Given an ultra-distribution $\varphi(s)$. For every λ and k the limit $\Phi(\lambda, z)$ of (45), if it exists, is said to be a *regular function* corresponding to λ, k and $\varphi(s)$.

A sequence $\{\varphi_n(s)\}$ of ultra-distributions converges to $\varphi(s)$, if to every $\lambda > 0$ there exist an integer k and regular functions $\Phi_n(\lambda, z)$, $\Phi(\lambda, z)$ corresponding to λ, k and $\varphi_n(s)$, $\varphi(s)$ respectively, such that $\Phi_n(\lambda, z)$ converges to $\Phi(\lambda, z)$ uniformly in every strip $-N < \eta < N$.

All properties of section 3 remain valid for the new convergence.

The theory of ultra-distributions extended in this way is equivalent with the theory of functionals on a space of entire function defined by authors mentioned in the introduction.

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Reçu par la Rédaction le 24. 10. 1962

STUDIA MATHEMATICA, T. XXIII. (1963)

Properties of the orthonormal Franklin system

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1. Introduction. The purpose of this paper is to present some properties of the orthonormal Franklin set and to indicate similarity of this system to another bases of the Banach space $C\langle 0,1\rangle$ of continuous functions on $\langle 0,1\rangle$. It was proved [6] a long time ago that the Franklin set forms a Schauder basis for $C\langle 0,1\rangle$. Not many more properties of these functions are known to be published. The other bases have been investigated in [6], [7], [1], [2] and [3]. Some applications of the author's results are given in [4]. Using the results of this paper one can get the same kind of applications for the Franklin functions. Some of these results were announced on the Conference of Functional Analysis held in Warszawa-Jabłonna in September 1960. Theorem 1 has a very simple proof. This theorem together with the Banach-Steinhaus theorem for sequences of linear operators gives a very simple proof of the Franklin result. The proofs of that result presented in [6] and [7] (p. 122-125) are very tedious.

2. Preliminaries and notation. The Haar functions are defined as follows:

$$\chi_1(t) \equiv 1 \quad \text{in} \quad \langle 0,1\rangle,$$

$$\chi_{2^n+1}(1) = -\sqrt{2^n},$$

$$\chi_{2^n+k}(t) = \begin{cases} \sqrt{2^n} & \text{in} \quad \left\langle \dfrac{k-1}{2^n}, \dfrac{2k-1}{2^{n+1}} \right), \\ -\sqrt{2^n} & \text{in} \quad \left\langle \dfrac{2k-1}{2^{n+1}}, \dfrac{k}{2^n} \right), \\ 0 \text{ elswhere in } \langle 0,1\rangle, & \end{cases} \tag{1}$$

where $n = 0, 1, \ldots;\ k = 1, 2, \ldots, 2^n$.

We shall define the Schauder functions using the Haar functions as follows:

$$\varphi_0(t) \equiv 1, \qquad \varphi_n(t) = \int_0^t \chi_n(\tau)\,d\tau, \qquad t \in \langle 0,1\rangle,\ n = 1, 2, \ldots \tag{2}$$

The Schmidt orthonormalization procedure applied to $\{\varphi_n\}$ leads to the Franklin functions

$$f_n(t) = \sum_{i=0}^{n} \lambda_{in}\varphi_i(t) \quad \text{with} \quad \lambda_{nn} > 0, \quad \text{for} \quad n = 0, 1, \ldots, \tag{3}$$

where the triangular matrix (λ_{in}) is uniquely determined.

In the sequel, the partial sums of the Fourier-Franklin series of a given continuous $x(t)$ will be denoted by $S_n[x]$.

The space of continuous functions on $\langle 0,1\rangle$ will be denoted by $C\langle 0,1\rangle$, and for $x\in C\langle 0,1\rangle$ we put $\|x\| = \max\limits_{0\leqslant t\leqslant 1}|x(t)|$. Let $\omega(\delta) = \omega(\delta; x)$ denote the modulus of continuity of x, i. e.

$$\omega(\delta) = \sup\{a : a = |x(t_1)-x(t_2)|,\ |t_1-t_2| \leqslant \delta,\ t_1, t_2\in\langle 0,1\rangle\}.$$

Moreover, we define

$$\|x\|_p = \Big(\int_0^1 |x(\tau)|^p\,d\tau\Big)^{1/p}, \quad 1 \leqslant p \leqslant \infty,$$

and

$$\|x\| = \|x\|_\infty \quad \text{for} \quad x\in C\langle 0,1\rangle.$$

The following theorem will be used later:

THEOREM A [1]. *If $x\in C\langle 0,1\rangle$, then*

$$x(t) = x(0) + \sum_{n=1}^{\infty}\Big[\int_0^1 \chi_n(\tau)\,dx(\tau)\Big]\varphi_n(t), \tag{4}$$

where the series is uniformly convergent in $\langle 0,1\rangle$.

Now, let us introduce the polynomials of the polygonal lines. Let $0 = t_0 < t_1 < \ldots < t_n = 1$ be a fixed partition of the interval $\langle 0,1\rangle$. Then the *polynomial* of the n-th degree is defined by

$$\varphi(t) = \frac{\xi_i - \xi_{i-1}}{t_i - t_{i-1}}(t - t_{i-1}) + \xi_{i-1} \tag{5}$$

for $t_{i-1} \leqslant t \leqslant t_i$, $i = 1, \ldots, n$, where $\{t_i\}$ is the given partition. Note that $\varphi(t_i) = \xi_i$ for $i = 0, \ldots, n$.

3. Polynomials of polygonal lines. First of all we shall prove in this section an inequality which will imply the Franklin's theorem from [6]. Then, an analogous inequality to the Bernstein-Zygmund classical one will be proved ([10], p. 11, (3.18)). Finally, an application of that Bernstein's inequality to the best approximations will be given.

Let us denote by $E_n(x)$ the *best approximation* of the continuous function $x\in C\langle 0,1\rangle$, by the n-th degree polynomials of polygonals, i. e.

$$E_n(x) = \inf\|x - \varphi\|,$$

where the infimum is taken over all φ defined by (5) with a fixed partition $\{t_i\}$.

THEOREM 1. *Let ψ_0 be a polynomial of the n-th degree ($n = 0, 1, \ldots$) determined for a given $x\in C\langle 0,1\rangle$ by the condition*

$$\|x - \psi_0\|_2 = \inf_{\xi_i}\|x - \varphi\|_2, \tag{6}$$

where φ is defined by (5). Then

$$\|\psi_0\| \leqslant 3\,\|x\|. \tag{7}$$

Proof. Let us put $\psi_0(t_i) = \xi_i^{(0)}$, $i = 0, \ldots, n$. Then we get from (6)

$$\left(\frac{\partial\,\|x-\varphi\|_2^2}{\partial\,\xi_i}\right)_{\substack{\xi_k = \xi_k^{(0)}\\ k=0,\ldots,n}} = 0,$$

i. e.

$$\frac{\delta_1}{3}\xi_0^{(0)} + \frac{\delta_1}{6}\xi_1^{(0)} = \frac{1}{\delta_1}\int_{t_0}^{t_1} x(\tau)(t_1-\tau)\,d\tau,$$

$$\frac{\delta_k}{6}\xi_{k-1}^{(0)} + \frac{\delta_k+\delta_{k+1}}{3}\xi_k^{(0)} + \frac{\delta_{k+1}}{6}\xi_{k+1}^{(0)}$$
$$= \frac{1}{\delta_k}\int_{t_{k-1}}^{t_k} x(\tau)(\tau - t_{k-1})\,d\tau + \frac{1}{\delta_{k+1}}\int_{t_k}^{t_{k+1}} x(\tau)(t_{k+1}-\tau)\,d\tau \quad \text{for} \quad k = 1, \ldots, n-1, \tag{8}$$

$$\frac{\delta_n}{6}\xi_{n-1}^{(0)} + \frac{\delta_n}{3}\xi_n^{(0)} = \frac{1}{\delta_n}\int_{t_{n-1}}^{t_n} x(\tau)(\tau - t_{n-1})\,d\tau,$$

where $\delta_i = t_i - t_{i-1}$, $i = 1, \ldots, n$. Now it is easy to see that equations (8) imply

$$|2\xi_0^{(0)} + \xi_1^{(0)}| \leqslant 3\,\|x\|,$$

$$\left|\frac{\delta_k}{\delta_k+\delta_{k+1}}\xi_{k-1}^{(0)} + \frac{\delta_{k+1}}{\delta_k+\delta_{k+1}}\xi_{k+1}^{(0)} + 2\xi_k^{(0)}\right| \leqslant 3\,\|x\|, \qquad k = 1, \ldots, n-1,$$

$$|\xi_{n-1}^{(0)} + 2\xi_n^{(0)}| \leqslant 3\,\|x\|;$$

hence $2\|\psi_0\| \leqslant 3\|x\| + \|\psi_0\|$, and this proves the theorem.

THEOREM 2 (The inequality of Bernstein-Zygmund type). *Let φ be given by (5). Then the inequality*

$$\|\varphi'\|_p \leqslant 4 \max_{1\leqslant i\leqslant n}\frac{1}{t_i - t_{i-1}}\|\varphi\|_p \tag{9}$$

holds for $1 \leqslant p \leqslant \infty$, $n = 1, 2, \ldots$

Proof. By easy computations one gets for $p < \infty$

$$\|\varphi'\|_p^p = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |\varphi'(t)|^p\,dt$$

$$= \sum_{i=1}^{n} \delta_i^{1-p} |\varphi(t_i) - \varphi(t_{i-1})|^p$$

$$\leqslant 2^p \sum_{i=1}^{n} \delta_i^{1-p} \left[\frac{|\varphi(t_i)|^2 + |\varphi(t_{i-1})|^2}{|\varphi(t_i)| + |\varphi(t_{i-1})|}\right]^p$$

$$\leqslant 2^p \sum_{i=1}^{n} \delta_i^{1-p} \left[\frac{2}{\delta_i} \int_{t_{i-1}}^{t_i} |\varphi(t)|\,dt\right]^p$$

$$\leqslant 4^p \sum_{i=1}^{n} \frac{1}{\delta_i^p} \int_{t_{i-1}}^{t_i} |\varphi(t)|^p\,dt$$

$$\leqslant \left(4 \max_{1\leqslant i\leqslant n} \frac{1}{\delta_i}\right)^p \int_0^1 |\varphi(t)|^p\,dt.$$

To complete the proof we pass with p to infinite.

Now, we are going to specify the partitions appearing in the definition of polynomials of any order. In the case of $n = 1$ the partition (t_0, t_1) is determined: $t_0 = 0$ and $t_1 = 1$. In the general case $(n > 1)$ there exists exactly one non-negative integer m such that $2^m < n \leqslant 2^{m+1}$. Let $n = 2^m + k$. Then we shall put

$$t_i = \begin{cases} \dfrac{i}{2^{m+1}} & \text{for } i = 0, \ldots, 2k, \\[2mm] \dfrac{i-k}{2^m} & \text{for } i = 2k+1, \ldots, n. \end{cases} \tag{10}$$

Note that a polynomial φ defined by (5) with $\{t_i\}$ defined in (10), according to (4), has a unique representation in the form

$$\varphi(t) = \sum_{i=0}^{n} \eta_i \varphi_i(t), \tag{11}$$

where φ_i are the Schauder functions defined in (2). In the sequel only polynomials of that type will be considered.

Theorem 1 can be generalized to the spaces $L^p\langle 0,1\rangle$ by the arguments given in [9], p. 12-13, and this result may be stated as

THEOREM 1'. *Let $x \in L^p\langle 0,1\rangle$. Then*

$$\|S_n[x]\|_p \leqslant 3\|x\|_p \quad \textit{for} \quad n = 1, 2, \ldots;\ 1 \leqslant p \leqslant \infty. \tag{12}$$

Theorem 2 can be stated, for the particular polynomials of type (11) in the form of

THEOREM 2'. *If $\psi_n = \varphi$ is given by* (11), *then*

$$\|\psi_n'\|_p \leqslant 8n\|\psi_n\|_p \quad \textit{for} \quad n = 0, 1, \ldots,$$

or better

$$\|\psi_n'\|_p \leqslant 8\cdot 2^m \|\psi_n\|_p \quad \textit{for} \quad n = 2, 3, \ldots;\ 2^m < n \leqslant 2^{m+1}, \tag{13}$$

where $1 \leqslant p \leqslant \infty$.

The following two theorems will concern the best approximation $E_n(x)$. The first one is an analogue of the Jackson theorem for the best approximations by trigonometric polynomials. The second one is a corresponding analogue of the S. Bernstein theorem (compare [8], p. 119 and p. 132).

THEOREM 3. *Let $x \in C\langle 0,1\rangle$. Then*

$$E_1(x) \leqslant \omega(1), \quad E_n(x) \leqslant 2\omega\left(\frac{1}{n}\right) \quad \textit{for} \quad n \geqslant 1, \tag{14}$$

or

$$E_n(x) \leqslant \omega\left(\frac{1}{2^m}\right) \quad \textit{for} \quad n \geqslant 2,\ 2^m < n \leqslant 2^{m+1}. \tag{14'}$$

Proof. Obviously

$$E_n(x) \leqslant \|x - \sigma_n(x)\|,$$

where

$$\sigma_n(x)(t) = x(0) + \sum_{i=1}^{n} \left[\int_0^1 \chi_i(\tau)\,dx(\tau)\right] \varphi_i(t),$$

and

$$\sigma_n(x)(t_i) = x(t_i)$$

for $\{t_i\}$ given by (10). This completes the proof.

THEOREM 4. *If for some $x \in C\langle 0,1\rangle$ and for a given α $(0 < \alpha < 1)$ the inequalities*

$$E_n(x) < \frac{M}{n^\alpha}, \quad n = 1, 2, \ldots\ (M = \text{const.})$$

hold, then

$$\omega(\delta) < c\delta^\alpha \quad \textit{for} \quad \delta \in (0, 1),$$

with some positive constant c.

The proof of this theorem is quite similar to that for the case of trigonometric polynomials (see e. g. [8], p. 132-135). Repeating that proof step by step we use the inequality (13) with $p=\infty$ instead of classical Bernstein's inequality. We shall leave out the details of the proof.

4. Properties of the single Franklin function. In the sequel we shall need the following result:

LEMMA 1. *The inequality*

$$\operatorname*{var}_{\langle 0,1\rangle} S_{n-1}[\sqrt{2^m}\varphi_n] < \tfrac{3}{4} \tag{15}$$

holds for $n=2,3,\ldots$, where the integer m is determined by $2^m < n \leqslant 2^{m+1}$.

Proof. It is very easy to check the case $n=2$ $(m=0)$. Indeed, in this case $S_1[\varphi_2]\equiv\frac{1}{4}$. Therefore, suppose $m\geqslant 1$. For the proof let us put

$$\psi = S_{n-1}[\sqrt{2^m}\varphi_n] \quad \text{and} \quad \eta_i = \psi(\bar{t}_i) \quad \text{for} \quad 0\leqslant i<n,$$

where

$$\bar{t}_i = \begin{cases} \dfrac{i}{2^{m+1}} & \text{for} \quad i=0,\ldots,2\nu-2, \\[2ex] \dfrac{i-\nu+1}{2^m} & \text{for} \quad i=2\nu-1,\ldots,n-1, \end{cases}$$

with $\nu = n-2^m$.

Obviously, η_i satisfy the equations (8) with $x=2^m\varphi_n$, and with n replaced by $n-1$. For this special function we get

$$\frac{1}{\delta_k^2}\int\limits_{\bar{t}_{k-1}}^{\bar{t}_k}\sqrt{2^m}\varphi_n(\tau)(\tau-\bar{t}_{k-1})\,d\tau = \frac{1}{\delta_k^2}\int\limits_{\bar{t}_{k-1}}^{\bar{t}_k}\sqrt{2^m}\varphi_n(\tau)(\bar{t}_k-\tau)\,d\tau$$

$$= \begin{cases} \dfrac{1}{8} & \text{for} \quad k=2\nu-1, \\[1ex] 0 & \text{for} \quad k\neq 2\nu-1,\ 1\leqslant k\leqslant n-1. \end{cases}$$

Thus, equations (8) will get the following forms:

The first case, $1<\nu<2^m$, $m>1$:

$$\begin{aligned} 2\eta_k+\eta_{k+1} &= 0, && k=0; \\ \eta_{k-1}+4\eta_k+\eta_{k+1} &= 0, && k=1,\ldots,2\nu-3; \\ \tfrac{1}{2}\eta_{k-1}+3\eta_k+\eta_{k+1} &= \tfrac{3}{4}, && k=2\nu-2; \\ \eta_{k-1}+4\eta_k+\eta_{k+1} &= \tfrac{3}{4}, && k=2\nu-1; \\ \eta_{k-1}+4\eta_k+\eta_{k+1} &= 0, && k=2\nu,\ldots,n-2; \\ \eta_{k-1}+2\eta_k &= 0, && k=n-1. \end{aligned} \tag{16.I}$$

The second case, $\nu=1$, $m\geqslant 1$:

$$\begin{aligned} 2\eta_k+\eta_{k+1} &= \tfrac{3}{4}, && k=0; \\ \eta_{k-1}+4\eta_k+\eta_{k+1} &= \tfrac{3}{4}, && k=1; \\ \eta_{k-1}+4\eta_k+\eta_{k+1} &= 0, && k=2,\ldots,n-2; \\ \eta_{k-1}+2\eta_k &= 0, && k=n-1. \end{aligned} \tag{16.II}$$

The third case, $\nu=2^m$, $m\geqslant 1$:

$$\begin{aligned} 2\eta_k+\eta_{k+1} &= 0, && k=0; \\ \eta_{k-1}+4\eta_k+\eta_{k+1} &= 0, && k=1,\ldots,n-3; \\ \tfrac{1}{2}\eta_{k-1}+3\eta_k+\eta_{k+1} &= \tfrac{3}{4}, && k=n-2; \\ \eta_{k-1}+2\eta_k &= \tfrac{3}{4}, && k=n-1. \end{aligned} \tag{16.III}$$

One can easily read the properties of η_i from equations (16). In each of these three cases we get

$$\begin{gathered} \eta_{2\nu-2} > -\eta_{2\nu-3} > \ldots > -\eta_1 > \eta_0 > 0, \\ \eta_{2\nu-1} > -\eta_{2\nu} > \ldots > (-1)^{n-1}\eta_{n-2} > (-1)^n\eta_{n-1} > 0. \end{gathered} \tag{17}$$

Moreover, from (16) we easily obtain

$$\begin{aligned} \eta_{2\nu-2} &> \eta_{2\nu-1} && \text{for} \quad \text{(16.I) and (16.II)}, \\ \eta_{2\nu-2} &< \eta_{2\nu-1} && \text{for} \quad \text{(16.III)}. \end{aligned} \tag{18}$$

Let us consider now the first case. Using (17) and (18) we get

$$\begin{aligned} \operatorname*{var}_{\langle 0,1\rangle}\psi = \eta_0-2\eta_1+2\eta_2-\ldots+2\eta_{2\nu-2}- \\ -2\eta_{2\nu}+2\eta_{2\nu+1}+\ldots+(-1)^{n-1}2\eta_{n-2}+(-1)^n\eta_{n-1}. \end{aligned} \tag{19}$$

Now, we multiply the k-th equation in (16.I) by $(-1)^{k+1}$ and then add the first $2\nu-1$ equations. The result we get is

$$\eta_0-2\eta_1+2\eta_2-\ldots-2\eta_{2\nu-3}+2\eta_{2\nu-2} = \tfrac{3}{4}+\tfrac{1}{2}\eta_{2\nu-3}-\eta_{2\nu-1}. \tag{20}$$

In the similar way, using the next $n-2\nu+1$ equations, we get

$$-2\eta_{2\nu}+2\eta_{2\nu+1}-\ldots+(-1)^{n-1}2\eta_{n-2}+(-1)^n\eta_{n-1} = \tfrac{3}{4}-3\eta_{2\nu-1}-\eta_{2\nu-2}. \tag{21}$$

Now, combining (19), (20) and (21) we obtain

$$\operatorname*{var}_{\langle 0,1\rangle}\psi = \tfrac{3}{4}+(\tfrac{3}{4}-4\eta_{2\nu-1}-\eta_{2\nu-2})+\tfrac{1}{2}\eta_{2\nu-3},$$

hence by the 2ν-th equation and by (17)

$$\operatorname*{var}_{\langle 0,1\rangle}\psi = \tfrac{3}{4}+\tfrac{1}{2}\eta_{2\nu-3}+\eta_{2\nu} < \tfrac{3}{4}.$$

A similar argument gives in the second case

$$\operatorname*{var}_{\langle 0,1\rangle} \psi = \tfrac{3}{4} - 3\eta_{2\nu-1} < \tfrac{3}{4},$$

and in the third case

$$\operatorname*{var}_{\langle 0,1\rangle} \psi = \tfrac{3}{4} + \eta_{2\nu-3} - \eta_{2\nu-1} < \tfrac{3}{4}.$$

This completes the proof of lemma 1.

The next lemma concerns the total variation of the Franklin function.

LEMMA 2. *Let $n = 2^m+\nu$, where $2^m < n \leqslant 2^{m+1}$. Then the inequality*

$$\operatorname*{var}_{\langle 0,1\rangle} f_n < \tfrac{7}{4} \operatorname*{var}_{\langle(\nu-1)/2^m,\nu/2^m\rangle} f_n \tag{22}$$

holds for $n = 2, 3, \ldots$

Proof. This inequality is obvious for $n = 2$, i. e. for $m = 0$. Let $m \geqslant 1$. The definition of f_n and (3) imply

$$f_n = \lambda_{nn}\{\varphi_n - S_{n-1}[\varphi_n]\}, \tag{23}$$

hence

$$\operatorname*{var}_{\langle 0,1\rangle} f_n \leqslant \lambda_{nn}\left\{\frac{1}{\sqrt{2^m}} + \operatorname*{var}_{\langle 0,1\rangle} S_{n-1}[\varphi_n]\right\}. \tag{24}$$

Note that uniqueness in (4) and in (3) gives

$$\begin{aligned}(25)\quad \lambda_{nn} &= \int_0^1 \chi_n(\tau)\,df_n(\tau)\\ &= \sqrt{2^m}\left[2f_n\left(\frac{2\nu-1}{2^{m+1}}\right) - f_n\left(\frac{\nu-1}{2^m}\right) - f_n\left(\frac{\nu}{2^m}\right)\right] \leqslant \sqrt{2^m}\operatorname*{var}_{\langle(\nu-1)/2^m,\nu/2^m\rangle} f_n.\end{aligned}$$

We shall complete the proof applying Lemma 1 to (24) and then majorizing λ_{nn} in (24) by the right side of (25).

The next lemma gives the points at which the absolute maximum of $|f_n|$ is attained.

LEMMA 3. *Let $n = 2^m+\nu$, $2^m < n \leqslant 2^{m+1}$. Then*

$$\|f_n\| = \begin{cases} -f_{2^m+\nu}(0) & \text{for } \nu = 1,\\ f_{2^m+\nu}\left(\dfrac{2\nu-1}{2^{m+1}}\right) & \text{for } 1 < \nu < 2^m,\\ -f_{2^m+\nu}(1) & \text{for } \nu = 2^m,\end{cases}$$

for $n \geqslant 3$.

Moreover, for $n = 0, 1, 2$ we have

$$\|f_0\| = f_0(t) \equiv 1,$$
$$\|f_1\| = -f_1(0) = f_1(1) = \sqrt{3},$$
$$\|f_2\| = -f_2(0) = f_2(\tfrac{1}{2}) = -f_2(1) = \sqrt{3}.$$

Proof. It is easy to get the last three equalities by direct computations:

$$\begin{aligned}&f_0(t) \equiv 1,\\ (26)\qquad &f_1(t) = \sqrt{3}(2t-1),\\ &f_2(t) = \begin{cases}\sqrt{3}(4t-1) & \text{for } t\in\langle 0,\tfrac{1}{2}\rangle,\\ \sqrt{3}(3-4t) & \text{for } t\in\langle \tfrac{1}{2},1\rangle.\end{cases}\end{aligned}$$

For $n \geqslant 3$ we shall get the proof from (17), (18), (23) and (16). According to (23) and (17) the only question is at which point of the three ones: $\bar{t}_{2\nu-2}$, $\frac{1}{2}(\bar{t}_{2\nu-2}+\bar{t}_{2\nu-1})$ and $\bar{t}_{2\nu-1}$, the maximum is attained. The functions f_n and $g = \sqrt{2^m}\varphi_n - \psi$, where $\psi = S_{n-1}[\sqrt{2^m}\varphi_n]$, attain the minimum and maximum at the same points. By the definitions of ψ (see the proof of Lemma 1) and g we have

$$g(\bar{t}_i) = -\eta_i \quad \text{for} \quad i = 2\nu-2,\ 2\nu-1,$$

$$g\left(\frac{\bar{t}_{2\nu-2}+\bar{t}_{2\nu-1}}{2}\right) = \frac{1-\eta_{2\nu-2}-\eta_{2\nu-1}}{2}.$$

In the first case ($2^m < \nu < 2^{m+1}$) we have, by Lemma 1 and (17), $-\eta_{2\nu-3} < \frac{1}{2}$, hence (16.I) implies

$$\tfrac{3}{4} - \tfrac{1}{2}\eta_{2\nu-3} = 3\eta_{2\nu-2} + \eta_{2\nu-1} < 1.$$

Therefore, the last inequality and (18) imply

$$g\left(\frac{\bar{t}_{2\nu-2}+\bar{t}_{2\nu-1}}{2}\right) > \eta_{2\nu-2} > \eta_{2\nu-1}.$$

This proves the lemma in the first case.

The other cases have very similar proofs. Indeed, let $\nu = 1$ and let $m \geqslant 1$. Then by (16.II) and by (18) we have $\frac{3}{4} = 2\eta_0 + \eta_1 < 3\eta_0$, hence $\eta_0 > \frac{1}{4}$. This implies $\eta_0 + \frac{3}{4} > 1$, i. e. $3\eta_0 + \eta_1 > 1$, hence $\eta_0 > \frac{1}{2}(1-\eta_0-\eta_1)$. Finally, let $\nu = 2^m$ and $m \geqslant 1$. Then, by (18), $\eta_{n-1} > \eta_{n-2}$ and, by (16.III), $\frac{3}{4} = \eta_{n-2} + 2\eta_{n-1} < 3\eta_{n-1}$, hence $\eta_{n-1} > \frac{1}{4}$. Therefore, $\eta_{n-1} + \frac{3}{4} = 3\eta_{n-1} + \eta_{n-2} > 1$ implies $\eta_{n-1} > \frac{1}{2}(1-\eta_{n-1}-\eta_{n-2})$, and this completes the proof.

Lemma 4. *Let $n \geqslant 2$, $2^m < n \leqslant 2^{m+1}$, and $n = 2^m + \nu$. Then*

$$f_n\left(\frac{2\nu-1}{2^{m+1}}\right) - f_n\left(\frac{\nu-1}{2^m}\right) > 0 \quad \textit{and} \quad f_n\left(\frac{2\nu-1}{2^{m+1}}\right) - f_n\left(\frac{\nu}{2^m}\right) > 0 .$$

Proof. For $1 < \nu < 2^m$ this lemma is implied by Lemma 3. If $\nu = 1$ we have, according to (18),

$$g\left(\frac{\bar{t}_0+\bar{t}_1}{2}\right) = \frac{1-\eta_0-\eta_1}{2} > \frac{1}{2} - \eta_0 .$$

Moreover, (17) and (16.II) imply $2\eta_0 < 2\eta_0 + \eta_1 = \frac{3}{4}$ hence $\eta_0 < \frac{1}{2}$. Therefore

$$g\left(\frac{\bar{t}_1+\bar{t}_0}{2}\right) > 0 .$$

In the third case, i. e. when $\nu = 2^m$, this lemma can be proved in a similar way.

Lemma 5. *Let n be any non-negative integer. Then*

$$\|f_n\|_p \leqslant 4\lambda_{nn}\|\varphi_n\|_p \quad \textit{and} \quad \lambda_{nn}\|\varphi_n\|_p \leqslant 3\|f_n\|_p$$

where $1 \leqslant p \leqslant \infty$; *and*

$$\|\varphi_0\|_p = 1 , \quad \|\varphi_1\|_p = (p+1)^{-1/p} ;$$

$$\|\varphi_n\|_p = \frac{1}{2(1+p)^{1/p}}\,\frac{1}{2^{m(1/2+1/p)}} \quad \textit{for} \quad n = 2^m+\nu,\ 2^m < n \leqslant 2^{m+1},\ m \geqslant 0 .$$

Proof. We shall leave out the easy computations leading to the values of the norms $\|\varphi_n\|_p$.

One gets the first inequality easily from (23) and (12). The second one is more difficult to prove. Let g be defined as in the proof of Lemma 3. Then we write the second inequality in the equivalent form

$$\|g\|_p \geqslant \tfrac{1}{3}\|\sqrt{2^m}\,\varphi_n\|_p .$$

Furthermore, by the proved part of Lemma 5 this inequality is equivalent to the following one:

$$\|g\|_p^p \geqslant \frac{1}{6^p}\cdot\frac{1}{p+1}\cdot\frac{1}{2^m} .$$

The last inequality is implied by the next one

$$\int\limits_{\bar{t}_{2\nu-2}}^{\bar{t}_{2\nu-1}} |g(\tau)|^p\,d\tau \geqslant \frac{1}{p+1}\cdot\frac{1}{2^m}\,\frac{1}{6^p} .$$

In order to check this inequality we easily find that

$$\int\limits_{\bar{t}_{2\nu-2}}^{\bar{t}_{2\nu-1}} |g(\tau)|^p\,d\tau = \frac{1}{p+1}\,\frac{1}{2^{m+1}}\left[\frac{\eta_{2\nu-2}^{p+1}+\eta^{p+1}}{\eta_{2\nu-2}+\eta} + \frac{\eta^{p+1}+\eta_{2\nu-1}^{p+1}}{\eta+\eta_{2\nu-1}}\right]$$

$$\geqslant \frac{1}{p+1}\,\frac{1}{2^{m+1}}\{[\min(\eta_{2\nu-2},\eta)]^p + [\min(\eta,\eta_{2\nu-1})]^p\}$$

$$\geqslant \frac{1}{p+1}\,\frac{1}{2^m}\left[\frac{\min(\eta_{2\nu-2},\eta)+\min(\eta,\eta_{2\nu-1})}{2}\right]^p ,$$

where $\eta = \frac{1}{2}(1-\eta_{2\nu-2}-\eta_{2\nu-1})$.

Now, we shall discuss this inequality seperately in each of three cases (16.I), (16.II) and (16.III) respectively.

First case. According to Lemma 3 we have $\eta > \eta_{2\nu-2}$ and $\eta > \eta_{2\nu-1}$. Therefore

$$\int\limits_{\bar{t}_{2\nu-2}}^{\bar{t}_{2\nu-1}} |g(\tau)|^p\,d\tau \geqslant \frac{1}{p+1}\,\frac{1}{2^m}\left(\frac{\eta_{2\nu-2}+\eta_{2\nu-1}}{2}\right)^p .$$

Now, by (16.I), we have

$$\tfrac{1}{2}\eta_{2\nu-3} + 3\eta_{2\nu-2} + \eta_{2\nu-1} = \tfrac{3}{4} ,$$

$$\eta_{2\nu-2} + 4\eta_{2\nu-1} + \eta_{2\nu} = \tfrac{3}{4} ;$$

hence

$$\frac{\eta_{2\nu-1}+\eta_{2\nu-2}}{2} = \frac{15}{88} - \frac{1}{11}\eta_{2\nu} - \frac{3}{44}\eta_{2\nu-3} .$$

Finally, (17) implies

$$\frac{\eta_{2\nu-2}+\eta_{2\nu-1}}{2} > \frac{15}{88} > \frac{1}{6} ,$$

and this gives the required inequality.

Second case. Again by Lemma 3 we have $\eta_0 > \eta$ and $\eta_0 > \eta_1$. Applying (16.II) and (17) one gets

$$\eta_0 + 4\eta_1 + \eta_2 = \tfrac{3}{4} ,$$

$$\eta_0 + 3\eta_1 + (\eta_1+\eta_2) = \tfrac{3}{4} ,$$

$$\eta_0 + 3\eta_1 \leqslant \tfrac{3}{4} < 1 ,$$

$$\eta_1 < \eta .$$

Thus $\eta_0 > \eta > \eta_1$, consequently

$$\int\limits_{\bar{t}_0}^{\bar{t}_1} |g(\tau)|^p \geqslant \frac{1}{p+1}\,\frac{1}{2^m}\left(\frac{\eta_1+\eta}{2}\right)^p .$$

Now, applying again (16.II) we get

$$\eta_1 = \tfrac{1}{7}(\tfrac{3}{4}-2\eta_2) \quad \text{and} \quad \eta_0 = \tfrac{9}{28}+\tfrac{1}{7}\eta_2,$$

hence, by (17),

$$\frac{\eta+\eta_1}{2} = \frac{11}{56}-\frac{3\,\eta_2}{28} > \frac{11}{56} > \frac{1}{6}.$$

Third case. Lemma 3 gives $\eta_{n-1} > \eta$ and $\eta_{n-1} > \eta_{n-2}$. Moreover, from (16.III) we get

$$\eta_{n-2}+2\eta_{n-1} = \tfrac{3}{4},$$
$$3\eta_{n-2} < \tfrac{3}{4},$$
$$\eta_{n-2} < \tfrac{1}{4}.$$

Combining the last inequality, (16.III) and (17) we get

$$\frac{1}{2}\eta_{n-3}+3\eta_{n-2}+\eta_{n-1} = \frac{3}{4},$$

$$3\eta_{n-2}+\eta_{n-1} < \frac{3}{4}+\frac{\eta_{n-2}}{2} < 1.$$

Therefore $\eta_{n-2} < \eta < \eta_{n-1}$ and

$$\int\limits_{t_{n-2}}^{t_{n-1}} |g(\tau)|^p d\tau \geqslant \frac{1}{p+1}\cdot\frac{1}{2^m}\left(\frac{\eta+\eta_{n-2}}{2}\right)^p.$$

Using once more (16.III) and (17) we easily find:

$$\eta_{n-2} = \tfrac{3}{20}-\tfrac{1}{5}\eta_{n-3},$$

$$\eta_{n-1} = \tfrac{3}{10}+\tfrac{1}{10}\eta_{n-3},$$

hence

$$\frac{\eta+\eta_{n-2}}{2} = \frac{17}{80}-\frac{3}{40}\eta_{n-3} > \frac{17}{80} > \frac{1}{6}.$$

Thus Lemma 5 is proved.

LEMMA 6. *Let $n \geqslant 2$, $2^m < n \leqslant 2^{m+1}$. Then*

$$\frac{\lambda_{nn}}{\sqrt{2^m}} \leqslant \operatorname{var} f_n \leqslant \frac{7}{4}\,\frac{\lambda_{nn}}{\sqrt{2^m}}$$

and $\operatorname{var} f_0 = 0$, $\operatorname{var} f_1 = 2\sqrt{3}$.

For the proof let us apply Lemma 4 and (25) to get

$$\frac{\lambda_{nn}}{\sqrt{2^m}} = \operatorname*{var}_{\langle(\nu-1)/2^m,\,\nu/2^m\rangle} f_n.$$

Now, the left inequality is obvious and the right one follows from Lemma 2. We leave out the computations for the cases $n = 0, 1$.

LEMMA 7. *Let n be a non-negative integer. Then*

$$\frac{1}{\|\varphi_n\|_2} \leqslant \lambda_{nn} \leqslant \frac{3}{\|\varphi_n\|_2}.$$

The values for $\|\varphi_n\|_2$ are given in Lemma 5.

Proof. The right inequality follows from Lemma 5 with $p = 2$. The left inequality can be proved by taking a scalar product of (23) with f_n; then

$$1 = (f_n, f_n) = \lambda_{nn}(f_n, \varphi_n) \leqslant \lambda_{nn}\|\varphi_n\|_2.$$

Thus, the proof is complete.

5. The main inequality. In this section we shall prove the following

THEOREM 5. *For $n \geqslant 0$ and $t \in \langle 0, 1\rangle$ we have*

$$\sum_{k=1}^{2^n} |f_{2^n+k}(t)| < 2^5\sqrt{3}\,\sqrt{2^n}.$$

Proof. We deduce from (12), with $p = \infty$, for any $x \in C\langle 0, 1\rangle$ the following inequality:

$$\Big\|\sum_{k=1}^{2^n}(f_{2^n+k}, x)f_{2^n+k}\Big\| \leqslant 6\,\|x\|.$$

Obviously, the last inequality remains true after replacing $\|\ \|$ by $\|\ \|_\infty$ and x by the following function of t:

$$\sum_{i=1}^{2^n}[\chi_{2^{n+1}+2i-1}(t)-\chi_{2^{n+1}+2i}(t)]r_i(u),$$

where u is a parameter, $u \in \langle 0, 1\rangle$, $r_i(u)$ denotes the i-th Rademacher function, i. e. $r_i(u) = \chi_2(2^{i-1}u)$. Thus, we obtain

$$\Big|\sum_{k=1}^{2^n}\Big(f_{2^n+k}, \sum_{i=1}^{2^n}(\chi_{2^{n+1}+2i-1}-\chi_{2^{n+1}+2i})r_i(u)\Big)f_{2^n+k}(t)\Big| \leqslant 6\sqrt{2^{n+1}}. \tag{27}$$

Now, we easily infer that

$$g_{2^n+k}(u) = \Big(f_{2^n+k}, \sum_{i=1}^{2^n}(\chi_{2^{n+1}+2i-1}-\chi_{2^{n+1}+2i})r_i(u)\Big)$$
$$= -\frac{1}{4\sqrt{2^{n+1}}}\sum_{i=1}^{2^n}\Big\{\Big[f_{2^n+k}\Big(\frac{2i-1}{2^{n+1}}\Big)-f_{2^n+k}\Big(\frac{2i-2}{2^{n+1}}\Big)\Big]+$$
$$+\Big[f_{2^n+k}\Big(\frac{2i-1}{2^{n+1}}\Big)-f_{2^n+k}\Big(\frac{2i}{2^{n+1}}\Big)\Big]\Big\}r_i(u).$$

Therefore, for almost all $u \in \langle 0, 1\rangle$ we have

$$|g_{2^n+k}(u)| \geqslant \frac{1}{4\sqrt{2^{n+1}}}\left|\left[f_{2^n+k}\left(\frac{2k-1}{2^{n+1}}\right)-f_{2^n+k}\left(\frac{2k-2}{2^{n+1}}\right)\right]+\right.$$

$$\left.+\left[f_{2^n+k}\left(\frac{2k-1}{2^{n+1}}\right)-f_{2^n+k}\left(\frac{2k}{2^{n+1}}\right)\right]\right|-$$

$$-\frac{1}{4\sqrt{2^{n+1}}}\sum_{\substack{i=1\\ i\neq k}}^{2^n}\left[\left|f_{2^n+k}\left(\frac{2i-1}{2^{n+1}}\right)-f_{2^n+k}\left(\frac{2i-2}{2^{n+1}}\right)\right|+\right.$$

$$\left.+\left|f_{2^n+k}\left(\frac{2i-1}{2^{n+1}}\right)-f_{2^n+k}\left(\frac{2i}{2^{n+1}}\right)\right|\right].$$

Applying Lemmas 2 and 4 we conclude that

$$|g_{2^n+k}(u)| \geqslant \frac{2}{4\sqrt{2^{n+1}}}\operatorname*{var}_{\langle (k-1)/2^n,\, k/2^n\rangle} f_{2^n+k}-\frac{1}{4\sqrt{2^{n+1}}}\operatorname*{var}_{\langle 0,1\rangle} f_{2^n+k}$$

$$=\frac{1}{4\sqrt{2^{n+1}}}\left\{\frac{1}{4}\operatorname*{var}_{\langle (k-1)/2^n,\, k/2^n\rangle} f_{2^n+k}+\left[\frac{7}{4}\operatorname*{var}_{\langle (k-1)/2^n,\, k/2^n\rangle} f_{2^n+k}-\operatorname*{var}_{\langle 0,1\rangle} f_{2^n+k}\right]\right\}$$

$$>\frac{1}{16\sqrt{2^{n+1}}}\operatorname*{var}_{\langle (k-1)/2^n,\, k/2^n\rangle} f_{2^n+k}$$

holds for almost all u, hence, again by Lemma 4, by (25) and by Lemma 7, it follows that

$$|g_{2^n+k}(u)| > \frac{1}{\sqrt{2}\,2^{n+4}}\lambda_{2^n+k,\,2^n+k} > \frac{1}{8}\sqrt{\frac{3}{2}}\,. \tag{28}$$

Now note that

$$\operatorname{sign} g_{2^n+k}(u) = -r_k(u) \tag{29}$$

holds for almost all u. Indeed, this follows immediately from Lemma 2 and from the definition of g_{2^n+k}. Finally, according to (29), for any fixed $t \in \langle 0, 1\rangle$ we find a $u \in (0, 1)$ such that $\operatorname{sign} f_{2^n+k}(t) = \operatorname{sign} g_{2^n+k}(u)$. We have used here the stochastic independence of the Rademacher functions. Therefore, for any u chosen in this way we obtain, by (27), (28) and (29), the required inequality:

$$6\sqrt{2^{n+1}} \geqslant \left|\sum_{k=1}^{2^n} g_{2^n+k}(u)f_{2^n+k}(t)\right| = \sum_{k=1}^{2^n}|g_{2^n+k}(u)|\,|f_{2^n+k}(t)|$$

$$> \frac{1}{8}\sqrt{\frac{3}{2}}\sum_{k=1}^{2^n}|f_{2^n+k}(t)|\,.$$

6. Approximation and absolute convergence. We shall now prove the following

THEOREM 6. *Let $x \in C\langle 0, 1\rangle$, $m \geqslant 0$, $n = 2^m+k$ and $2^m < n \leqslant 2^{m+1}$. Then*

$$\|x - S_{2^m+k}\| \leqslant 4\omega\left(\frac{1}{2^m}\right).$$

Proof. It is sufficient to apply (12) with $p = \infty$ and (14′) to get

$$\|x - S_{2^m+k}\| \leqslant \|x - E_{2^m+k}(x)\| + \|S_{2^m+k}[x - E_{2^m+k}(x)]\|$$

$$\leqslant 4\|x - E_{2^m+k}(x)\| \leqslant 4\omega\left(\frac{1}{2^m}\right).$$

COROLLARY. *Let $n = 2^m+k$, $m \geqslant 0$, $2^m < n \leqslant 2^{m+1}$. Then for every $x \in C\langle 0, 1\rangle$ we have*

$$|a_{2^m+k}(x)| \leqslant \frac{12\sqrt{3}}{\sqrt{2^m}}\,\omega\left(\frac{1}{2^m}\right)$$

where $a_n(x) = \int\limits_0^1 x(\tau) f_n(\tau)\, d\tau$.

To prove the corollary we shall note that, by Lemmas 5 and 7, we get

$$\|f_{2^m+k}\| \geqslant \frac{1}{3}\lambda_{nn}\|\varphi_{2^m+k}\| \geqslant \frac{1}{3}\,\frac{\|\varphi_{2^m+k}\|}{\|\varphi_{2^m+k}\|_2} = \frac{\sqrt{3}}{3}\sqrt{2^m}\,.$$

Thus, Theorem 6 together with the last inequality imply

$$|a_{2^m+k}(x)| \leqslant \frac{\|x - S_{2^m+k}\| + \|x - S_{2^m+k-1}\|}{\|f_{2^m+k}\|}$$

$$\leqslant \frac{4\omega(1/2^m) + 4\omega(1/2^{m-1})}{(\sqrt{3}/3)2^{m/2}} \leqslant \frac{12\sqrt{3}}{2^{m/2}}\,\omega\left(\frac{1}{2^m}\right).$$

THEOREM 7. *Let $x \in C\langle 0, 1\rangle$ satisfy the inequality*

$$|a_{2^m+k}(x)| \leqslant \frac{M}{2^{m(1/2+a)}} \qquad (M > 0)$$

for some a, $0 < a \leqslant 1$, and for $m \geqslant 0$, $k = 1, \ldots, 2^m$. Then

$$\|x - S_{2^m+k}(x)\| < \frac{M\cdot 2^6\sqrt{3}}{2^a - 1}\cdot\frac{1}{2^{am}}$$

holds for $k = 1, \ldots, 2^m$; $m = 0, 1, \ldots$

Proof. Since x is continuous, we have

$$x=\sum_{n=0}^{\infty}a_nf_n=S_{2^m+k}(x)+\sum_{r=2^m+k+1}^{\infty}a_rf_r.$$

Now, by assumptions,

$$\|x-S_{2^m+k}(x)\|\leqslant M\sum_{p=m}^{\infty}\frac{1}{2^{p(1/2+a)}}\Big\|\sum_{l=1}^{2^p}|f_{2^p+l}(t)|\Big\|,$$

hence, by Theorem 5,

$$\|x-S_{2^m+k}\|\leqslant\sqrt{3}\cdot M\cdot 2^5\sum_{p=m}^{\infty}\frac{1}{2^{pa}}<\frac{M\cdot 2^6\sqrt{3}}{2^a-1}\cdot\frac{1}{2^{am}},$$

and the theorem is proved.

COROLLARY. *Let $0<a<1$. Then the following three conditions are equivalent*:

(i) $\|x-S_n(x)\|=O\left(\frac{1}{n^a}\right)$ *as* $n\to\infty$;

(ii) $\omega(\delta)=O(\delta^a)$ *as* $\delta\to 0_+$;

(iii) $|a_n(x)|=O\left(\frac{1}{n^{a+\frac{1}{2}}}\right)$ *as* $n\to\infty$.

Proof. Note that (i) implies (ii) by Theorem 4, (ii) implies (iii) by the Corollary to Theorem 6, and (iii) implies (i) by Theorem 7.

Finally we shall say a few words about the absolute convergence of the Fourier-Franklin series. There can be established, by the results of this paper, theorems similar to those corresponding ones for the Fourier-Haar series (compare [5]). Here, we shall state only two of them.

THEOREM 8. *Let $x\in C\langle 0,1\rangle$. If*

$$\sum_{m=1}^{\infty}\omega\left(\frac{1}{2^m}\right)<\infty,$$

then

$$\sum_{n=0}^{\infty}|a_n(x)f_n(t)|$$

converges uniformly in $\langle 0,1\rangle$.

THEOREM 9. *Let $x\in C\langle 0,1\rangle$, and let*

$$\sum_{m=1}^{\infty}2^{m/2}\omega\left(\frac{1}{2^m}\right)<\infty.$$

Then

$$\sum_{n=0}^{\infty}|a_n|<\infty.$$

7. Remarks. There are still several properties of the Franklin functions to be known, for instance: Lebesgue functions, convergence almost everywhere, local properties of the Fourier-Franklin expansions, Franklin system as an unconditional basis in L_p, $p>1$, etc. The author believes that most of the properties of Haar functions could be established also for Franklin functions. The results of this paper can be used to prove the main result of [4].

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Reçu par la Rédaction le 19. 12. 1962