On the differentiability of weak solutions of certain non-elliptic equations

by

H. MARCINKOWSKA (Warszawa)

Lax [7] has given the method for studying the differentiability of weak solutions of elliptic equations of order $2m$ with the aid of Hilbert spaces $H_p$ ($p$ being an arbitrary integer). The purpose of this paper is to adapt the theory of Lax to some classes of non-elliptic equations. This can be made with the aid of Hilbert spaces $H_{p,q}$ ($p$, $q$ are arbitrary integers) which will be defined in Chapter 1. In chapter 2 we consider the regularity properties of these spaces, when the indices are sufficiently large. In chapter 3 the differentiability theorem for certain non-elliptic equations is given. As a special case we obtain some results concerning the regularity of weak solutions of elliptic equations depending on a parameter.

1. The norms $\|\cdot\|_{u,0}$ and related Hilbert spaces

1.1. Our definition (and the definition of the spaces $H_{-m}$ given by Lax) is based on the following theorem concerning Banach spaces:

**Theorem A.** Let $X_u$ and $X_s$ be two reflexive Banach spaces such that

1. $X_u$ is a dense subset of the space $X_s$,
2. $\|\cdot\|_u \geq \|\cdot\|_s$ for all $x$ in $X_u$ ($\|\cdot\|_u$ and $\|\cdot\|_s$ denote the norms in the spaces $X_u$ and $X_s$ respectively).

Let $X^*_u$ be a space conjugate to $X_u$ (that is the space of all continuous linear functionals on $X_u$). For $y \in X^*_u$ put

$$
\|y\| = \sup_{x \in X_u, \|x\|_u < 1} |y(x)|
$$

and let $X_-$ be the completion of $X^*_u$ in the norm $\|\cdot\|_-$. Then the space $X_-$ is isometrically isomorphic with the space $X^*_u$, and so is the space $X_+$ with regard to the space $X^*_u$, the latter isomorphism being given by the correspondence

$$
X^*_u \ni x \leftrightarrow \varepsilon X_+,
$$
when \( l(y) = y(x) \) for all \( y \in X_- \). When we set \( b(x, y) \overset{df}{=} y(x) \) for \( y \in X_- \) and \( x \in X_+ \), then the generalized Schwarz inequality

\[
|b(x, y)| \leq \|z\| |y|_-
\]

holds.

This theorem can be proved by using the arguments contained in the paper of Lax [7].

1.2. The two-indices norms shall be first defined for infinitely differentiable functions in some domain \( \Omega \) of the Euclidean space \( \mathbb{R}^m \); then we obtain the related Hilbert spaces with the aid of completion. We suppose the domain \( \Omega \) to be the product of two domains: \( \Omega \) of the space \( \mathbb{R}^m \), and \( \Omega \) of the space \( \mathbb{R}^n \). We denote by \( z = (z_1, \ldots, z_n) \) the point of the space \( \mathbb{R}^n \), and by \( y = (y_1, \ldots, y_m) \) the point of the space \( \mathbb{R}^m \). The class of all complex-valued functions which are infinitely differentiable in \( \Omega \) and whose all derivatives are square summable in \( \Omega \), will be denoted by \( C_0^\infty(\Omega) \). By \( B \) we denote a linear subset of the class \( C_0^\infty(\Omega) \) containing the class \( C_0^\infty(\Omega) \), which has the following properties:

1. For each function \( \varphi \in C_0^\infty(\Omega) \) or \( y \in C_0^\infty(\Omega) \) and for each \( u \in B \) the functions \( \varphi u \) and \( y u \) are also in \( B \).

Let \( B_k \) be the subset of the class \( B \) consisting of all functions \( w(x, y) \) which vanish for \( x \in \partial \Omega - \bar{K} \) and \( y \in \Omega \), when \( \bar{K} \) is a compact contained in \( \Omega \). Then \( B_k \) has the same meaning when the roles of \( x \) and \( y \) are interchanged.

In the sequel the letters \( m, k \) will denote non-negative integers and \( p, q \) arbitrary integers. The derivative \( \frac{\partial^{|\alpha|} u}{\partial x^1 \cdots \partial x^m} \), \( (|\alpha| = a_1 + \cdots + a_m) \) will be denoted briefly by \( D_{\alpha}u \), and, analogously, \( D_{\beta}u \) will denote the derivative \( \frac{\partial^{|\beta|} u}{\partial y^1 \cdots \partial y^m} \), \( (|\beta| = b_1 + \cdots + b_n) \).

1.5. We first define the spaces \( H_{\alpha, k}(\Omega, B) \). Let

\[
H_{\alpha, k} \overset{df}{=} \sum_{\alpha \in \mathbb{N}^m} \sum_{k \in \mathbb{Z}} |D_{\alpha} u|^2 \|(\cdot)\|_0
\]

for all \( u \in B \), and let \( H_{\alpha, k}(\Omega, B) \) be the completion of the class \( B \) in the norm \( \|(\cdot)\|_k \). To each element \( u \) of the space \( H_{\alpha, k}(\Omega, B) \) and to each \( \beta \) \( (0 \leq |\beta| \leq k) \) corresponds the strong derivative \( D_{\beta}u \). Defined as the limit

\[
\frac{D_{\beta}u}{(\cdot)_{\beta}}(\cdot)
\]

(\( \beta \) is a domain of the Euclidean space) denotes the class of all functions infinitely differentiable in \( \beta \) and having a compact support contained in \( \beta \).

in \( L^2(\Omega) \) of the sequence \( \{ D_{\beta}u_n \} \) when \( u_n \) belongs to \( B \) and \( \|u_n - u\|_{\alpha, k} \to 0 \).

The same arguments as used by Friedrichs [3] show, that the correspondence

\[
H_{\alpha, k}(\Omega, B) \ni u \to D_{\beta}u \in L^2(\Omega)
\]

is a one-to-one linear and continuous mapping, which leaves invariant the elements of \( B \). Therefore the space \( H_{\alpha, k}(\Omega, B) \) may be treated as a subset of \( L^2(\Omega) \), when each element is identified with its strong derivative of order zero. It is a Hilbert space with the scalar product

\[
(u, v)_{\alpha, k} \overset{df}{=} \sum_{\beta \in \mathbb{N}^m} \sum_{|\beta| \leq k} (D_{\beta}u, D_{\beta}v)_{\Omega, 0}
\]

the derivatives being taken in the strong sense.

Lemma 1. The class \( B \) is dense in \( B_k \).

Proof (1). It is sufficient to show, that an arbitrary function \( u \) belonging to \( B \) can be approximated with functions of the class \( B_k \), with respect to the norm \( \|(\cdot)\|_k \). Let \( \varphi \in C_0^\infty(\Omega) \) be a function satisfying the conditions

\[
\begin{align*}
1^a & \ 0 \leq \varphi(x) \leq 1, \\
2^a & \ \varphi(x) = 1 \text{ for } x \text{ lying in some compact } \beta \text{ contained in } \Omega,
\end{align*}
\]

write

\[
u_k(x, y) = \varphi(x) u(x, y), \quad (x, y) \in \Omega.
\]

Then \( u_k \in B_k \) and

\[
\|u - u_k\|_{\alpha, k} \overset{df}{=} \sum_{\beta \in \mathbb{N}^m} \sum_{|\beta| \leq k} \int_{\Omega} |1 - \varphi(x)|^2 (D_{\beta}u(x, y))^2 \, dx dy
\]

\[
\leq \sum_{\beta \in \mathbb{N}^m} \sum_{|\beta| \leq k} \int_{\Omega} (D_{\beta}u(x, y))^2 \, dx dy
\]

From the square-summability of \( D_{\beta}u \) follows, that the last sum may be arbitrarily small for suitable \( \beta \), q. e. d.

We now define for \( u \in L^2(\Omega) \) the norm \( \|u\|_{\alpha, k} \) as the norm \( \|\cdot\| \) described in theorem 1.2. When \( H_{\alpha, k}(\Omega, B) \) is taken as the space \( X_+ \), and \( L^2(\Omega) \) as the space \( X_- \) and \( X_- \). The corresponding space \( X_- \) is denoted by \( H_{\alpha, k}(\Omega, B) \). From theorem 1 it follows that on the product \( H_{\alpha, k}(\Omega, B) \times H_{\alpha, k}(\Omega, B) \) the bilinear form \( b_{\alpha, k}(w, v) \) can be defined, having the property

\[
b_{\alpha, k}(w, v) = (u, v)_{\alpha, k}
\]

for \( u, v \in L^2(\Omega) \). Because of the density of the class \( C_0^\infty(\Omega) \) in \( L^2(\Omega) \) it is also dense in \( H_{\alpha, k}(\Omega, B) \).

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(1) This proof has been suggested to the author by Prof. S. Lojasiewicz. The proof given previously by the author was more complicated.
1.4. Now we set
\[ \|u\|_{H_{\alpha}} = \sum_{\alpha < \gamma \leq \max} \|D^\alpha u\|_{L^2} \]
for \(u \in B\) and we define \(H_{\alpha}(\Omega, B)\), as the completion of the class \(B\) in the norm \(\|\cdot\|_{H_{\alpha}}\). An analogous reasoning as in the proof of lemma 1 shows, that \(B_{\alpha}\) is dense in \(B\) in the norm \(\|\cdot\|_{H_{\alpha}}\) and therefore also with respect to the norm \(\|\cdot\|_{H_{\alpha-\beta}}\). For each \(u\) belonging to \(H_{\alpha}(\Omega, B)\) and for each \(\gamma, \beta\) \((0 \leq |\gamma| \leq m, 0 \leq |\beta| \leq k)\) the strong derivative \(D_\gamma D_\beta u\) may be defined as the limit in \(L^2(\Omega)\) of \(D_\gamma D_\beta u_n\), when \(u_n\) is a sequence of functions of the class \(B\) approximating \(u\) in the norm \(\|\cdot\|_{H_{\alpha}}\). When we identify each \(u \in H_{\alpha}(\Omega, B)\) with its strong derivative of order zero, the space \(H_{\alpha}(\Omega, B)\) can be considered as a subset of \(L^2(\Omega)\) (namely the set of all functions square summable in \(\Omega\), which have strong derivatives to the order \(\alpha\) with respect to \(x\) and \(\partial x\) with respect to \(y\)).

**Lemma 2.** The space \(H_{\alpha}(\Omega, B)\) may be mapped in an one-to-one linear and continuous manner into the space \(H_{\alpha}(\Omega, B, v)\); this mapping leaves invariant the functions of the class \(B\).

**Proof.** A system \(\{w\}\) of elements of the space \(H_{\alpha}(\Omega, B)\) (\(w = (a_1, \ldots, a_m) 0 \leq |a| \leq m\) having the following properties corresponds to each element \(u\) of \(H_{\alpha}(\Omega, B)\):
1. when \(u_k \in B\) is a sequence approximating \(u\) in the norm \(\|\cdot\|_{H_{\alpha}}\), then \(\sum_{\alpha, \beta} \|D_\gamma D_\beta u_k - D_\gamma D_\beta u\|_{L^2} \to 0.
2. \(\|w\|_{H_{\alpha}} = \sum_{\alpha, \beta} \|D_\gamma D_\beta u\|_{L^2}\).

The mapping is given by the correspondence \(u \to w\) and it will be proved that from \(u \to w\) it follows that \(w \to 0\) for \(0 \leq |a| \leq m\). For an arbitrary function \(\varphi \in B\), we have after integration by parts
\[ (D_\gamma D_\beta u, \varphi)_{L^2(\Omega)} = (u_k, (\xi)\varphi)_{L^2(\Omega)} \]
and in the limit
\[ (w, \varphi) = (u, (\xi)\varphi) \]
the last brackets being taken in the sense of the duality between the spaces \(H_{\alpha}(\Omega, B)\) and \(H_{\alpha-\beta}(\Omega, B, \pi)\). From the last equality and from lemma 1 it follows, that \(w^\alpha = 0 (0 \leq |\alpha| \leq m)\) when \(u \to 0\), q. e. d.

According to lemma 2 the space \(H_{\alpha}(\Omega, B)\) may be treated as a subset of \(H_{\alpha}(\Omega, B)\) when \(u\) is identified with \(w^\alpha\). Especially in the case \(q = 0\), the element \(w\) is called strong derivative in the norm \(\|\cdot\|_{H_{\alpha}}\) with respect to \(\alpha\) of order \(\alpha\) and can be denoted by \(D_\alpha\), when there is no danger of misunderstanding. The spaces \(H_{\alpha}(\Omega, B)\) are Hilbert spaces with the scalar product
\[ (u, v)_{H_{\alpha}} = \sum_{\alpha, \beta \leq \max} (D_\alpha D_\beta u, D_\alpha D_\beta v)_{L^2(\Omega)} \]
in particular for \(q = k\)
\[ (u, v)_{H_{\alpha}} = \sum_{\alpha, \beta \leq \max} (D_\alpha D_\beta u, D_\alpha D_\beta v)_{L^2(\Omega)} \]
(the derivatives are taken in the strong sense).

1.5. The space \(H_{\alpha}(\Omega, B)\) is defined as the space \(X_{\alpha}\), which is given by theorem 1 when one puts \(H_{\alpha}(\Omega, B)\) as \(X_{\alpha}\), \(H_{\alpha}(\Omega, B)\) as \(X_{\alpha}\), and \(H_{\alpha}(\Omega, B)\) as \(X_{\alpha}\). It is isometrically isomorphic to the adjoint space of the Hilbert space \(H_{\alpha}(\Omega, B)\) and therefore is a Hilbert space. A consequence of theorem 1 is the following

**Theorem 1.** On the product \(H_{\alpha}(\Omega, B) \times H_{\alpha}(\Omega, B)\) the binear form \(b_{\alpha}\) having the following properties can be defined:
1. \(b_{\alpha}(u, v) = (u, v)_{L^2(\Omega)}\) for all \(u, v\) when \(u \in \alpha, v \in H_{\alpha}(\Omega, B)\).
2. The generalised Schwarz inequality
\[ |b_{\alpha}(u, v)| \leq \|u\|_{H_{\alpha}} \cdot \|v\|_{H_{\alpha}} \]
holds for all \(u \in H_{\alpha}(\Omega, B)\) and \(v \in H_{\alpha}(\Omega, B)\).
3. The isometric correspondence
\[ H_{\alpha}(\Omega, B) \ni u \mapsto (u, v)_{L^2(\Omega)}, \alpha \in H_{\alpha}(\Omega, B) \]
gives an isometric mapping of \(H_{\alpha}(\Omega, B)\) on \(H_{\alpha}(\Omega, B)\).

1.6. Definition 1. Let \(\|\cdot\|_{p}\) and \(\|\cdot\|_{q}\) be two norms of Banach type defined on a linear set \(X\) and satisfying the inequality \(\|w\|_{p} \leq \|w\|_{q}\) for all \(w \in X\). We say they are compatible on \(X\) (1), if each sequence \(u_n \in X\) which is fundamental in both norms and tends to zero in one of the norms \(\|w\|_{p}\) or \(\|w\|_{q}\), and this mapping leaves invariant the elements of the set \(X\). Therefore the \(\|w\|_{q}\) completion can be treated as a dense subset of the \(\|w\|_{q}\) completion.

Let \(X_{1}\) and \(X_{2}\) be two Banach spaces such, that \(X_{1}\) is a dense subset of \(X_{2}\) and \(\|w\|_{1} \geq \|w\|_{2}\) for all \(w \in X_{2}\). Because of the density each linear functional on \(X_{2}\) is uniquely determined by its restriction to the set \(X_{1}\) and this restriction is evidently continuous in the norm \(\|\cdot\|_{1}\) so

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(1) In Russian correspondence (see [5]).
is a linear functional on $X_1$. Denote by $\| \cdot \|_0$ and $\| \cdot \|_0'$ the norms in corresponding adjoint spaces.

$$\|u\|_0 = \sup_{\|w\|_0' \leq 1} \langle u, w \rangle, \quad \|u\|_0' = \sup_{\|w\|_0 \leq 1} \langle u, w \rangle.$$  

Then the inequality $\|u\|_0 \leq \|u\|_0'$ holds for all $u \in X_1^*$ and it may be proved in a simple way, that the norms $\| \cdot \|_0$ and $\| \cdot \|_0'$ are compatible on $X_1^*$.

**Lemma 3.** For $p_1 \geq p_2$ and $q_1 \geq q_2$ the inequality

$$\|u\|_{p_1, q_1} \leq \|u\|_{p_2, q_2}$$

holds for all $u \in B$, the norms $\| \cdot \|_{p_1, q_1}$ and $\| \cdot \|_{p_2, q_2}$ are compatible on $B$.

**Proof.** The inequality (1) follows immediately from the definition of the norms $\| \cdot \|_{p, q}$. We shall prove the compatibility of the norms.

In the case when $p_j$ and $q_j$ ($j = 1, 2$) are non-negative it is evident because we identify each element of the space $H_{p_j}(Q, B)$ with its strong derivative of order zero. Therefore $H_{p_j}(Q, B)$ is a dense subset of $H_{p_j}(Q, B)$ ($m \geq m_1, k_1 \geq k_2$) and from the preceding remarks it follows, that the norms $\| \cdot \|_{p_1, q_1}$ and $\| \cdot \|_{p_2, q_2}$ are compatible on the class $B$ (considered as the set of linear functionals on $H_{p_j}(Q, B)$).

As both spaces $H_{p_j, q_j}(Q, B)$ ($j = 1, 2$) are the completions of $B$ in the corresponding norms, we have the dense embedding $H_{p_j, q_j}(Q, B) \subset H_{p_j, q_j}(Q, B)$.

A similar reasoning proves that the norms $\| \cdot \|_{p_j, q_j}$ and $\| \cdot \|_{p_j, q_j}$ are compatible; thus $H_{p_j, q_j}(Q, B)$ is a dense subset of $H_{p_j, q_j}(Q, B)$ (from this follows the compatibility of the norms $\| \cdot \|_{p_1, q_1}$ and $\| \cdot \|_{p_2, q_2}$ on the class $B$). Therefore $H_{p_1, q_1}(Q, B)$ is also a dense subset of $H_{p_1, q_1}(Q, B)$.

Let $u_\alpha$ be a sequence of functions of the class $B$ fundamental in both norms $\| \cdot \|_{p, q}$ and $\| \cdot \|_{p, q}$ ($\alpha \geq k_2$) and let $\|u_\alpha\|_{p, q} \rightarrow 0$ for $s \rightarrow \infty$. Then for $0 \leq |\alpha| \leq s$ the sequence $\{D^\alpha u_\alpha\}$ is fundamental in the both norms $\| \cdot \|_{p, q}$ and $\| \cdot \|_{p, q}$ and $\|D^\alpha u_\alpha\|_{p, q} \rightarrow 0$. So $\|D^\alpha u_\alpha\|_{p, q} \rightarrow 0$, because the norms $\| \cdot \|_{p, q}$ and $\| \cdot \|_{p, q}$ are compatible, and therefore $\|u_\alpha\|_{p, q} \rightarrow 0$. So the norms $\| \cdot \|_{p, q}$ and $\| \cdot \|_{p, q}$ are compatible on $B$. From this follows the dense inclusion $H_{p_1, q_1}(Q, B) \subset H_{p_1, q_1}(Q, B)$ and as a consequence, the compatibility of the norms $\| \cdot \|_{p_1, q_1}$ and $\| \cdot \|_{p_1, q_1}$ on the class $B$.

So the lemma is proved and we have also verified.

**Theorem 2.** For $p_1 \geq p_2$ and $q_1 \geq q_2$ the space $H_{p_1}(Q, B)$ may be treated as a dense subset of the space $H_{p_2}(Q, B)$. The inequality (1) holds for all $u \in H_{p_1}(Q, B)$.

Let $p_1 \geq p$ and $q_1 \geq q$. According to what has been stated above we have the embeddings

$$H_{p_1}(Q, B) \subset H_{p}(Q, B), \quad H_{p}(Q, B) \subset H_{p_1}(Q, B).$$

Let $u \in H_{p_1}(Q, B)$, $v \in H_{p_1}(Q, B)$ and let $(u_\alpha)$ and $(v_\beta)$ be the corresponding approximating sequences of smooth functions

$$\|u_\alpha - u\|_{p_1} \rightarrow 0, \quad \|v_\beta - v\|_{p_1} \rightarrow 0.$$

From theorem 1 and lemma 3 it follows that $b_{\alpha, \beta}(u, v) = \lim_{\alpha, \beta \rightarrow \infty} b_{\alpha, \beta}(u, v)$; so for fixed $v \in H_{p_1}(Q, B)$ the form $b_{\alpha, \beta}(\cdot, v)$ is a restriction to the space $H_{p_1}(Q, B)$ of the form $b_{\alpha, \beta}(\cdot, v)$. (Evidently the roles of $\alpha$ and $\beta$ may be interchanged). Therefore we can omit the index and in the sequel we shall write simply $(u, v)$ instead of $b_{\alpha, \beta}(u, v)$.

From the definition of the norms $\| \cdot \|_{p, q}$ can be obtained in a simple manner.

**Lemma 4.** The inequality

$$\|u\|_{p, q} \geq ||D^\alpha u||_{p, q}$$

holds for

$$\begin{cases}
B & \text{when} |\alpha| \leq p, |\beta| \leq q, \\
B_{\alpha, \beta} & \text{when} |\alpha| > p, |\beta| \leq q, \\
B_{\beta, \alpha} & \text{when} |\alpha| \leq p, |\beta| > q, \\
C_{\alpha, \beta} & \text{when} |\alpha| > p, |\beta| > q.
\end{cases}$$

When $\Omega$ is the $N$-dimensional cube and $B$ is the class of all functions which belong to $C^\infty(\Omega)$ and are periodic, with $\Omega$ as the period parallelogram, then the inequality (2) holds for all $u \in B$ without any restriction concerning the support.

2. Some differentiability properties of the spaces $H_{p, q}(Q, B)$.

2.1. The present chapter contains some inequalities concerning the norms $\| \cdot \|_{p, q}$ which are similar to the inequalities for the norms $\| \cdot \|_{p, q}$ obtained by Ehrlich [1]. From these inequalities follows and analogue to the Sobolev Lemma for the spaces $H_{p, q}(Q, B)$.

We make the following assumptions concerning the domains $\Omega$ ($j = 1, 2$) (see [1] and [8]):
There are positive constants $A$ and $t_4$ (depending on $\Omega, n, \alpha$) such that the inequality

$$|u|_{\bar{A}^k} \leq At^{n-k-1}\left(\sum_{k=0}^{\infty} \text{ess} \sup_{x \in \Omega} |u|_{\bar{A}^{k+1}}^{n-k-1}\right)^{1/(n-k-1)}$$

$$0 \leq k \leq n; 0 \leq l \leq n$$

holds for $u \in P^\alpha(\Omega)$ and $t \geq t_4$.

With the aid of similar estimates, as used by Ehrling [1], can be also proved

**Lemma 6.** There exists a positive constant $A$ (depending on $\Omega, |\alpha|, |\beta|$) such that for $u \in P^\alpha(\Omega)$

$$\sup_{x \in \Omega} |D_x^\alpha D_y^\beta \text{sup}_{x \in \Omega} u(x, y)| < A u|_{\bar{A}^1}$$

$$\text{and } u|_{\bar{A}^1}$$

In the inequality (5) the roles of $x$ and $y$ may be interchanged.

2.2. We suppose now that $B$ is a subset of $P^\alpha(\Omega)$. The following two lemmas show that the functions belonging to the space $H_{\alpha, \beta}(\Omega, B)$ with sufficiently large indices have some regularity properties analogous to those given by Sobolev’s Lemma in the case of the space $H_\alpha$.

**Lemma 7.** Let $u \in H_{\alpha, \beta}(\Omega, B)$ for some $|\alpha| > R/2, k > S/2$. There exists a function $u \in P^\alpha(\Omega)$ such that the inequalities

$$D_x^\alpha D_y^\beta u(x, y) = D_x^\alpha D_y^\beta u_1(x, y)$$

hold almost everywhere in $\Omega$. So the space $H_{\alpha, \beta}(\Omega, B)$ may be treated as a subset of $P^\alpha(\Omega)$.

**Proof.** According to the remarks of the section 1.6. and to lemma 6 it is sufficient to show, that the norms $||u||_{H_{\alpha, \beta}(\Omega, B)}$ and $||u||_{H_\alpha}$ are compatible on $B$, where

$$||u||_{H_{\alpha, \beta}(\Omega, B)} = \sup_{x \in B} |D_x^\alpha D_y^\beta u(x, y)|.$$

Let $(u_n) \subset B$ be a sequence fundamental in both norms and tending to zero in the norm $||u||_{H_{\alpha, \beta}(\Omega, B)}$, and $||u||_{H_\alpha}$.
plesness of the space $H_{m,n}(Q, B)$ it is a square summable function $u$ such that $\|u_n - u\|_{m, n} \to 0$. But

$$\left| \int \int u_n(x, y) dxdy - \int \int u(x, y) dxdy \right| \leq \left| \int \int |u_n(x, y) - u(x, y)| dxdy \right|$$

and the integral on the right is not greater than $|Q|^{1/2} \|u_n - u\|_{m, n}$. Therefore

$$\int \int |u(x, y)| dxdy = 0$$

and so $u(x, y)$ vanish almost everywhere in $Q$, q.e.d.

As a consequence of inequality (5) we obtain (with $A_1 = |Q|^{1/2} A^{1/2}$)

$$\sup_\mathcal{F} \int \int D^2_{k} D^2_{l} u(x, y) dxdy \leq A_1 \|u\|_{m, n}$$

for $u \in P^m(Q)$. A similar reasoning as in the proof of lemma 7 shows that the norms $\|\cdot\|_{m, n} = \|\cdot\|_{m, n}$ are compatible on $B$, where

$$\|u\|_{m, n} = \sup_\mathcal{F} \int \int D^2_{k} u(x, y) dxdy$$

From this it follows

**Lemma 8.** Let $u \in H_{m,n}(Q, B)$ $(m \geq 0, n \geq 0)$. For each $a$(0 $\leq |a| \leq m)$ there exists a function $w^a \in P^{m-|a|-1}(Q)$ such that the quantities

$$\int \int D^2_{k} D^2_{l} u(x, y) dxdy = D^2_{k} w^a(x, y) \quad (0 \leq |a| \leq m, 0 \leq |b| < k - S(2))$$

hold almost everywhere in $Q$. So for $0 \leq |a| \leq m$ the functions

$$D^2_{k} u(x, y)$$

may be treated as belonging to the class $P^{m-|a|-1}(Q)$ and the derivation of order $\beta$ with respect to $y$ $(0 \leq |\beta| < k - S(2))$ can be made under the sign of integral, when this last derivation is taken in the strong sense.

5. Application of the spaces $H_{m,n}(Q, B)$ to the study of weak solutions of some non-elliptic equations

5.1. Let $A$ be the class of all differential operators defined in $Q$, which can be expressed in the form

$$L u = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^2_{\alpha} (a_{m, n} D^2_{\alpha} D^2_{n} u) + \sum_{|\beta| \leq m} (-1)^{|\beta|} D^2_{\beta} (b_{m, n} D^2_{\beta} D^2_{n} u) \quad (m, n \geq 0)$$

for sufficiently differentiable $u$ and which satisfies the following assumptions concerning the coefficients:

1° $a_{m, n}$ and $b_{m, n}$ are complex-valued functions infinitely differentiable in $Q$ and having all derivatives bounded in $Q$,

2° $a_{m, n}(x, y) = \bar{a}_{m, n}(x, y)$ for $(x, y) \in \Omega$,

3° let $\xi$ be the system of complex numbers $\xi_{m, n}(|a| = m, |\beta| = n)$; when one puts

$$Q(x, y; \xi) = \sum_{|\beta| \leq m} a_{m, n}(x, y) \xi_{m, n} \bar{a}_{m, n}(x, y)$$

there exists a positive constant $\delta$ such that the inequality

$$Q(x, y; \xi) \geq \delta \sum_{|\beta| \leq m} |\xi_{m, n}|^2$$

holds everywhere in $Q$.

The expression on the right of (6) shall be called *canonical form* of the operator $L$. From assumption 3° it follows that the operators of class $A$ are not elliptic in $Q$. In the special case $u = 0$ operator $L$ has the canonical form

$$L u = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^2_{\alpha} (a_{m, n} D^2_{\alpha} D^2_{n} u) + \sum_{|\beta| \leq m} (-1)^{|\beta|} D^2_{\beta} (b_{m, n} D^2_{\beta} D^2_{n} u)$$

It is elliptic in $Q$ and its coefficients depend on the parameter $y \in \mathbb{R}$. So the study of operators belonging to $A$ gives us some information about the elliptic operators depending on a parameter.

Denote by $A_e$ the differential operator $I - \sum_{r=1}^{\infty} \lambda^r |\partial^2_{x} \xi|^{2}$; $A_e$ has the same meaning, when $x$ is replaced by $y$. Simple calculations show that

$$(A_e)^r u = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \lambda^r |\alpha| D^2_{\alpha} u$$

and similarly for $(A_e)^r (r = 1, 2, \ldots)$. From the definition of class $A$ follows in a simple manner

**Lemma 9.** When $L \in A$, the formal adjoint operator $L^*$ and all the products of $L$ with $A_e$ and $A_e$ are also in $A$.  

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3.2. In the following we apply the Hilbert spaces defined in chapter 1 to the study of the weak solution of equation

\[ L_0 w = 0, \]

when \( L \) belongs to \( A \) and \( r \) is an element of some space \( H_{r+2} \). Our procedure is similar to the method used by Lax [7] in the case of an elliptic operator. We start with some energetic inequalities, which are analogous to the well-known inequality for elliptic operators given by Gårding [4].

**Lemma 10.** Let \( L \) be an operator of class \( A \) with \( m, n > 0 \); so each of the differential expressions \( A_L^r A_L^s L_{r+s} u, A_L^r L_{r+s} A_L^s u, L_{r+s} A_L^r A_L^s u, L_{r+s} L_{r+s} A_L^s u \) \((0 \leq r \leq \tau, 0 \leq s \leq \tau, u \in P^m(\Omega))\) can be brought to the canonical form

\[ \begin{align*}
\sum & (-1)^{m+s+r+s+D_0^r D_0^s (A_{\omega, r,s} A_0 D_0^r D_0^s u) + \\
& + \sum_{|s|=\tau, |r|=\tau} (-1)^{|r|+|s|} D_0^r D_0^s (e_{\omega, r,s} A_0 D_0^r D_0^s u).
\end{align*} \]

Denote by \( I_{\tau,r}(u) \) the corresponding Dirichlet integral and let \( \Omega \) satisfy the assumptions of section 2.1. There are some positive constants \( t_1, c \) (depending on \( \Omega, L, r, \tau \)) such that the inequality

\[ |L_0 u| \geq c \| u \|_{H_{1+r, 1+r, \tau}} \quad \text{for } u \in P^m(\Omega), \quad 0 \leq r \leq \tau, 0 \leq s \leq \tau \]

holds, when the functions \( \text{Re} b_{w, \alpha} \) \((|\alpha| = m)\), \( \text{Re} b_{w, \beta} \) \((|\beta| = n)\) and \( \text{Re} b_{w, \gamma} \) have a lower bound in \( \Omega \) exceeding \( t_1 \).

**Proof.** Let \( I_{\tau,r}(u) \) be the Dirichlet integral corresponding to the first sum in (8). According to the condition 3 (section 3.1) we have

\[ |I_{\tau,r}(u)| \leq \frac{\sum}{|s|=\tau, |r|=\tau} (-1)^{|r|+|s|} D_0^r D_0^s (e_{\omega, r,s} A_0 D_0^r D_0^s u). \]

The second sum can be presented in the form

\[ \begin{align*}
\sum & (-1)^{m+s+r+s} \left( \int_{|\alpha|=m} D_0^{r+s} D_0^{s+r} \left( \text{Re} b_{w, \alpha} A_0 D_0^{r+s} D_0^{s+r} u \right) \right) + \\
& + \sum_{|s|=\tau, |r|=\tau} (-1)^{|r|+|s|} D_0^r D_0^s (e_{\omega, r,s} A_0 D_0^r D_0^s u).
\end{align*} \]

When the coefficients \( \omega_{w, \alpha} \) do not depend on the functions \( \text{Re} b_{w, \alpha} \) \((|\alpha| = m)\), \( \text{Re} b_{w, \beta} \) \((|\beta| = n)\), \( \text{Re} b_{w, \gamma} \) (they depend only on the derivatives of these functions of order at most \( r \) with respect to \( x \), and at most \( s \) with respect to \( y \)). Denoting by \( I_{\tau}(u) \) the Dirichlet integral corresponding to the sum \( \sum(1) \) in (11) \((j = 1, 2, 3, 4)\) we obtain

\[ |I_{\tau,r}(u)| \geq t_1 \left( \| u \|_{H_{1+r, 1+r, \tau}} + \sum_{|s|=\tau, |r|=\tau} \| D_0^{r+s} D_0^{s+r} u \|_{L^2(\Omega)} + \sum_{|s|=\tau, |r|=\tau} \| D_0^{r+s} D_0^{s+r} u \|_{L^2(\Omega)} \right). \]

From this and similar estimates for \( I_{\tau,r}(u) \) and \( I_{\tau}(u) \) follows

\[ |I_{\tau}(u)| \leq t_1 \left( \| u \|_{H_{1+r, 1+r, \tau}} + \| u \|_{H_{1+s, 1+s, \tau}} + \| u \|_{H_{1+r+s, 1+r+s, \tau}} \right). \]

The remaining integral \( I_{\tau}(u) \) can be estimated with the aid of inequality (9)

\[ |I_{\tau}(u)| \leq \sum_{|s|=\tau, |r|=\tau} \| D_0^{r+s} D_0^{s+r} u \|_{L^2(\Omega)} \leq \psi(t) (t \| u \|_{H_{1+r, 1+r, \tau}} + t \| u \|_{H_{1+s, 1+s, \tau}} + t \| u \|_{H_{1+r+s, 1+r+s, \tau}}) \quad (t \geq t_0), \]

when \( \psi(t) \to 0 \) as \( t \to \infty \). Suppose \( t_1 \geq t_0 \); so from (10), (12) and (13) we obtain for \( t \geq t_1 \)

\[ |I_{\tau}(u)| \geq (d - \psi(t)) \| u \|_{H_{1+r, 1+r, \tau}} + t_1 \| u \|_{H_{1+s, 1+s, \tau}} + t \| u \|_{H_{1+r+s, 1+r+s, \tau}}. \]

Let

\[ \psi(t) \leq \frac{d - 1}{2} \]

for \( t \geq t_1 \). So for \( t \geq \max(t_1, t_0) \)

\[ |I_{\tau}(u)| \geq \frac{d}{2} \| u \|_{H_{1+r, 1+r, \tau}} + \frac{t_1}{2} \| u \|_{H_{1+s, 1+s, \tau}} + \frac{t}{2} \| u \|_{H_{1+r+s, 1+r+s, \tau}} \]

and according to the estimate (3) we get the inequality (9), q. e. d.

Using similar arguments the following two lemmas can be proved:

**Lemma 11.** Let \( L \) be an operator of class \( A \) with \( m > 0, n = 0 \) (so it is an elliptic operator depending on a parameter and its canonical form is given by formula (6a)). We suppose that \( \Omega \) satisfies all the assumptions of section 2.1 and we denote by \( I_{\tau}(u) \) the Dirichlet integral corresponding to the canonical form of the operator \( \Delta_L^r \) or \( \Delta_L^r \). There are some positive constants \( t_1, c \) (depending on \( \Omega, L, r_1 \)) such that

\[ |I_{\tau}(u)| \geq c \| u \|_{H_{1+r, 1+r, \tau}} \quad (u \in P^m(\Omega), \quad 0 \leq r \leq \tau), \]

when the function \( \text{Re} b_{w, \alpha} \) has lower bound in \( \Omega \) exceeding \( t_1 \).
Lemma 12. We suppose that all the assumptions of lemma 11 are true; let \( L_{\alpha}(w)(0 \leq r \leq r_0, 0 \leq s \leq s_0) \) have the same meaning as in lemma 10. So there are positive constants \( t_1, \epsilon \) (also depending on \( \Omega, L, r_0, s_0 \)) such that
\[
\|u\|_{\alpha, r, s} \geq \epsilon \|u\|_{\alpha, r_0, s_0}(u \in P_0(\Omega), 0 \leq r \leq r_0, 0 \leq s \leq s_0)
\]
when the functions \( \text{Re} a_m, (|a| = m) \) and \( \text{Re} b_m \) have a lower bound in \( \Omega \) exceeding \( t_1 \).

Remark. Simple calculation shows that the inequalities (9), (14) and (15) are true in the case \( L = \Delta^2 \Delta^2 (m, n > 0) \). More generally, when \( a_{m,n} = b_{m,n} = 0 \) for \( a \neq a' \) or \( b \neq b' \) and the remaining coefficients have a positive lower bound in \( \Omega \), \( L \) satisfies the energetic inequality
\[
\|u\| \geq \epsilon \|u\|_{\alpha, r_0, s_0},
\]
although the assumption that some coefficients are large may be not satisfied. \( I(u) \) denotes the Dirichlet integral corresponding to the canonical form of \( L \).

The inequalities (9), (14) and (15) can be brought to a different form when we suppose that the coefficients \( a_{m,n} \) and \( b_{m,n} \) (or \( a_m \) and \( b_m \)) and the function \( u \) satisfy such boundary conditions that after the integration by parts the boundary integrals vanish. We obtain then the estimate
\[
\|L_{\alpha}(u, u)\| \geq \epsilon \|u\|_{\alpha, r_0, s_0}(u \in P_0(\Omega), 0 \leq r \leq r_0, 0 \leq s \leq s_0),
\]
when \( L_{\alpha} \) denotes some of the operators \( \Delta^2 \Delta^2 L, \Delta^2 \Delta^2 L, \Delta^2 \Delta^2 L, \Delta^2 \Delta^2 L \).

3.3. In this and in next section we suppose that \( \Omega \) is the \( N \)-dimensional cube defined by inequalities \( |x| < a \) (\( i = 1, \ldots, R \)), \( |y| < a \) (\( j = 1, \ldots, S \)). Let \( B_p \) be the class of all functions infinitely differentiable in the whole space \( \mathbb{R}^N \) and periodic with \( \Omega \) as the period-parallelogram. Our purpose is a study of periodic weak solutions (see definition 2) of equation (7) with the aid of the spaces \( H_{\alpha, \beta}(\Omega, B_p) \). We begin with the following differential inequality:

**Lemma 13.** Let \( L \) be an operator of class \( \Lambda \) with coefficients \( a_{m,n} \) and \( b_{m,n} \) (or \( a_m \) and \( b_m \)) belonging to \( B_p \). We suppose that the inequality (17) is true. So
\[
\|u\|_{\alpha, \beta} \leq \epsilon \|L u\|_{\alpha, \beta, \alpha, \beta}
\]
in particular
\[
\|u\|_{\alpha, \beta} = 0
\]
From inequality (17) it follows in the limit that
\[
\|u\|_{\alpha, \beta} = \epsilon \|u\|_{\alpha, \beta, \alpha, \beta}
\]
therefore \( u = 0 \) and according to theorem 1 the functional \( I \) has the norm zero, also vanish identically on \( H_{\alpha, \beta}(\Omega, B_p) \). Thus we have proved that \( \Gamma \) is dense in \( H_{\alpha, \beta}(\Omega, B_p) \).

Let now \( f \) be an arbitrary function of class \( B_p \) and \( \epsilon \) a positive number. When we apply what has been just proved to the operator \( \Delta^2 \Delta^2 L \) it follows that there exists a function \( f \) such that
\[
\|\Delta^2 \Delta^2 f - \Delta^2 \Delta^2 L f\|_{\alpha, \beta, \alpha, \beta} < \epsilon.
\]
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From lemma 13 applied to the operator \( \mathcal{A}_x^* \mathcal{A}_x \) it follows that
\[
\|f_x - Lf\|_{H_{-\mu - \nu}^0(\Omega, B_x)} \leq c \|\mathcal{A}_x^* \mathcal{A}_x f_x - Lf\|_{H_{-\mu - \nu}^0(\Omega, B_x)}
\]
and therefore \( \Gamma \) is dense in the space \( H_{-\mu - \nu}^0(\Omega, B_x) \) \( \mu, \nu > 0 \), q.e.d.

5.4. Definition 2. Let \( u, v \) be two elements of a space \( H_{\mu,\alpha}(\Omega, B_x) \) and let \( L \) be a differential operator with coefficients belonging to \( B_x \). We say that \( u \) is the \textit{periodic weak solution} of the equation
\[
Lu = v
\]
if the equality \( (u, L^* \varphi) = (v, \varphi) \) holds identically for \( \varphi \in B_x \).

The following theorem is analogous to the differentiability theorem of Lax [7] for elliptic equations.

**Theorem 3.** Let \( \Omega \) the \( N \)-dimensional cube and \( L \) an operator of class \( \Lambda \) satisfying inequality (17) with coefficients belonging to \( B_x \). We suppose that \( u \) is the periodic weak solution of equation (7) lying in a (sufficiently large) space \( H_{\mu,\alpha}(\Omega, B_x) \). When \( v \) is an element of \( H_{\mu + 2m,\alpha + 2m}(\Omega, B_x) \), then \( u \) is in \( H_{\mu,\alpha}(\Omega, B_x) \).

**Proof.** From the generalized Cauchy inequality we obtain applying lemma 13 to the operator \( L^* \) (when we suppose that \( \gamma_1 \) and \( \gamma_m \) are sufficiently large)
\[
\|L^* \varphi, u\| \leq c \|\varphi\|_{\Omega, B_x} \|L^* \varphi\|_{H_{-\gamma_1 - \gamma_m - 2m}^0(\Omega, B_x)}.
\]

So the linear functional \( \mathcal{I}(\varphi) = (\varphi, u) \) is bounded on the dense subset \( \Gamma \) of the space \( H_{-\gamma_1 - \gamma_m - 2m}(\Omega, B_x) \) and therefore can be prolonged uniquely to the linear functional on the whole space. From theorem 1 it follows that \( u \) belongs to \( H_{\mu,\alpha}(\Omega, B_x) \), q.e.d.

It follows from theorem 3 and lemmas 7 and 8 that \( u \) has some differentiability properties in the classical sense, when the numbers \( p + 2m \) and \( q + 2n \) are non-negative and at least one of them is sufficiently large. In the special case \( s = 0 \), from theorem 3 follows the differentiability of periodic weak solutions of elliptic equations depending on a parameter (according to the remarks in section 3.1).

**Bibliography**


Reçu par l'éditeur le 21.1.1962