STUDIA MATHEMATICA, T. XXII. (1962)

A comparison of weak and strong continuity

by

J. V. RYFF (Stanford, Cal.)

It is always interesting to learn the degree to which a property given in the weak topology of a Banach space B is valid in the strong (norm) topology. In this paper we shall be concerned with functions $t \to x(t)$ from a subset of the real line taking values in a fixed Banach space, in short, vector-valued functions.

By and large, if for each continuous linear functional f in the conjugate space B^*, the function $f(x(t))$ as a complex-valued function has property π, then $x(t)$ is said to be *weakly-π*. An exception to this will be found in the definition of weak differentiability. Not only is it required that $d(x(t))/dt$ exist for each $f \in B^*$ but further, there must be some element ξ (usually written $x'(t)$) in B such that $df(x(t))/dt = f(\xi)$. In reflexive spaces there is no problem, otherwise the existence of ξ is not guaranteed by the existence of the derivatives alone. In the following resumé we shall try to give an account of some more familiar properties of vector-valued functions which have meaning in both the weak and strong sense.

(a) If $x(t)$ is weakly differentiable in a measurable set E to $x'(t)$, then $x(t)$ is strongly differentiable to $x'(t)$ almost everywhere ([1], p. 193).

(b) If $x(t)$ is of weak bounded variation it is also of bounded variation but not necessarily of strong bounded variation (see [5], p. 59-60, for definitions and proof).

(c) If $x(t)$ is weakly measurable and if, except for a set of zero measure, the range of $x(t)$ is separable, then $x(t)$ is strongly measurable ([5], p. 72-73). In particular, when B is separable they are equivalent (see also [2]).

(d) If $x(\zeta)$ is a weakly holomorphic vector-valued function in a domain D of the complex plane, then $x(\zeta)$ is holomorphic in the strong sense ([5], p. 92-93). We shall give a sharper statement of this later.

It appears that a rather interesting possibility has yet to be considered: If $x(t)$ is weakly continuous on an interval $[a, b]$ are there points of strong continuity?

1. Weakly continuous vector-valued functions. In this section $x(t)$ will be a weakly continuous vector-valued function from an interval $[a, b]$ to a Banach space B. The main result is then

THEOREM. *The set of points of strong discontinuity of $x(t)$ forms a set of first category.*

The proof is modeled after one given by Boas ([3], p. 99-102) with modifications necessary because of the infinite dimensional range.

LEMMA. *$x(t)$ cannot be strongly discontinuous at each point in $[a, b]$.*

Proof. By contradiction. If t is a point of strong discontinuity then the images of arbitrarily small neighborhoods of t do not shrink to $x(t)$. Thus, to each t in $[a, b]$ there is an integer n such that the diameter of the image of each neighborhood of t is at least $1/n$. Let D_n be the collection of those t in $[a, b]$ with this property. Each D_n is closed, for if t_0 is an accumulation point of D_n, then any neighborhood of t_0 is a neighborhood of some t in D_n. Moreover

$$[a, b] = \bigcup_{n=1} D_n$$

since each point t is supposed to be a point of strong discontinuity. By Baire's theorem some D_n contains an interval, say I_0. The range of $x(t)$ is separable (consider the closed linear extension of the points $x(r)$, where r is rational) so that we may cover the range of $x(t)$ by a denumerable number of closed spheres S_k centered at points $x(t_k)$ having diameter less than $1/n$. Denote by H_k the inverse image of the intersection of the sphere S_k with the range of $x(t)$. The H_k cover $[a, b]$ and so cover I_0. On the other hand, no H_k can cover any subinterval of I_0. By Baire's theorem, some H_k must be dense in a subinterval of I_0. If $\{t_\nu\}$ is a sequence in this H_k converging to t', then $x(t')$ belongs to the corresponding sphere S_k. If not, $x(t')$ would not belong to the convex closure of the points $\{x(t_\nu)\}$. But by the weak continuity of $x(t)$, the point $x(t')$ is a weak limit of the sequence $\{x(t_\nu)\}$, and such limits always belong to the convex closure of the sequence. This proves that H_k is closed and, since it is dense in a subinterval of I_0, it must actually cover this subinterval. This gives the contradiction.

The proof of the theorem is now fairly easy. Let D_n be the set of points t in $[a, b]$ such that a sequence $\{t_\nu\}$ can be found with $t_\nu \to t$ and $\|x(t_\nu)-x(t)\| \geqslant 1/n$, $n = 1, 2, \ldots$ The sets D_n cover the points of strong discontinuity of $x(t)$. If the closure $\bar{D}_n$ of some D_n contained an interval, then by our lemma we can find a point of strong continuity t_0 in $\bar{D}_n$. For $\varepsilon = 1/2n$ determine $\delta > 0$ such that $|t-t_0| < \delta$ implies $\|x(t)-x(t_0)\| < 1/2n$. Take $t \in D_n$ to satisfy $|t-t_0| < \delta/2$ and then find t_ν such that $|t_\nu - t| < \delta/2$ and

$$\|x(t_\nu)-x(t)\| \geqslant \frac{1}{n}. \tag{1}$$

Then

$$\|x(t_\nu)-x(t)\| \leqslant \|x(t_\nu)-x(t_0)\| + \|x(t_0)-x(t)\| < \|x(t_\nu)-x(t_0)\| + \frac{1}{2n}.$$

But $|t_\nu - t_0| < \delta$, gives $\|x(t_\nu)-x(t_0)\| < 1/2n$, contradicting inequality (1).

It is fitting that we include here an example which seems to have about as many strong discontinuities as possible. The interval will be $[0,1]$ and the Banach space is to be the real Hilbert space l_2.

First we define a sequence $\psi_n(t)$ of continuous real-valued functions on $[0,1]$ in the following manner:

If $C_1 = (\frac{1}{3}, \frac{2}{3})$, take $\psi_1(t)$ to be zero outside of C_1 and $\psi_1(\frac{1}{2}) = 1$. Then complete $\psi_1(t)$ linearly. That is, for $\frac{1}{3} \leqslant t \leqslant \frac{2}{3}$

$$\psi_1(t) = 1 - 6\,|t-\tfrac{1}{2}|.$$

Next, for $C_{2,1} = (\frac{1}{9}, \frac{2}{9})$ and $C_{2,2} = (\frac{7}{9}, \frac{8}{9})$ define $\psi_2(t)$ to vanish in the complement of these intervals, set

$$\psi_2(\tfrac{1}{6}) = \psi_2(\tfrac{5}{6}) = 1$$

and complete as a linear function. With $C_{3,1} = (\frac{1}{27}, \frac{2}{27}), \ldots, C_{3,4} = (\frac{25}{27}, \frac{26}{27})$ define $\psi_3(t)$ to be zero outside these intervals, equal to unity at the midpoint of each C_{3k}, and linear otherwise. Thus, at each stage in removing middle thirds in the construction of the Cantor set we define a continuous function $\psi_n(t)$ which peaks to unity at the midpoints of each removed interval and vanishes outside. Now set

$$\varphi_1(t) = \psi_1(t)$$

$$\cdots\cdots\cdots$$

$$\varphi_n(t) = \frac{1}{2^{n-1}}\psi_1(t) + \ldots + \frac{1}{2}\psi_{n-1}(t) + \psi_n(t).$$

Each $\varphi_n(t)$ is continuous, bounded by unity and vanishes on the Cantor set. If t_{k_n} is the midpoint of the interval C_{k_n} removed in the construction of the Cantor set we have

$$\varphi_m(t_{k_n}) = 0 \qquad (m < k),$$

$$\varphi_m(t_{k_n}) = \frac{1}{2^{m-k}} \qquad (1 \leqslant k \leqslant m).$$

Define the function $x(t)$ from $[0,1]$ to l_2 by

$$x(t) = (\varphi_1(t), \ldots, \varphi_m(t), \ldots).$$

It is necessary to show that for each t this sequence belongs to l_2. But the largest values the $\varphi_m(t)$ can take are at the midpoints t_{k_n}. If τ is such a point, we have in fact

$$\sum_{n=1}^{\infty} \varphi_n(\tau)^2 = \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}}\right)^2 = \tfrac{4}{3}.$$

On the other hand, if $a = (a_1, a_2, \ldots, a_m, \ldots)$ is an arbitrary element of l_2, the function

$$\langle a, x(t)\rangle = \sum_{k=1}^{\infty} a_k \varphi_k(t)$$

is a continuous function of t, so that $x(t)$ is weakly continuous on [0,1] ($\langle\ \rangle$ represents the inner product in l_2).

We now demonstrate that $x(t)$ is (strongly) discontinuous at each point c of the Cantor set. At these points $x(c)$ is the zero sequence. Choose a sequence $\{t_\nu\}$ of midpoints converning to c. Then

$$\|x(t_\nu) - x(c)\| = \|x(t_\nu)\| = \frac{2}{\sqrt{3}}.$$

Accordingly, strong continuity must fail at each such point. If, instead, we extracted middle fourths or less it would be possible to construct sets with measure arbitrarily close to unity. Defining a sequence $\varphi_n(t)$ as we did for the Cantor ternary set it is clear that one may construct weakly continuous functions which are strongly discontinuous on sets of measure $1-\varepsilon$, for any $0 < \varepsilon < 1$.

Whether the result can be sharpened to strong discontinuities only on nowhere dense perfect sets in place of sets of first category is not known.

2. Weak continuity with respect to a fundamental set. Following Alexiewicz ([1], p. 186), define a subset B_0^* of the conjugate space to be *fundamental for* B, if for any $\varepsilon > 0$ and x in B there exist elements $f_1, \ldots, f_n$ in B_0^* and real numbers $a_1, \ldots, a_n$ such that

$$\|f\| = 1, \quad |f(x)| \geqslant \|x\| - \varepsilon \quad \text{where} \quad f = a_1 f_1 + \ldots + a_n f_n.$$

This means that $\|x\|$ can be found by taking the supremum of $|f(x)|$ over those functionals of unit norm in the real linear extension of B_0^*. A function $x(t)$ is called B_0^*-*weakly continuous* at t_0 if $f(x(t))$ is continuous at t_0 for each f in B_0^*. One may define B_0^*-weakly differentiable functions analogously. Although a weakly differentiable function is strongly differentiable almost everywhere, Gelfand ([4], p. 265) gives an example of a B_0^*-weakly differentiable function which is strongly differentiable nowhere. We shall show by example that this complete breakdown occurs in the case of B_0^*-weak continuity. This makes all the more interesting

the fact that if, for any set B_0^* fundamental for B, $f(x(\zeta))$ is holomorphic for ζ in some domain D and all f in B_0^*, then $x(\zeta)$ is itself holomorphic in the strong sense (see [5], p. 92-94, and [2], p. 67). The illustration we give is of an elementary nature.

If a Banach space Y is known to be the conjugate space of a Banach space X, then the image of X in $Y^* = X^{**}$ under the canonical embedding will be fundamental for Y. With this in mind, we take the Banach space M of essentially bounded functions on the interval [0,1] and identify the space L_1 of integrable functions on [0,1] with a subset of M^* fundamental for M. Each function $\varphi(s)$ in L_1 determines a continuous linear functional F_φ on M according to the equation

$$F_\varphi(m) = \int_0^1 m(s)\varphi(s)\,ds$$

for each function $m(s)$ in M.

Define a function x_t from the interval $[0, 1]$ to the space M by setting x_t equal to the characteristic function of the interval $[0, t]$. That is,

$$x_t(s) = \begin{cases} 1 & \text{for} \quad 0 \leqslant s \leqslant t, \\ 0 & \text{for} \quad t < s \leqslant 1. \end{cases}$$

For each function $\varphi(s)$ in L_1 we then have

$$F_\varphi(x_t) = \int_0^1 x_t(s)\varphi(s)\,ds = \int_0^t \varphi(s)\,ds,$$

a continuous function of t. But clearly, as a vector-valued function, x has no points of strong continuity.

References

[1] A. Alexiewicz, *On differentiation of vector-valued functions*, Studia Mathematica 11 (1950), p. 185-196.

[2] — *On certain "weak" properties of vector-valued functions*, ibidem 17 (1958), p. 65-68.

[3] R. Boas, *A Primer of Real Functions*, Carus Mathematical Monographs 13, New York 1960.

[4] I. Gelfand, *Abstrakte Functionen und lineare Operatoren*, Matiematiczeskii Sbornik 4 (46) (1938), p. 235-286.

[5] E. Hille and R. Phillips, *Functional Analysis and Semi-Groups*. American Mathematical Society Colloquium Publications 31, Providence 1957.

STANFORD UNIVERSITY

Reçu par la Rédaction le 14. 12. 1961