,dX$ whenever $m(E) = s$. For such a set $E \subset E^n$ let $E_1 \subset E$ be the subset on which $\hat{f} \geqslant 0$ and $E_2 = E - E_1$. Then, as before,

$$\int\limits_E |\hat{f}| = \int\limits_{E_1} \hat{f} - \int\limits_{E_2} \hat{f} = -\int\limits_{E^n} f\hat{\chi}_{E_1} + \int\limits_{E^n} f\hat{\chi}_{E_2} \leqslant \int\limits_0^\infty f^*(t)\hat{\chi}_{E_1}^*(t)\,dt + \int\limits_0^\infty f^*(t)\hat{\chi}_{E_2}^*(t)\,dt.$$

But,

$$\int_0^\infty f^*(t)\hat{\chi}^*_{E_1}(t)\,dt = \lim_{\substack{a\to 0\\ b\to\infty}} \{f^*(b)\,b\hat{\chi}^{**}_{E_1}(b) - f^*(a)\,a\hat{\chi}^{**}_{E_1}(a)\} - \int_0^\infty t\hat{\chi}^{**}_{E_1}(t)\,df^*(t).$$

Since f belongs to $L^p(E^n)$ there exists a constant $A>0$ such that $f^*(t)\leqslant At^{-1/p}$. Thus,

$$f^*(t)t\hat{\chi}^{**}_{E_1}(t)\leqslant \frac{A\|\Omega\|}{2}t^{-1/p}\int_0^t \sinh^{-1}\frac{2m(E_1)}{s}\,ds\to 0 \quad \text{as} \quad t\to\infty.$$

Hence,

$$\int_0^\infty f^*(t)\hat{\chi}^*_{E_1}(t)\,dt \leqslant -\int_0^\infty t\hat{\chi}^{**}_{E_1}(t)\,df^*(t)$$

$$\leqslant -\frac{\|\Omega\|}{2}\int_0^\infty\left\{\int_0^t \sinh^{-1}\left(\frac{2m(E_1)}{s}\right)ds\right\}df^*(t) = \frac{\|\Omega\|}{2}\cdot\int_0^\infty f^*(t)\sinh^{-1}\frac{2m(E_1)}{t}\,dt.$$

A similar estimate holds for $\int_0^\infty f^*\hat{\chi}^*_{E_2}$; consequently

$$\int_E |\hat{f}| \leqslant \frac{\|\Omega\|}{2}\int_0^\infty f^*(t)\left\{\sinh^{-1}\frac{2m(E_1)}{t}+\sinh^{-1}\frac{2m(E_2)}{t}\right\}dt$$

$$\leqslant \|\Omega\|\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{m(E_1)+m(E_2)}{t}\right)dt$$

$$= \|\Omega\|\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)dt$$

and the theorem is proved.

4. The conjugate function. If a periodic function f of period 2π belongs to $L^1(0,2\pi)$ its conjugate function is defined by the principal value integral

$$\bar{f}(x) = -\frac{1}{\pi}\lim_{\varepsilon\to 0+}\left[\int_{-\pi}^{-\varepsilon}+\int_{\varepsilon}^{\pi}\right]\frac{f(x-t)}{2\tan(\frac{1}{2}t)}\,dt.$$

It is well-known (see [4], p. 131) that $\bar{f}(x)$ exists a. e. in $[-\pi,\pi)$ and in analogy to property (7), that the distribution function of $\bar{\chi}_E$, when E is a measurable subset of $[-\pi,\pi)$, satisfies

$$\lambda(y) = 4\tan^{-1}\left\{\frac{\sin(\frac{1}{2}|E|)}{\sinh \pi y}\right\} \qquad \text{(see [2])}.$$

Consequently

$$\bar{\chi}^*_E(t) = \frac{1}{\pi}\sinh^{-1}\left\{\frac{\sin(\frac{1}{2}|E|)}{\tan(\frac{1}{4}t)}\right\}.$$

Thus, the same argument that was used in proving Theorem 1 gives us the following analog to that result:

THEOREM 4. *If $f\in L^1(0,2\pi)$, then*

$$s\bar{f}^{**}(s)\leqslant \frac{2}{\pi}\int_0^{2\pi} f^*(t)\sinh^{-1}\left\{\frac{\sin(\frac{1}{4}x)}{\tan(\frac{1}{4}t)}\right\}dt \leqslant \frac{2}{\pi}\int_0^{2\pi} f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)dt,$$

where $0<s\leqslant 2\pi$.

As in § 2, it immediately follows from this theorem that the transformation $f\to\bar{f}$ is bounded from L^p into L^p, $1<p<\infty$. Another important consequence is the following theorem of Zygmund (see [6]):

THEOREM 5. *If $f\in L\log^+L$ (that is, if $\int_0^{2\pi}|f|\log^+|f|<\infty$), then $\bar{f}\in L^1(0,2\pi)$. Moreover*

$$\int_0^{2\pi}|\bar{f}(t)|\,dt \leqslant \frac{2}{\pi}\int_0^{2\pi} f^*(t)\sinh^{-1}\left(\cot\frac{t}{4}\right)dt = \frac{2}{\pi}\int_0^{2\pi} f^*(t)\log\cot\left(\frac{t}{8}\right)dt.$$

Proof. We have, by Theorem 4,

$$\int_0^{2\pi}|\bar{f}(t)|\,dt = \int_0^{2\pi}\bar{f}^*(t)\,dt = 2\pi\bar{f}^{**}(2\pi)\leqslant \frac{2}{\pi}\int_0^{2\pi} f^*(t)\sinh^{-1}\left(\cot\frac{t}{4}\right)dt.$$

But we may use the identity

$$\sinh^{-1}\left(\cot\frac{t}{4}\right) = \log\cot\frac{t}{8}$$

to get the last integral of Theorem 5. But an elementary argument shows that the last integral of Theorem 5 is finite if and only if $|f|\log^+|f|$ is integrable. This proves Theorem 5.

We remark that the inequality of Theorem 5 is a best possible inequality in the sense that there is a function for which the equality holds. In fact, let

$$f(x) = \begin{cases} 1 & \text{if} \quad 0\leqslant x<\pi,\\ -1 & \text{if} \quad \pi\leqslant x<2\pi,\end{cases}$$

$$f^*(x) = 1 \quad \text{if} \quad 0<t<2\pi.$$

But $\bar{f} = 2\bar{\chi}_E$ where $E = [0, \pi)$,

$$\int_0^{2\pi} |\bar{f}(t)|\,dt = 2\int_0^{2\pi} \bar{\chi}_E^*(t)\,dt = \frac{2}{\pi}\int_0^{2\pi} \sinh^{-1}\left(\frac{\sin\frac{1}{2}\pi}{\tan\frac{1}{4}t}\right)dt$$

$$= \frac{2}{\pi}\int_0^{2\pi} \sinh^{-1}\left(\cot\frac{t}{4}\right)dt = \frac{2}{\pi}\int_0^{2\pi} f^*(t)\sinh^{-1}\left(\cot\frac{t}{4}\right)dt.$$

Bibliography

[1] M. Riesz, *Sur les fonctions conjugées*, Mathematische Zeitschrift 27 (1927), p. 218-244.

[2] E. M. Stein and G. Weiss, *An extension of a Theorem of Marcinkiewicz and some of its applications*, J. of Math. and Mechanics 8 (1959), p. 263-284.

[3] E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Oxford 1937.

[4] A. Zygmund, *Trigonometric Series*, Cambridge 1959.

[5] — *On singular integrals*, Rendiconti di Matematica 16 (1957), p. 468-505.

[6] — *Sur les fonctions conjuguées*, Fund. Mat. 13 (1929), p. 284-303.

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