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Entire functions in B_0-algebras containing dense division algebras

by

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Let R be a B_0^*-*algebra*, i. e. locally convex metric linear algebra, in which operations of addition and multiplication are continuous with respect to both arguments. If R is moreover complete we say that R is B_0-*algebra*. In this note we shall consider only B_0^*-algebras over complex scalars. Without loss of generality we can assume that R possesses the unit e, i. e. such an element e that $ex = xe = x$ for all x belonging to R.

The topology in a B_0^*-algebra R may be introduced by means of a denumerable sequence of pseudonorms satisfying

(1) $$\|x\|_i \leqslant \|x\|_{i+1}, \quad i = 1, 2, \ldots,$$

and

(2) $$\|xy\|_i \leqslant \|x\|_{i+1}\|y\|_{i+1}$$

([9], Theorem 24). A sequence x_n tends to x if and only if

$$\lim_n \|x_n - x\|_i = 0, \quad i = 1, 2, \ldots$$

If we can choose a sequence of pseudonorms satysfying (1) in such a way that we can replace (2) by

(2') $$\|xy\|_i \leqslant \|x\|_i\|y\|_i,$$

then we say that R is a *multiplicatively convex algebra*, briefly *m-convex algebra* [1], [4].

We say that a B_0^*-algebra is a *division algebra* if each element x is invertible, i. e. there is an element x^{-1} such that $xx^{-1} = x^{-1}x = e$. If the operation of the inversion $x \to x^{-1}$ is continuous, then the given algebra is isomorphic to the algebra of complex numbers [2].

If a division B_0^*-algebra is m-convex, then it is isomorphic to the algebra of complex numbers ([4], Proposition 2.8).

If a division B_0^*-algebra is complete, i. e. is a B_0-algebra, then it is isomorphic to the algebra of complex numbers [10]. But in the general case of B_0^*-algebras this is not true. First counter-example has been constructed by Williamson [8].

Let

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an arbitrary entire function of the complex variable z. We say that the function $\varphi(x)$ is *determined* in a B_0-algebra R if for each x belonging to R the series $\sum_{n=0}^{\infty} a_n x^n$ is convergent.

If in a B_0-algebra R there are determined all entire functions, then R is m-convex [5]. It implies that if in the completion $\bar{R}$ of a division B_0^*-algebra R all entire functions are determined, then R is isomorphic to the algebra of complex numbers. It is not known, whether it is possible to replace the assumption that all entire functions are determined in the completion $\bar{R}$ by the assumption that all entire functions are determined in R.

The following question arises: is a division B_0^*-algebra R isomorphic to the algebra of complex numbers, if in its completion $\bar{R}$ there is determined some entire function (for example e^x).

In this paper a negative answer to this question is given. It follows from

THEOREM. *For every entire function*

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$$

there exists a commutative B_0-algebra R_φ containing a dense division B_0^*-algebra non-isomorphic to the algebra of complex numbers, such that the series $\sum_{n=0}^{\infty} a_n x^n$ is convergent for each $x \in R_\varphi$.

Obviously, if B_0-algebra contains a division algebra non-isomorphic to the algebra of complex numbers, then there are no multiplicative linear functionals. Hence we have the following

COROLLARY. *For each entire function*

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n,$$

there is a commutative B_0-algebra R_φ such that the function $\varphi(x)$ is determined in R_φ and in this algebra there are no multiplicative linear functionals.

This corollary contains the negative answer to Problem 1 of paper [5].

The proof of the theorem is based on the following construction which is an extension of the construction of Williamson [8] on the one hand, and of the construction used in paper [5] in the proof of the Proposition 2.5, on the other hand.

Let $f(u)$ be a non-negative function determined for $u \geqslant 0$, increasing, equal to 0 in 0 and tending to infinity at infinity. Let $f(u)$ satisfy condition Δ_2, i. e. there is such a constant C that $f(2u) \leqslant Cf(u)$ (¹). Without loss of generality we can assume that C is an integer.

Let

$$a_{p,n} = \begin{cases} e^{pf(|n|)} & \text{for} \quad n = -1, -2, \ldots \\ 1 & \text{for} \quad n = 0, \\ e^{-f(n)/p} & \text{for} \quad n = 1, 2, \ldots \end{cases}$$

$(p = 1, 2, \ldots)$.

Let $L(a_{p,n})$ denote the space of all sequences $x = \{x_n\}$ of complex numbers such that

$$\|x\|_p = \sum_{n=-\infty}^{\infty} a_{p,n} |x_n| < +\infty.$$

$L(a_{p,n})$ is a B_0-space with respect to the topology induced by the pseudonorms $\|x\|_p$ (see [3] and also [6]).

LEMMA 1. *If we determine multiplication in $L(a_{p,n})$ as convolution, i. e. $z = x * y$, $x = \{x_n\}$, $y = \{y_n\}$, $z = \{z_n\}$, $z_n = \sum_{k=-\infty}^{\infty} x_k y_{n-k}$, then $L(a_{p,n})$ is a B_0-algebra.*

Proof. First, we shall show that for each p there is such an r that

$$a_{p,n+m} \leqslant a_{r,n} a_{r,n} \tag{3}$$

for all n and m.

Let $n, m \geqslant 0$. Then

$$a_{p,n+m} = e^{-f(n+m)/p} = e^{-f(n+m)/2p} e^{-f(n+m)/2p} \leqslant e^{-f(n)/p} e^{-f(m)/p} = a_{2p,n} a_{2p,m}.$$

Let $n, m < 0$. Then

$$a_{p,n+m} = e^{pf(|n+m|)} \leqslant e^{Cp[f(|n|)+f(|m|)]} = a_{Cp,n} a_{Cp,m}.$$

Let $n \geqslant 0$, $m < -n$. Then

$$a_{p,n+m} = e^{pf(|n+m|)} \leqslant e^{pf(|m|)} = e^{2pf(|m|)} e^{-f(n)/2p} e^{f(n)/2p} e^{-pf(|m|)} \leqslant a_{2p,n} a_{2p,m},$$

because

$$e^{f(n)/2p - pf(|m|)} \leqslant e^{(1/2p - p)f(n)} \leqslant 1.$$

Let $n \geqslant 0$, $-n \leqslant m < -n/2$. Then $a_{p,n+m} \leqslant 1$ but

$$a_{Cp,n} a_{Cp,m} = e^{-f(n)/Cp} e^{Cp f(|m|)} = e^{Cp f(|m|) - f(n)/Cp}$$
$$\geqslant e^{Cp f(|m|) - f(2|m|)/Cp} \geqslant e^{(Cp - 1/p) f(|m|)} \geqslant 1.$$

(¹) We have obviously $f(m+n) < C[f(m)+f(n)]$ for $m, n > 0$.

Hence $a_{p,n+m} \leqslant a_{Cp,n} a_{Cp,m}$.

Let $n \geqslant 0$, $-n/2 \leqslant m \leqslant 0$. Then $n+m \geqslant n/2$ and

$$a_{p,n+m} = e^{-f(n+m)/p} \leqslant e^{-f(n/2)/p} \leqslant e^{-f(n)/Cp} = a_{Cp,n},$$

hence $a_{p,n+m} \leqslant a_{Cp,n} a_{Cp,m}$, because $a_{Cp,m} \geqslant 1$.

Therefore for $r = \max(C, 2)p$ we have

$$a_{p,n+m} \leqslant a_{r,n} a_{r,m} \tag{3}$$

for all n and m.

Now, we can estimate the pseudonorms of the convolution:

$$\|z\|_p = \sum_{n=-\infty}^{\infty} a_{p,n} \Big| \sum_{k=-\infty}^{\infty} x_k y_{n-k} \Big| \leqslant \sum_{n,k=-\infty}^{\infty} a_{p,n} |x_k| \, |y_{n-k}|$$

$$\leqslant \sum_{n,k=-\infty}^{\infty} a_{r,k}\, a_{r,n-k} |x_k| \, |y_{n-k}| = \sum_{k=-\infty}^{\infty} a_{r,k} |x_k| \sum_{m=-\infty}^{\infty} a_{r,m} |y_m| = \|x\|_r \|y\|_r .$$

Hence $L(a_{p,n})$ is a B_0-algebra, q. e. d.

Let M be a set of functions, analytic in the set $\{z: 0 < |z| < r_x\}$, where r_x depends on the function $x(z)$ and such that $\lim_{z\to 0} x(z)$ exists, finite or infinite. The set M is a division algebra with respect to the pointwise multiplication. By $\tilde{M}$ we denote the set of all sequences $\{\ldots, 0, \ldots, 0, x_{-k}, \ldots, x_0, \ldots, x_n, \ldots\}$ of coefficients of expansions of the functions $x(z) \in M$ in the Laurent series

$$x(z) = \sum_{n=-k}^{\infty} x_n z^n .$$

Obviously, the multiplication in $\tilde{M}$ is determined by convolution, and $\tilde{M}$ is a division algebra.

LEMMA 2. *If $f(n)/n \to \infty$, then $\tilde{M} \subset L(a_{p,n})$.*

Proof. Obviously it is enough to prove that the series

$$\sum_{n=0}^{\infty} a_{p,n} |x_n| \tag{5}$$

is convergent for each p. But $x(z)$ is analytic in the set $\{z: 0 < |z| < r_x\}$ whence the series

$$\sum_{n=0}^{\infty} x_n \left(\frac{r_x}{2}\right)^n$$

is convergent. Since $f(n)/n \to \infty$, $a_{p,n} < (r_x/2)^n$, for sufficiently large n, therefore series (5) is convergent.

It is easy to see that $\tilde{M}$ is dense in $L(a_{p,n})$. In the example of Williamson $f(n)$ was equal to $(n+1)\ln(n+1)$.

LEMMA 3. *Let $M_1, M_2, \ldots$ be a sequence of positive real numbers such that*

$$\frac{M_n}{n} \to \infty . \tag{6}$$

Then there exists such an infinite-dimensional commutative B_0-algebra R, containing a dense division algebra, that for each complex sequence $\{b_n\}$, $n = 0, 1, 2, \ldots$, satisfying

$$\sum_{n=0}^{\infty} |b_n| e^{M_n} < \infty \tag{7}$$

and for each $x \in R$, the series $\sum_{n=0}^{\infty} b_n x^n$ is convergent in R.

Proof. Let

$$\Gamma(n) = \inf_{k \geqslant n} \sqrt{n M_n}.$$

Then $\Gamma(n)$ is increasing and

$$\frac{\Gamma(n)}{M_n} \to 0, \quad \frac{\Gamma(n)}{n} \to \infty \quad (n \to \infty).$$

From Lemmas 2.1 and 2.2 of paper [5] it follows that there is a function $\Omega(u)$ such that

1. $\Omega(u)$ is a convex function,
2. $\Omega(u)/u$ is increasing and tends to infinity when u tends to infinity.
3. If $i_j \geqslant 0$ and $\sum_{j=1}^{n} i_j = k$, then

$$\Omega(k) \leqslant 8 \sum_{j=1}^{n} \Omega(i_j) + \Gamma(n).$$

If in the definition of $L(a_{p,n})$ we put $f(n) = \Omega(n)$, we obtain the algebra which satisfies the conclusion of Lemma 3.

Property 3 implies that $\Omega(u)$ satisfies condition Δ_2 for $u \geqslant 1$, hence Properties 1 and 3 and Lemma 1 imply that $L(a_{p,n})$ is a B_0-algebra.

Lemma 2 and Property 2 imply that $L(a_{p,m})$ contains a dense division algebra.

Let $x = \{x_n\}$ be an arbitrary element of $L(a_{p,n})$. Let

$$x = y + z, \quad y = \{y_n\}, \quad z = \{z_n\},$$

where

$$y_n = \begin{cases} x_n & \text{for } n \geqslant 0, \\ 0 & \text{for } n < 0, \end{cases} \qquad z_n = \begin{cases} 0 & \text{for } n \geqslant 0 \\ x_n & \text{for } n < 0. \end{cases}$$

Obviously $\|x\|_p = \|y\|_p + \|z\|_p$.

From Property 2 it follows that

$$\Omega\Big(\sum_{j=1}^{n} i_j\Big) \geqslant \sum_{j=1}^{n} \Omega(i_j).$$

Hence $\|y^n\|_p \leqslant \|y\|_p^n$. On the other hand, in the same way as in Proposition 2.5 of paper [5] we can show that $\|z^n\|_p \leqslant e^{p\Gamma(n)}\|z\|_{8p}^n$. Therefore

$$\|x^n\|_p = \|(y+z)^n\|_p = \Big\|\sum_{k=0}^{n}\binom{n}{k} y^k z^{n-k}\Big\|_p \leqslant \sum_{k=0}^{n}\binom{n}{k}\|y^k\|_{p+1}\|z^{n-k}\|_{p+1},$$

$$\sum_{k=0}^{n}\binom{n}{k} e^{p\Gamma(n)}\|y\|_{p+1}^k\|z\|_{8(p+1)}^{n-k} \leqslant e^{p\Gamma(n)}(\|y\|_{p+1}+\|z\|_{8(p+1)})^n$$
$$\leqslant e^{p\Gamma(n)}(\|y\|_{8(p+1)}+\|z\|_{8(p+1)})^n = e^{p\Gamma(n)}\|x\|_{8(p+1)}^n.$$

And in the same way as in Proposition 2.5 of paper [5] we end the proof of the lemma, q. e. d.

As a corollary we obtain the proof of the theorem. In fact, if

$$\varphi(z) = \sum_{n=0}^{\infty} b_n z^n,$$

then it is easy to construct such a sequence of positive reals that (6) and (7) hold. We can thus define R_φ as algebra R constructed in lemma 3.

The radical of a B_0-algebra R is defined as the set $A_1 = \{x$: for each $y \in R$ the element $(e+yx)$ is invertible$\}$.

In m-convex commutative algebras the radical is equal to the set $A_2 = \{x$: for each number t the element $(e+tx)$ is invertible$\}$.

Obviously, $A_1 \subset A_2$, but if R contains a division algebra R^* non-isomorphic to the algebra of complex numbers, then $A_1 \neq A_2$.

Indeed, $A_1 \cap R^* = \{0\}$, because if $x \in R^*$, putting $y = -x^{-1}$ we obtain $e+yx = 0$ and it is not invertible. On the other hand, if $x \in R^*$ and $x \neq te$ for all t, then $e+tx \in R^*$ and it is not equal to 0, hence it is invertible and $A_2 \cap R \neq \{0\}$. Therefore $A_1 \neq A_2$.

In an m-convex commutative algebra the set A_2 is equal to the set $A_3 = \{x$: for each t the series $\sum_{n=0}^{\infty} t^n x^n$ is convergent$\}$.

Obviously $A_3 \subset A_2$, but if a B_0-algebra R contains a division algebra R^* non-isomorphic to the algebra of complex numbers, then $A_2 \neq A_3$. Indeed, let $x \neq te$ for all t and x belongs to R^*, then x and x^{-1} obviously belong to A_2. On the other hand,

$$\|x^n\|_{p+1}\|x^{-n}\|_{p+1} \geqslant \|e\|_p$$

whence x and x^{-1} cannot simultaneously belong to A_3.

In the algebra $L(a_{p,n})$, also $A_3 \neq A_1$, because the element

$$x = \{x_n\}, \quad \text{where} \quad x_n = \begin{cases} 1 & \text{for} \quad n = 1, \\ 0 & \text{otherwise}, \end{cases}$$

belongs to A_3. But we do not know the answer to the following

Problem 1. If a B_0-algebra R contains a division algebra non-isomorphic to the algebra of complex numbers, is there at least one element $x \in A_3$?

The preceding considerations give a negative answer to a part of the questions raised in Problem 5 of paper [7], but further remains open

Problem 2. If x belongs to the radical of a commutative B_0-algebra R, must the series $\sum_{n=0}^{\infty} y^n x^n$ be convergent for each $y \in R$?

With radical it is also connected

Problem 3. Let a B_0-algebra contain a dense division algebra. Is the radical of this algebra equal to $\{0\}$?

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