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The minimum modulus of a linear operator and its use in spectral theory

by

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1. Introduction. Let X be a normed linear space (with either real or complex scalars) and let T be a linear operator whose domain $D(T)$ and range $R(T)$ lie in X. Here the adjective "linear" describes the purely algebraic properties of additivity and homogeneity; a linear operator need not be continuous.

We assume throughout that X and $D(T)$ contain some non-zero vectors. Let S be the surface of the unit ball in $D(T)$:

$$S = \{x : \|x\| = 1, x \in D(T)\}.$$

Then we define

$$N(T) = \sup_{x \in S} \|Tx\|, \quad \mu(T) = \inf_{x \in S} \|Tx\|.$$

We call $\mu(T)$ the *minimum modulus* of T. It is a non-negative real number. The value of $N(T)$ may be $+\infty$. It is finite if and only if T is continuous. If T is continuous and $D(T) = X$, $N(T)$ is more usually denoted by $\|T\|$.

If λ is any scalar and I is the identity operator on X, $\lambda I - T$ is a linear operator with the same domain as T. The general intent of our paper is to exhibit the usefulness of the minimum modulus $\mu(\lambda I - T)$ as an instrument for studying the spectrum of T. Let $\sigma(T)$ denote the spectrum of T. By definition, $\lambda \in \sigma(T)$ if one or more of the following assertions is true:

(a) $\lambda I - T$ has no inverse;
(b) $\lambda I - T$ has an inverse, but $(\lambda I - T)^{-1}$ is a discontinuous operator;
(c) the range of $\lambda I - T$ is not dense in X.

The set of all scalars not in $\sigma(T)$ is called the *resolvent set* of T, denoted by $\varrho(T)$. In studying topological questions related to $\sigma(T)$ and $\varrho(T)$, we use the topology of the real line or of the complex plane, depending on whether X is a real or a complex space.

For studying the structure of $\sigma(T)$ it is convenient to make use of the classification of operators into *states*, as described in Taylor [6],

p. 235-236. See also Taylor and Halberg [7]. Let X and Y be normed linear spaces with the same scalar field, and let A be a linear operator (not necessarily continuous) with domain in X and range in Y. There are nine possible states for A, denoted by

$$\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3, \quad \mathrm{II}_1, \mathrm{II}_2, \mathrm{II}_3, \quad \mathrm{III}_1, \mathrm{III}_2, \mathrm{III}_3.$$

The symbols I, II, III refer to $R(A)$:

I. $R(A) = Y$,

II. $\overline{R(A)} = Y$, but $R(A) \neq Y$,

III. $\overline{R(A)} \neq Y$.

The symbols 1, 2, 3 refer to the inverse of A:

1. A has a continuous inverse;
2. A has a discontinuous inverse;
3. A has no inverse.

Then A is in state III_1 if $\overline{R(A)} \neq Y$ and A has a continuous inverse. Likewise for the meanings of the other states.

If X', Y' are the spaces conjugate to X and Y, respectively, they are Banach spaces. If A is a linear operator with domain dense in X and range in Y, there is a uniquely defined operator A' (the conjugate of A) with domain in Y' and range in X'. We can then speak about various possible states for A'. A study of the relationships between the states of A and the states of A' has been made. This was done by Taylor and Halberg [7] for the case in which A is continuous and $D(A) = X$; these results were extended to the general case by Goldberg [2]. These studies have a bearing on our present investigations of $\sigma(T)$.

Returning now to a consideration of T and its spectrum, we see that $\lambda \in \varrho(T)$ if the state of $\lambda I - T$ is I_1 or II_1; otherwise $\lambda \in \sigma(T)$. We can then subdivide $\sigma(T)$ into seven parts, according to the states of $\lambda I - T$. These states are customarily grouped as follows:

$\lambda \in C\sigma(T)$ (continuous spectrum) if and only if the state of $\lambda I - T$ is I_2 or II_2;

$\lambda \in R\sigma(T)$ (residual spectrum) if and only if the state of $\lambda I - T$ is III_1 or III_2;

$\lambda \in P\sigma(T)$ (point spectrum) if and only if the state of $\lambda I - T$ is I_3, II_3, or III_3.

As a matter of notation we shall denote by $\mathrm{II}_2\sigma(T)$ the set of scalars λ such that the state of $\lambda I - T$ is II_2. Likewise for the sets $\mathrm{I}_3\sigma(T)$, $\mathrm{III}_1\sigma(T)$, and so on. In a similar way we define $\mathrm{I}_1\varrho(T)$ and $\mathrm{II}_1\varrho(T)$; these two sets have $\varrho(T)$ as their union.

We shall now describe briefly the contents of the paper. Concerning the minimum modulus, the basic lemmas are Lemmas 2.3 and 2.4, from

which we are able to deduce (in section 3) that $\varrho(T)$ and $\mathrm{III}_1\sigma(T)$ are open, and that $\mu(\lambda I - T) = 0$ if λ is in the boundary of $\sigma(T)$. The result about $\varrho(T)$ is not new, of course. The other results are new in the generality here presented, with no hypothesis that X is complete or that T is closed. If $\lambda_0 \in \varrho(T)$, the *resolvent radius* of λ_0 may be defined as the least upper bound of real numbers r such that $\lambda \in \varrho(T)$ if $|\lambda - \lambda_0| < r$. When X is complete and T is closed, this resolvent radius is the reciprocal of the spectral radius of $(\lambda_0 I - T)^{-1}$, and it may be expressed as a limit with the aid of the minimum modulus (Theorem 3.5). Likewise, if $\lambda_0 \in \mathrm{III}_1\sigma(T)$, we can define the III_1-radius of λ_0. If T is continuous, with $D(T) = X$, or if X is complete and T is closed, the III_1-radius of λ_0 is not less than $\sup_n \{\mu[(\lambda_0 I - T)^n]\}^{1/n}$. These assertions follow from Theorems 3.8 and 3.9. Other related results are also given in section 3.

In section 4 we use the theory of conjugate operators to show that $\mathrm{I}_3\sigma(T)$ is an open set when T is a closed and densely defined operator in a complete space X. In fact, $\mathrm{I}_3\sigma(T) = \mathrm{III}_1\sigma(T')$. We can also use the theory of conjugate operators to estimate the resolvent radius or III_1-radius of a point, with different hypotheses from those in section 3.

In section 5 we show how to get an equation of a locus containing the boundary of the spectrum of a bounded linear operator in Hilbert space (Theorem 5.1). We also show that, if A is an operator of a special sort called *positive seminormal*, its residual spectrum is empty.

The concluding section is devoted to a detailed study of $\sigma(A)$ in several special cases of a certain class of operators. In the most interesting case we get a spectrum which is a circular disk, composed of an inner open disk, which is $\mathrm{III}_1\sigma(A)$, and two concentric annular rings, which are $\mathrm{III}_2\sigma(A)$ and $\mathrm{II}_2\sigma(A)$, respectively.

2. Some general lemmas.

All of the operators considered in this section are assumed to be linear and to have their domains and ranges in X.

LEMMA 2.1. *The operator T has a continuous inverse T^{-1} if and only if $\mu(T) > 0$. In that case*

$$N(T^{-1}) = \frac{1}{\mu(T)}. \tag{2.1}$$

Proof. We know from the definition of $\mu(T)$ that it is the largest real number k such that $\|Tx\| \geqslant k\|x\|$ for each x in $D(T)$. When T has a continuous inverse T^{-1}, $N(T^{-1})$ is the smallest real number C such that $\|T^{-1}y\| \leqslant C\|y\|$ for each y in $R(T)$. The lemma follows easily from these facts; we omit further details.

LEMMA 2.2. *When T_1 and T_2 have the same domain, then*

$$|\mu(T_1) - \mu(T_2)| \leqslant N(T_1 - T_2). \tag{2.2}$$

Proof. We may assume $N(T_1-T_2) < \infty$, the result being trivial otherwise. For any x in the common domain, with $\|x\| = 1$, we have

$$\|T_1x\| - \|T_2x\| \leqslant \|T_1x - T_2x\| \leqslant N(T_1-T_2),$$

whence

$$\mu(T_1) - \|T_2x\| \leqslant N(T_1-T_2).$$

Then

$$\mu(T_1) - N(T_1-T_2) \leqslant \|T_2x\|,$$

and so

$$\mu(T_1) - N(T_1-T_2) \leqslant \mu(T_2), \quad \text{or} \quad \mu(T_1) - \mu(T_2) \leqslant N(T_1-T_2).$$

In this result we can exchange T_1 and T_2. We then get (2.2), because $N(T_2-T_1) = N(T_1-T_2)$.

LEMMA 2.3. *If T and T_0 have the same domain, and if $N(T-T_0) < \mu(T_0)$, then $\mu(T) > 0$ and $\overline{R(T)}$ is not a proper subset of $\overline{R(T_0)}$.*

Proof. We know that $\mu(T_0) - \mu(T) \leqslant N(T-T_0)$. Hence, by the hypothesis,

$$0 < \mu(T_0) - N(T-T_0) \leqslant \mu(T).$$

Now suppose, contrary to the assertion, that $\overline{R(T)}$ is a proper subset of $\overline{R(T_0)}$. We can choose a real number Θ so that

$$\frac{N(T-T_0)}{\mu(T_0)} < \Theta < 1.$$

By a lemma due to F. Riesz (Theorem 3.12-E in Taylor [6]) there exists an element y_0 in $\overline{R(T_0)}$ such that $\|y_0\| = 1$ and $\|y-y_0\| \geqslant \Theta$ if $y \in \overline{R(T)}$. Choose $y_n = T_0x_n$ so that $y_n \to y_0$. Then

$$\mu(T_0)\|x_n\| \leqslant \|T_0x_n\| \leqslant \|T_0x_n - y_0\| + \|y_0\|,$$

and so

$$\|x_n\| \leqslant \frac{1}{\mu(T_0)}[\|y_n - y_0\| + 1].$$

But also,

$$\Theta \leqslant \|y_0 - Tx_n\| \leqslant \|y_0 - T_0x_n\| + \|T_0x_n - Tx_n\|,$$

$$\Theta \leqslant \|y_0 - y_n\| + N(T-T_0)\|x_n\|,$$

so that

$$\Theta \leqslant \|y_0 - y_n\| + \frac{N(T-T_0)}{\mu(T_0)}[\|y_n - y_0\| + 1].$$

Letting $n \to \infty$, we obtain

$$\Theta \leqslant \frac{N(T-T_0)}{\mu(T_0)} < \Theta,$$

a contradiction. This completes the proof.

LEMMA 2.4. *If T and T_0 have the same domain, and if $N(T-T_0) < \frac{1}{2}\mu(T_0)$, then $\overline{R(T_0)}$ is not a proper subset of $\overline{R(T)}$ and $\overline{R(T)}$ is not a proper subset of $\overline{R(T_0)}$.*

Proof. The hypothesis implies $\mu(T_0) > 0$ and $N(T-T_0) < \mu(T_0)$. Hence Lemma 2.3 can be applied, and we get part of the desired conclusion. We also have

$$\mu(T_0) - \mu(T) \leqslant N(T-T_0).$$

Combining this with the hypothesis that $2N(T-T_0) < \mu(T_0)$, we see that $N(T-T_0) < \mu(T)$. We can now apply Lemma 2.3 with the roles of T_0 and T exchanged. The conclusion is that $\overline{R(T_0)}$ is not a proper subset of $\overline{R(T)}$.

Under the conditions of Lemma 2.4, we see that either both $R(T)$ and $R(T_0)$ are dense in X or neither is dense in X.

3. Applications to spectral theory. Let T be a fixed linear operator with domain and range in X. For each scalar λ we define

$$\Phi(\lambda) = \mu(\lambda I - T). \tag{3.1}$$

The function Φ is thus a function with non-negative real values. It is continuous. In fact,

$$|\Phi(\lambda_1) - \Phi(\lambda_2)| \leqslant |\lambda_1 - \lambda_2|. \tag{3.2}$$

This inequality follows directly from Lemma 2.2, for

$$(\lambda_1 I - T)x - (\lambda_2 I - T)x = (\lambda_1 - \lambda_2)x,$$

and so

$$N[(\lambda_1 I - T) - (\lambda_2 I - T)] = |\lambda_1 - \lambda_2|. \tag{3.3}$$

Since Φ is continuous, the set of λ's for which $\Phi(\lambda) > 0$ is an open set. This set is the same as the set of λ's for which $\lambda I - T$ has a continuous inverse. We may divide it into the two parts

$$\varrho(T) = \mathrm{I}_1\varrho(T) \cup \mathrm{II}_1\varrho(T), \quad \text{and} \quad \mathrm{III}_1\sigma(T).$$

Each of these parts is open, as we shall see.

THEOREM 3.1. *The resolvent set $\varrho(T)$ is open. If $\lambda_0 \in \varrho(T)$, then $\lambda \in \varrho(T)$ for each λ such that $|\lambda - \lambda_0| < \Phi(\lambda_0)$.*

Proof. It suffices to prove the second assertion. By hypothesis $\Phi(\lambda_0) > 0$ and $R(\lambda_0 I - T)$ is dense in X. By (3.3) and Lemma 2.3 we see that $|\lambda - \lambda_0| < \Phi(\lambda_0)$ implies that $\Phi(\lambda) > 0$ and that $\overline{R(\lambda I - T)} = X$, which means that $\lambda \in \varrho(T)$.

Next we prove a preliminary result: If $\lambda_0 \in \mathrm{III}_1\sigma(T)$ and if $|\lambda - \lambda_0| < \frac{1}{2}\Phi(\lambda_0)$, then $\lambda \in \mathrm{III}_1\sigma(T)$, and hence $\mathrm{III}_1\sigma(T)$ is open. For, by (3.3) we have

$$N[\lambda I - T) - (\lambda_0 I - T)] < \tfrac{1}{2}\mu(\lambda_0 I - T),$$

and we can apply Lemma 2.4 to conclude that $\overline{R(\lambda_0 I - T)}$ is not a proper subset of $\overline{R(\lambda I - T)}$. Since, by hypothesis, $\overline{R(\lambda_0 I - T)} \neq X$, we must also have $\overline{R(\lambda I - T)} \neq X$. We can also conclude that $\Phi(\lambda) > 0$, and hence that $\lambda \in \mathrm{III}_1\sigma(T)$. In fact, we can see from (3.2) that $\Phi(\lambda) \geqslant \Phi(\lambda_0) - |\lambda - \lambda_0| > \frac{1}{2}\Phi(\lambda_0) > 0$.

Presently we shall get a better result about the size of an open circular neighbourhood of λ_0 lying in $\mathrm{III}_1\sigma(T)$. But first we observe the following theorem about points on the boundary of $\sigma(T)$. We denote this boundary by $\partial\sigma(T)$.

THEOREM 3.2. *If $\lambda \in \partial\sigma(T)$, then $\Phi(\lambda) = 0$, i. e. $\mu(\lambda I - T) = 0$.*

Proof. The set $\{\lambda : \Phi(\lambda) > 0\}$ is open, because Φ is continuous. It is the union of the disjoint open sets $\varrho(T)$ and $\mathrm{III}_1\sigma(T)$. Now, a point λ of $\partial\sigma(T)$ must be in $\sigma(T)$, because $\sigma(T)$ is closed. But such a λ cannot be an interior point of $\sigma(T)$, so it cannot be in $\mathrm{III}_1\sigma(T)$. Hence, necessarily, $\Phi(\lambda) = 0$ if $\lambda \in \partial\sigma(T)$.

Next we have a result comparable to Theorem 3.1:

THEOREM 3.3. *If $\lambda_0 \in \mathrm{III}_1\sigma(T)$ and if $|\lambda - \lambda_0| < \Phi(\lambda_0)$, then $\lambda \in \mathrm{III}_1\sigma(T)$.*

Proof. From (3.2) we have

$$\Phi(\lambda) \geqslant \Phi(\lambda_0) - |\lambda - \lambda_0| > 0$$

if $|\lambda - \lambda_0| < \Phi(\lambda_0)$. Let E be the set $\{\lambda : |\lambda - \lambda_0| < \Phi(\lambda_0)\}$. We see that E is the union of the sets $E \cap \varrho(T)$ and $E \cap \mathrm{III}_1\sigma(T)$. Since E is connected, and the sets $\varrho(T)$, $\mathrm{III}_1\sigma(T)$ are open, one of the two constituent sets of E must be empty. In this case $E \cap \mathrm{III}_1\sigma(T)$ is not empty. Therefore $E \subset \mathrm{III}_1\sigma(T)$.

The argument used in the foregoing proof can be used to establish the following proposition:

THEOREM 3.4. *Any connected open subset of the set $\{\lambda : \Phi(\lambda) > 0\}$ lies entirely in $\varrho(T)$ or entirely in $\mathrm{III}_1\sigma(T)$.*

It may be demonstrated by examples that each of the sets

$$\mathrm{I}_2\sigma(T),\ \mathrm{II}_2\sigma(T),\ \mathrm{III}_2\sigma(T),\ \mathrm{I}_3\sigma(T),\ \mathrm{II}_3\sigma(T),\ \mathrm{III}_3\sigma(T)$$

can fail to be open. Thus, of the seven sets comprising $\sigma(T)$ in this system of classification, the only one of which it may be asserted that it is open, no matter how X and T are chosen, is $\mathrm{III}_1\sigma(T)$. This situation is altered somewhat if we place restrictions on T and X. If X is complete and if T is closed, then $\mathrm{I}_2\sigma(T)$ is empty, and therefore open, as a result of a well-known theorem (Theorem 4.2-I in Taylor [6]). Also, as we shall see in Theorem 4.2, $\mathrm{I}_3\sigma(T)$ is open if X is complete and T is a closed operator with domain dense in X.

For an example in which $\mathrm{I}_2\sigma(T)$ is not open, let X be the subspace of l consisting of those points $x = (\xi_1, \xi_2, \ldots)$ in l for which $\xi_n = 0$ if n is sufficiently large (i. e. $\xi_n = 0$ if $n > N$, where N depends on x). Let T have domain X, with

$$T(\xi_1, \xi_2, \ldots) = (\xi_1, \tfrac{1}{2}\xi_2, \tfrac{1}{3}\xi_3, \ldots).$$

Then $\mathrm{I}_2\sigma(T)$ consists of the single point $\lambda = 0$. The reader may easily prove the validity of this assertion.

Consider now a point λ_0 for which $\Phi(\lambda_0) > 0$. We define the Φ-radius of λ_0 to be $+\infty$ if $\Phi(\lambda) > 0$ for every λ. Otherwise we define the Φ-radius of λ_0 to be the distance from λ_0 to the set $\{\lambda : \Phi(\lambda) = 0\}$. We know by Theorems 3.1 and 3.3 that the Φ-radius of λ_0 is not less than $\Phi(\lambda_0)$. Also, we know by Theorem 3.4 that if r_0 is the Φ-radius of λ_0, the open circular disk $\{\lambda : |\lambda - \lambda_0| < r_0\}$ lies entirely in $\varrho(T)$ or entirely in $\mathrm{III}_1\sigma(T)$.

If λ_0 is in $\varrho(T)$, we shall speak of the Φ-radius of λ_0 as *the resolvent radius* of λ_0. If λ_0 is in $\mathrm{III}_1\sigma(T)$, we shall call the Φ-radius *the III_1-radius of λ_0*.

We shall investigate ways of estimating the Φ-radius of a point where the value of Φ is positive. In doing this we shall need to deal with powers of the operator T. If n is a positive integer greater than 1, we define T^n inductively in the obvious way, with $D(T^n)$ consisting of those x in $D(T)$ such that $Tx, \ldots, T^{n-1}x$ are also in $D(T)$. Likewise, if $P(\lambda)$ is any polynomial in λ of degree n, we define $P(T)$ in the obvious way, with $D(T^n)$ as the domain of $P(T)$.

We observe the following fact: If A and B are linear operators in X, and if AB is the operator defined by $(AB)x = A(Bx)$, with $D(AB)$ consisting of those x in $D(B)$ such that $Bx \in D(A)$, then

$$\mu(AB) \geqslant \mu(A)\mu(B). \tag{3.4}$$

For, if $\|x\| = 1$ and $x \in D(AB)$, then

$$\|ABx\| \geqslant \mu(A)\|Bx\| \geqslant \mu(A)\mu(B).$$

From this it is clear that $\mu(A^n) \geqslant [\mu(A)]^n$, whence

$$[\mu(A^n)]^{1/n} \geqslant \mu(A). \tag{3.5}$$

There are examples to show that the sequence $\mu(A)$, $[\mu(A^2)]^{1/2}$, $[\mu(A^3)]^{1/3}, \ldots$ can be strictly increasing. (Take $a_n = n/(n+1)$ in case II in section 6).

We now give a theorem on the determination of the resolvent radius of a point in $\varrho(T)$.

THEOREM 3.5. *Suppose $\lambda_0 \in \varrho(T)$, where X is complete and T is closed. Then the resolvent radius of λ_0 is r_0, where*

$$r_0 = \lim_{n\to\infty} \{\mu[\lambda_0 I - T)^n]\}^{1/n}. \tag{3.6}$$

Proof. The assumptions on X and T are used to guarantee that we can use the spectral mapping theorem (Theorem 5.71-A in Taylor [6]). Let $B = (\lambda_0 I - T)^{-1}$, and let $r_\sigma(B)$ be the spectral radius of B. If we define $f(\lambda)$ by

$$f(\lambda) = \frac{1}{\lambda_0 - \lambda},$$

it is easy to see that the operational calculus yields $f(T) = B$. By the spectral mapping theorem $\sigma(B) = f[\sigma_\sigma(T)]$. From this we see that $r_\sigma(B)$ is the supremum of $|f(\lambda)|$ as λ varies over $\sigma_e(T)$. This is the reciprocal of the infimum of $|\lambda - \lambda_0|$ as λ varies over $\sigma_e(T)$. If $\sigma_e(T)$ consists of ∞ alone, $r_\sigma(B) = 0$. Otherwise, we see that $\sigma(T)$ is not empty, and $r_\sigma(B)$ is the reciprocal of the distance from λ_0 to $\sigma(T)$. This distance must in fact be the distance from λ_0 to $\partial\sigma(T)$, and also the distance from λ_0 to the set on which $\Phi(\lambda) = 0$. Hence we see that the resolvent radius of λ_0 is $r_0 = [r_\sigma(B)]^{-1}$. To obtain formula (3.6) we observe that $B^n = [(\lambda_0 I - T)^n]^{-1}$, so that, by (2.1),

$$\|B^n\| = \{\mu[(\lambda_0 I - T)^n]\}^{-1}.$$

We then get (3.6) from the standard formula for a spectral radius, namely $r_\sigma(B) = \lim_{n\to\infty} \|B^n\|^{1/n}$.

Next we prove two lemmas which will be useful in subsequent arguments.

LEMMA 3.6. *Let A be a continuous linear operator in X, with $D(A) = X$. (We do not assume that X is complete). Suppose that $\mu(\lambda^n I - A^n) > 0$ for some n, where $n > 1$. Then also $\mu(\lambda I - A) > 0$.*

Proof. We can write

$$\lambda^n I - A^n = B_n(\lambda I - A), \tag{3.7}$$

where

$$B_n = \lambda^{n-1} I + \lambda^{n-2} A + \ldots + A^{n-1}. \tag{3.8}$$

Then, if $\|x\| = 1$, we have

$$0 < \mu(\lambda^n I - A^n) \leqslant \|(\lambda^n I - A^n)x\| \leqslant \|B_n\| \|(\lambda I - A)x\|,$$

from which it follows that $B_n \neq 0$ and that

$$\mu(\lambda I - A) \geqslant \frac{\mu(\lambda^n I - A^n)}{\|B_n\|} > 0.$$

LEMMA 3.7. *Suppose that A is a linear operator with domain and range in X, and suppose $\varrho(A)$ contains a point β such that $R(\beta I - A) = X$. (We do not assume A continuous, or that X is complete.) Suppose that, for some λ and some $n > 1$, we have $\mu(\lambda^n I - A^n) > 0$. Then $\mu(\lambda I - A) > 0$.*

Proof. It may be remarked that the hypothesis implies that A is closed. (See Theorems 4.2-B and 4.2-C in Taylor [6]). Also, if T is closed and X is complete, we shall have $R(\lambda I - T) = X$ for every λ in $\varrho(T)$. But we do not assume that X is complete. Let $S = (A - \beta I)^{-1}$. Observe that S is continuous and that $D(S) = X$. We first prove an auxiliary proposition. Suppose a and $\{x_k\}$ are such that $x_k \in D(A)$, $\|x_k\| = 1$, and $(aI - A)x_k \to 0$ as $k \to \infty$. Then, for each non-negative integer m,

$$\liminf_{k\to\infty} \|S^m x_k\| > 0. \tag{3.9}$$

This is true if $m = 0$. Clearly $a \in \sigma(A)$, so $a \neq \beta$. If $m > 0$, $S^m x_k \in D(A)$, and we can write

$$(\beta - a) S^m x_k = (\beta I - A) S^m x_k - (aI - A) S^m x_k = -S^{m-1} x_k - S^m (aI - A) x_k;$$

$$|\beta - a| \|S^m x_k\| \geqslant \|S^{m-1} x_k\| - \|S^m (aI - A) x_k\|.$$

Since S^m is continuous, it is clear that $S^m(aI - A)x_k \to 0$ as $k \to \infty$, and therefore

$$|\beta - a| \liminf_{k\to\infty} \|S^m x_k\| \geqslant \liminf_{k\to\infty} \|S^{m-1} x_k\|.$$

From this we see inductively that (3.9) is true.

We now proceed with the proof of Lemma 3.7. As in the proof of Lemma 3.6 we write the formula (3.7), with B_n defined by (3.8). Now, a polynomial in t can be written as a polynomial (of the same degree) in $t - \beta$. In this way we see that B_n can be expressed in the form

$$B_n = c_0 I + c_1(A - \beta I) + \ldots + c_{n-1}(A - \beta I)^{n-1}.$$

Now suppose that $\mu(\lambda I - A) = 0$. From this we shall deduce that $\mu(\lambda^n I - A^n) = 0$, thus proving the lemma. By the hypothesis there exists a sequence $\{x_k\}$ such that $x_k \in D(A)$, $\|x_k\| = 1$, and $(\lambda I - A)x_k \to 0$. Let $y_k = S^{n-1} x_k$. Then

$$(\lambda^n I - A^n) y_k = B_n(\lambda I - A) S^{n-1} x_k = B_n S^{n-1} (\lambda I - A) x_k.$$

But
$$B_n S^{n-1} = c_0 S^{n-1} + c_1 S^{n-2} + \ldots + c_{n-1} I,$$
so that $B_n S^{n-1}$ is a continuous operator. It follows that

(3.10) $$\lim_{k\to\infty} (\lambda^n I - A^n) y_k = 0.$$

But, by the auxiliary proposition, $\liminf_{k\to\infty} \|y_k\| > 0$. It then follows from (3.10) that
$$\lim_{k\to\infty} (\lambda^n I - A^n)\left(\frac{y_k}{\|y_k\|}\right) = 0,$$
and hence that $\mu(\lambda^n I - A^n) = 0$. This contradiction finishes the proof.

The next two theorems deal with estimates of the Φ-radius of a point λ_0 for which $\Phi(\lambda_0) > 0$. The results are less precise than the results in Theorem 3.5, but we do not assume that X is complete, and we use different methods.

THEOREM 3.8. *Suppose $D(T) = X$ and let T be continuous. Suppose $\Phi(\lambda_0) > 0$. Then the Φ-radius of λ_0 is not less than r_1, where*

(3.11) $$r_1 = \{\sup_n \mu[(\lambda_0 I - T)^n]\}^{1/n}.$$

Let E be the set $\{\lambda : |\lambda - \lambda_0| < r_1\}$. Then E lies either all in $\varrho(T)$ or all in $\mathrm{III}_1\sigma(T)$.

Proof. Observe that $r_1 > 0$, by (3.5). If $\lambda \in E$ we have $|\lambda - \lambda_0| < \mu[(\lambda_0 I - T)^n]^{1/n}$ for some n. Then $|\lambda - \lambda_0|^n < \mu[(\lambda_0 I - T)^n]$. Let $A = \lambda_0 I - T$, $a = \lambda - \lambda_0$. If $\|x\| = 1$ we have
$$\|(a^n I - A^n)x\| \geqslant \|A^n x\| - |a^n| \geqslant \mu(A^n) - |a^n| > 0,$$
and so $\mu(a^n I - A^n)x > 0$. By Lemma 3.6 we conclude that $\mu(aI - A) > 0$, which is the same as $\mu(\lambda I - T) > 0$, because $aI - A = T - \lambda I$. The conclusion now follows from Theorem 3.4.

THEOREM 3.9. *Suppose $\varrho(T)$ contains a point β such that $R(\beta I - T) = X$. (This time we do not assume T continuous or $D(T) = X$). Suppose $\Phi(\lambda_0) > 0$. Then the Φ-radius of λ_0 is not less than r_1, where r_1 is given by (3.11), and the conclusion about the set E is just the same as in Theorem 3.8.*

Proof. Define $A = \lambda_0 I - T$. Observe that $\beta I - T = -[(\lambda_0 - \beta)I - A]$, so that $\lambda_0 - \beta \in \varrho(A)$ and $R[(\lambda_0 - \beta)I - A] = X$. We can then proceed just as in the proof of Theorem 3.8, except that we appeal to Lemma 3.7 instead of to Lemma 3.6.

When we try to extend Theorem 3.5 by weakening the assumptions on T and X, we run into difficulties with the method. However, by using a different method we can prove the following theorem, in which we do not assume X complete or T closed. We denote the completion of X by $\hat{X}$.

THEOREM 3.10. *Suppose $\lambda_0 \in \varrho(T)$. Let B be the unique continuous linear extension of $(\lambda_0 I - T)^{-1}$ to all of $\hat{X}$. Let E be the set $\{\lambda : |\lambda - \lambda_0| r_\sigma(B) < 1\}$. Then $E \subset \varrho(T)$. Thus the resolvent radius of λ_0 is not less than the reciprocal of $r_\sigma(B)$.*

Proof. If $\lambda \in E$ and $\lambda \neq \lambda_0$, we see that $-(\lambda - \lambda_0)^{-1} \in \varrho(B)$. Let C be defined on $\hat{X}$ by
$$C = \frac{1}{\lambda - \lambda_0} B \left[B + \frac{I}{\lambda - \lambda_0}\right]^{-1}.$$

We shall show that $C(\lambda I - T)x = x$ when $x \in D(T)$. From this will follow the inequality $1 \leqslant \|C\| \mu(\lambda I - T)$, whence $E \subset \varrho(T)$ by Theorem 3.4. If $y = (\lambda I - T)x$, we can write $y = (\lambda - \lambda_0)(\lambda_0 I - T)^{-1}(\lambda_0 I - T)x + (\lambda_0 I - T)x$, or
$$y = (\lambda - \lambda_0)\left[(\lambda_0 I - T)^{-1} + \frac{I}{\lambda - \lambda_0}\right](\lambda_0 I - T)x.$$

Then, since B is an extension of $(\lambda_0 I - T)^{-1}$, we can write
$$Cy = \frac{1}{\lambda - \lambda_0} B \left[B + \frac{I}{\lambda - \lambda_0}\right]^{-1} (\lambda - \lambda_0)\left[B + \frac{I}{\lambda - \lambda_0}\right](\lambda_0 I - T)x.$$

On simplification, this becomes
$$Cy = B(\lambda_0 I - T)x = x,$$
which is what we wished to show.

If we attempt to use the foregoing method to estimate the Φ-radius of a point λ_0 in $\mathrm{III}_1\sigma(T)$, we meet the difficulty that the operator $(\lambda_0 I - T)^{-1}$ may not have a continuous extension to all of $\hat{X}$. The usual way to extend a continuous linear operator whose domain of definition is a linear manifold M not dense in X is to use a projection having as its range the closure of M in $\hat{X}$. But there may be no continuous projection of this sort. For this reason the next theorem contains a specific assumption which rules out the aforementioned difficulty.

THEOREM 3.11. *Suppose $\lambda_0 \in \mathrm{III}_1\sigma(T)$, and suppose that B is a continuous linear operator mapping $\hat{X}$ into $\hat{X}$, such that B is an extension of $(\lambda_0 I - T)^{-1}$. Then the III_1-radius of λ_0 is not less than the reciprocal of $r_\sigma(B)$.*

The proof is just like the proof of Theorem 3.10.

If X is a Hilbert space, continuous extensions of $(\lambda_0 I - T)^{-1}$ will always be available. In this case we should attempt to choose B in Theorem

3.11 so that $r_\sigma(B)$ is as small as possible, so as to get better estimates of the III_1-radius of λ_0.

We conclude this group of theorems with the following result:

THEOREM 3.12. *Let X be a normed linear space. Suppose A is a continuous linear operator in X with $D(A) = X$. Suppose B is a continuous linear extension of A^{-1}, with $D(B) = X$. (This involves the assumption that A has a continuous inverse, but says nothing about $R(A)$). Finally, suppose that*

$$\limsup_{n\to\infty} \|A^n B^n\|^{1/n} \leqslant 1. \tag{3.12}$$

Then

$$r_\sigma(B) = \lim_{n\to\infty} [\mu(A^n)]^{-1/n}. \tag{3.13}$$

Proof. The fact that B is an extension of A^{-1} is expressed by $BA = I$. It follows that $B^n A^n = I$. Then $\|x\| \leqslant \|B^n\| \|A^n x\|$ for each x, whence $1 \leqslant \|B^n\| \mu(A^n)$, and $[\mu(A^n)]^{-1/n} \leqslant \|B^n\|^{1/n}$. Thus

$$\limsup_{n\to\infty} [\mu(A^n)]^{-1/n} \leqslant \limsup_{n\to\infty} \|B^n\|^{1/n} = r_\sigma(B). \tag{3.14}$$

(The familiar relation between the sequence $\{\|B^n\|^{1/n}\}$ and $r_\sigma(B)$ is true, even when the space X is incomplete, as is shown in the dissertation of Gindler [1], Corollary 3.3). Now let $P_n = A^n B^n$. Then $\|P_n x\| \geqslant \mu(A^n) \|B^n x\|$ for each x, and so

$$\|B^n\|^{1/n} \leqslant [\mu(A^n)]^{-1/n} \|P_n\|^{1/n}.$$

From this and (3.12) follows the relation

$$\lim_{n\to\infty} \|B^n\|^{1/n} \leqslant \liminf_{n\to\infty} [\mu(A^n)]^{-1/n}.$$

Because of (3.14) we then have (3.13).

4. The conjugate of a densely defined operator. Let A be a linear operator with domain in X and range in Y, and let it be densely defined (i. e. let $D(A)$ be dense in X). Then the conjugate operator A' is uniquely defined. It is a closed linear operator with domain in Y' and range in X'. The domain of A' consists of all y' in Y' such that $y'(Ax)$ is continuous as a function of x on $D(A)$; then $A'y' = x'$ is defined by $x'(x) = y'(Ax)$. We are going to need the following theorem:

THEOREM 4.1. (a) *If Y is complete and if the state of A is I_3, the state of A' is III_1.* (b) *If X is complete, if A is closed, and if the state of A' is III_1, then the state of A is I_3.*

The truth of (a) may be read from the first state diagram in a paper by Goldberg (page 72 in Goldberg [2]). The truth of (b) may be read from the second state diagram, on page 78 in the same paper by Goldberg. For the convenience of readers we outline the main steps in reaching conclusions (a) and (b). The details, or references to them, can be found in Goldberg's paper. If Y is complete, $R(A) = Y$ implies that A' has a continuous inverse. This, with the fact that A' is closed, implies that $R(A')$ is closed. Then, from the fact that A has no inverse, it can be proved that $R(A') \neq X$, and hence $R(A')$ is not dense in X'. This disposes of (a). For (b) the most difficult part of the proof is in showing that, when X is complete, if A is closed, and if A' has a continuous inverse, then $R(A) = Y$. It also follows that, if A has an inverse, A^{-1} is continuous. Then, with the assumption that $R(A')$ is not dense in X', one can prove that A has no inverse.

From Theorem 4.1 we can conclude the following

THEOREM 4.2. *Suppose X is a complete space and that T is a closed and densely defined linear operator in X. Then λ is in $\mathrm{I}_3\sigma(T)$ if and only if λ is in $\mathrm{III}_1\sigma(T')$. Therefore $\mathrm{I}_3\sigma(T)$ is an open set.*

Proof. We apply Theorem 4.1, with $X = Y$ and $A = \lambda I - T$. We know from section 3 that $\mathrm{III}_1\sigma(T')$ is an open set.

The foregoing theorem is not true if we omit the assumption that X is complete. Consider the following example. Let X be the subspace of l^2 consisting of elements $x = (\xi_1, \xi_2, \ldots)$ such that $\xi_n = 0$ for all except a finite set of values of n. Let $D(T) = X$ and define $T(\xi_1, \xi_2, \ldots) = (2\xi_2, 3\xi_3, 4\xi_4, \ldots)$. Then T is closed and $R(T) = X$, but it may be verified that the only eigenvalue of T is $\lambda = 0$. Hence $\mathrm{I}_3\sigma(T)$ consists of the single point $\lambda = 0$. In this case we can identify X' with l^2, and T' is defined by $T'(\xi_1, \xi_2, \ldots) = (0, 2\xi_1, 3\xi_2, \ldots)$, with $D(T')$ consisting of elements for which $\sum_{n=1}^{\infty} (n+1)^2 |\xi_n|^2 < \infty$. It is readily seen that $\mu(T') = 2$. Also, if $\lambda \neq 0$, we can solve the infinite system of linear equations represented by $(\lambda I - T')x = y$, and in this way we can see that $(\lambda I - T')^{-1}$ is never a bounded operator defined on all of l^2. Therefore $\varrho(T')$ is empty in this case. In fact $\lambda I - T'$ is in the state III_1 for every λ, so that $\mathrm{III}_1\sigma(T')$ is the same as $\sigma(T')$, which is the entire complex plane. The assertion that $\lambda I - T'$ is in the state III_1 for every λ can be proved directly by showing that $\mu(\lambda I - T') > 0$ for every λ; once this is done, we can conclude that the state of $\lambda I - T'$ is III_1, for we have seen that $\varrho(T')$ is empty.

A different argument could be given, as follows: Taking $X = l^2$, let T_1 be the operator in X with $D(T_1)$ consisting of all $x = (\xi_1, \xi_2, \ldots)$ such that $\Sigma n^2 |\xi_n|^2 < \infty$, and $T_1(\xi_1, \xi_2, \ldots) = (2\xi_2, 3\xi_3, 4\xi_4, \ldots)$. Here again we can identify X' with l^2, and it turns out that T_1' is the same as the T' we have already considered. Now it turns out that $\lambda I - T_1$ is in state

I_3 for every λ. Since X is complete and T_1 is closed, as well as densely defined, we can use Theorem 4.2 to conclude that the state of $\lambda I-T'$ is III_1 for every λ.

The direct demonstration that $\mu(\lambda I-T')>0$ in this example is not entirely simple when $\lambda\neq 0$. For the following argument we are indebted to Professor E. G. Straus, of the University of California, Los Angeles. We present it as an interesting illustration of technique in estimating a minimum modulus. Suppose E_n is the projection operator defined by

$$E_n(\xi_1,\xi_2,\ldots)=(\xi_1,\xi_2,\ldots,\xi_n,0,0,\ldots).$$

Then $\|x\|^2=\|E_nx\|^2+\|(I-E_n)x\|^2$. Let n be chosen as the first integer such that $n+2>|\lambda|\sqrt{2}$. Now $\|T'x\|^2=\|T'E_nx\|^2+\|T'(I-E_n)x\|^2$, and it is easy to see that $\|T'(I-E_n)x\|\geqslant(n+2)\|(I-E_n)x\|$. If $x\in D(T')$ and $\|x\|=1$, we must have either $\|(I-E_n)x\|^2\geqslant 1/2$ or $\|E_nx\|^2\geqslant 1/2$. In the first of these two cases we have $\|T'x\|\geqslant(n+2)/\sqrt{2}$, and therefore

$$\|\lambda x-T'x\|\geqslant\|T'x\|-|\lambda|\geqslant\frac{n+2}{\sqrt{2}}-|\lambda|>0.$$

In the second case, where $\|E_nx\|^2\geqslant 1/2$, let the maximum of $|\xi_1|,|\xi_2|,\ldots$ be $|\xi_k|$. Then we certainly have $|\xi_k|^2\geqslant 1/2n$, because

$$|\xi_1|^2+\ldots+|\xi_n|^2\geqslant\tfrac{1}{2}.$$

From the expression for $\|\lambda x-T'x\|^2$ we now select those terms which involve ξ_k. If $k=1$ they are

$$|\lambda|^2|\xi_1|^2+|\lambda\xi_2-2\xi_1|^2\geqslant|\lambda|^2|\xi_1|^2\geqslant\frac{|\lambda|^2}{2n}.$$

If $k>1$, these terms are

$$|\lambda\xi_k-k\xi_{k-1}|^2+|\lambda\xi_{k+1}-(k+1)\xi_k|^2.$$

This sum can be written in the form

$$|\xi_k|^2|\lambda-k\alpha_k|^2+|\xi_k|^2|\lambda\beta_k-(k+1)|^2,$$

where

$$\alpha_k=\frac{\xi_{k-1}}{\xi_k}\quad\text{and}\quad\beta_k=\frac{\xi_{k+1}}{\xi_k}.$$

Since $|\alpha_k|\leqslant 1$ and $|\beta_k|\leqslant 1$, we see that

$$|\lambda-k\alpha_k|^2+|\lambda\beta_k-(k+1)|^2\geqslant\tfrac{1}{4}.$$

For, if $|\lambda-k\alpha_k|<1/2$, then $|\lambda|<k+1/2$, and

$$|\lambda\beta_k-(k+1)|\geqslant k+1-|\lambda\beta_k|\geqslant\tfrac{1}{2}.$$

Thus, when $\|E_nx\|^2\geqslant 1/2$, we see that $\|\lambda x-T'x\|^2$ is not less than the smaller of the numbers $|\lambda|^2/2n$, $1/8n$.

Combining all the results, we see that $\mu(\lambda I-T')$ is not less than the smallest of the numbers

$$\frac{n+2}{\sqrt{2}}-|\lambda|,\qquad\frac{|\lambda|}{\sqrt{2n}},\qquad\frac{1}{2\sqrt{2n}},$$

where n is chosen so that the first number is positive.

Presently we shall need the following result:

LEMMA 4.3. *If B is a linear operator with $D(B)$ and $R(B)$ in X, and if $D(B)$ is dense in X, then $R(B')=X'$ if and only if B has a continuous inverse.*

For this proposition see Hille-Phillips [4], Theorem 2.11.15. The space X need not be complete. See also Theorem 4.7-A in Taylor [6]; it is not necessary to have B continuous or $D(B)=X$.

By the use of conjugate operators we can give the following generalization of Theorem 3.8:

THEOREM 4.4. *Suppose that $D(T^n)$ is dense in X for $n=1,2,\ldots$. Suppose that $\Phi(\lambda_0)>0$. If $A=\lambda_0I-T$, suppose that $(A^n)'=(A')^n$ for $n=1,2,\ldots$ Then the Φ-radius of λ_0 is not less than $\sup_n[\mu(A^n)]^{1/n}$.*

Proof. We remark at the outset that $D(A^n)=D(T^n)$; therefore, since $D(T^n)$ is dense in X, the conjugate of A^n is well-defined. Under these conditions $(A^n)'$ is an extension of $(A')^n$, and it can happen that the domain of $(A')^n$ is a proper subset of the domain of $(A^n)'$. (See the example given later in the paper). But we have explicitly assumed that this does not occur.

Now suppose that $|\lambda-\lambda_0|<[\mu(A^n)]^{1/n}$ for some value of n. We wish to show that $\Phi(\lambda)=\mu(\lambda I-T)>0$. We write $\alpha=\lambda-\lambda_0$. Then, as in the proof of Theorem 3.8, we conclude that $\mu(\alpha^nI-A^n)>0$. From Lemma 4.3 we then conclude that the range of $(\alpha^nI-A^n)'$ is all of X'. Next, we note that

$$(4.1)\qquad(\alpha^nI-A^n)'=\alpha^nI-(A^n)'=\alpha^nI-(A')^n=(\alpha I-A')S,$$

where

$$S=\alpha^{n-1}I+\alpha^{n-2}A'+\ldots+(A')^{n-1}.$$

Note that we use I for the identity operator in X' as well as for the identity operator in X. From (4.1) it is clear that the range of $(\alpha^nI-A^n)'$, which is X', is contained in the range of $\alpha I-A'$. Therefore this latter range is X' also. Applying Lemma 4.3 again, we conclude that $\alpha I-A$ has a continuous inverse, i. e. that $\mu(\alpha I-A)=\mu(\lambda I-T)>0$.

Here is an example of an operator T such that $(T^2)'$ is a proper extension of $(T')^2$. Let X be l^2, and let $D(T)$ consist of all elements $x=\{\xi_n\}$

such that $\xi_n = 0$ when n exceeds some N which depends on x. Let $T(\xi_1, \xi_2, \xi, \ldots) = (\xi_1, 0, 2\xi_2, 0, 3\xi_3, 0, \ldots)$. We can identify X' with l^2. If we calculate $(T')^2$ and $(T^2)'$, it turns out that $D\{(T^2)'\}$ is identifiable with the set of all $\{a_n\}$ in l^2 such that

$$\sum_{n=1}^{\infty} n^2(2n-1)^2|a_{4n-3}|^2 < \infty,$$

whereas $D\{(T')^2\}$ is identifiable with the subset of $D\{(T^2)'\}$ for which

$$\sum_{n=1}^{\infty} n^2|a_{2n-1}|^2 < \infty.$$

The latter condition places the further restriction

$$\sum_{n=1}^{\infty} (2n)^2|a_{4n-1}|^2 < \infty$$

on the elements of $D\{(T^2)'\}$, and so $(T^2)'$ is a proper extension of $(T')^2$.

5. Operators in Hilbert space. In this section we suppose that X is a complex Hilbert space. If A is a bounded linear operator on X, the adjoint A^* is another operator of the same kind, while the conjugate A' is a bounded linear operator on the conjugate space X'. There is a norm-preserving linear isomorphism between X and X', however, and by means of this it may be shown that the operators A^* and A' have the same state. We also know that $\sigma(A^*)$ is the set of complex conjugates of the points of $\sigma(A)$; therefore, λ is a boundary point of $\sigma(A)$ if and only if $\bar{\lambda}$ is a boundary point of $\sigma(A^*)$.

Now suppose that H is a bounded selfadjoint operator on X, and let us define $m(H)$ and $M(H)$ as follows:

$$m(H) = \inf_{\|x\|=1} (Hx, x), \qquad M(H) = \sup_{\|x\|=1} (Hx, x). \tag{5.1}$$

We observe the properties

$$m(-H) = -M(H), \qquad m(cH) = cm(H) \quad \text{if} \quad c > 0, \tag{5.2}$$

$$m(H_1)+m(H_2) \leqslant m(H_1+H_2), \tag{5.3}$$

$$m(cI+H) = c+m(H) \text{ if } c \text{ is real.} \tag{5.4}$$

The minimum modulus is related to the function m; the following relations are evident:

$$\mu(A) = \sqrt{m(A^*A)}, \qquad \mu(A^*) = \sqrt{m(AA^*)}. \tag{5.5}$$

Now, by Theorem 3.2, if $\lambda \epsilon \partial\sigma(A)$, we see from (5.5) that $m(B) = 0$, where $B = (\lambda I - A)^*(\lambda I - A)$. Furthermore, we shall have $\bar{\lambda} \epsilon \partial\sigma(A^*)$, so that $\mu[(\lambda I - A)^*] = 0$, whence $m(C) = 0$, where $C = (\lambda I - A)(\lambda I - A)^*$. Let us compute $m(B)$ and $m(C)$ in more convenient forms. We write $\lambda = re^{i\Theta}$, where $r \geqslant 0$ and Θ is real. Then we define

$$J_\Theta = \tfrac{1}{2}(e^{-i\Theta}A + e^{i\Theta}A^*).$$

It then turns out that

$$B = r^2I - 2rJ_\Theta + A^*A, \qquad C = r^2I - 2rJ_\Theta + AA^*.$$

Therefore, by (5.4),

$$m(B) = r^2 + m[-2rJ_\Theta + A^*A], \tag{5.7}$$

$$m(C) = r^2 + m[-2rJ_\Theta + AA^*]. \tag{5.8}$$

We can now state the following theorem:

THEOREM 5.1. *If $\lambda \epsilon \partial\sigma(A)$, where $\lambda = re^{i\Theta}$, then*

$$r^2 = M[2rJ_\Theta - A^*A] \tag{5.9}$$

and

$$r^2 = M[2rJ_\Theta - AA^*]. \tag{5.10}$$

The proof follows from (5.7), (5.8), and (5.2), because of the fact that $m(B) = m(C) = 0$.

If the locus of points $re^{i\Theta}$ satisfying (5.9) could be determined, this knowledge would be useful in the following way. Let the locus be S, and let D be a maximal connected subset of the complement of S. Then D must be either all in $\sigma(A)$ or all in $\varrho(A)$. For otherwise, since D is connected and $\varrho(A)$ is open, there would be an accumulation point of $D \cap \varrho(A)$ in $D \cap \sigma(A)$, and this point would be a point of $\partial\sigma(A)$ not in S.

From Theorem 5.1 we may deduce the following result:

THEOREM 5.2. *If either $A^*A = I$ or $AA^* = I$, then $\partial\sigma(A)$ lies on the circumference $\{\lambda: |\lambda| = 1\}$. Under these conditions, if A is not unitary, $\sigma(A)$ is the full disk $\{\lambda: |\lambda| \leqslant 1\}$.*

Proof. If either $A^*A = I$ or $AA^* = I$, we see from (5.2) and (5.4) that $r^2 = 2rM(J_\Theta) - 1$, or

$$1 + r^2 = 2rM(J_\Theta). \tag{5.11}$$

But $A^*A = I$ is equivalent to $\|Ax\| = \|x\|$ for all x, because $(A^*Ax, x) = (Ax, Ax)$. Hence $A^*A = I$ implies $\|A\| = 1$, whence $\|A^*\| = 1$ also, because $\|A\| = \|A^*\|$. Likewise, $AA^* = I$ implies $\|A^*\| = \|A\| = 1$. Thus, since $M(H) \leqslant \|H\|$ for any selfadjoint H, we see that either $A^*A = I$

or $AA^* = I$ implies $M(J_\Theta) \leqslant \|J_\Theta\| \leqslant \frac{1}{2}(\|A\| + \|A^*\|) = 1$. From (5.11) it follows that $1+r^2 \leqslant 2r$, or $(1-r)^2 \leqslant 0$, whence $r = 1$. This proves the first assertion. If A is not unitary 0 must be in $\sigma(A)$ and $\sigma(A^*)$. Since $\|A\| = 1$, we know that points outside the unit circle are in $\varrho(A)$. Therefore every ray through 0 must intersect the unit circumference in a point of $\partial\sigma(A)$, and the interior of the unit circle must all be in $\sigma(A)$.

Actually, we can prove a theorem somewhat like Theorem 5.2, but in a Banach space setting. Instead of using Theorem 5.1, we use Theorems 3.1 and 3.3.

THEOREM 5.3. *Let X be a Banach space. Let A be an isometric operator*: $\|Ax\| = \|x\|$ *for each* x. *Then, if* $R(A) \neq X$, $\sigma(A) = \{\lambda : |\lambda| \leqslant 1\}$. *If* $R(A) = X$, $\sigma(A) \subset \{\lambda : |\lambda| = 1\}$.

Proof. From the hypothesis we see that $\|A\| = 1$. Then $\lambda \in \varrho(A)$ if $|\lambda| > 1$. Also, $\mu(A) = 1$, and hence A has a continuous inverse. It is easy to see that $R(A)$ is closed. Therefore the point $\lambda = 0$ is in $\mathrm{III}_1\sigma(A)$ if $R(A) \neq X$, and in $\varrho(A)$ if $R(A) = X$. In the first case, $\lambda \in \mathrm{III}_1\sigma(A)$ if $|\lambda| < 1$, by Theorem 3.3. In the second case, $\lambda \in \varrho(A)$, by Theorem 3.1. The conclusions of the theorem now follow.

In a 1957 paper [5] C. R. Putnam asserted the inequality

$$\{r - M(J_\Theta)\}^2 \leqslant \{M(J_\Theta)\}^2 - \{\mu(A^*)\}^2, \tag{5.12}$$

under the assumptions that $\lambda = re^{i\Theta}$ is a point of $\partial\sigma(A)$ and that A is what he called a *positive seminormal operator*. This means that $m[AA^* - A^*A] \geqslant 0$. He then deduced that $\sigma(A) = \{\lambda : |\lambda| \leqslant 1\}$ if A is isometric but not unitary. Our Theorems 5.1 and 5.2. have been proved as a result of our examination of Putnam's paper. But we emphasize that no use of the concept of seminormality is needed in our proofs. Moreover, in Theorem 5.1 we have equalities in (5.9) and (5.10), whereas Putnam asserted only the less precise result of the inequality in (5.12). Actually, (5.12) can be deduced from (5.10), and a corresponding inequality, with $\mu(A)$ in place of $\mu(A^*)$, is deduced in the same way from (5.9). In fact, from (5.9) and (5.2) we have

$$r^2 + m[A^*A - 2rJ_\Theta] = 0,$$

and so, by (5.3),

$$0 \geqslant r^2 + m(A^*A) + m(-2rJ_\Theta).$$

By (5.5) this can be written

$$0 \geqslant r^2 - 2rM(J_\Theta) + \{\mu(A)\}^2.$$

On completing the square we then have

$$\{r - M(J_\Theta)\}^2 \leqslant \{M(J_\Theta)\}^2 - \{\mu(A)\}^2. \tag{5.13}$$

From (5.13) we deduce that

$$M(J_\Theta) - \sqrt{\{M(J_\Theta)\}^2 - \{\mu(A)\}^2} \leqslant r. \tag{5.14}$$

If $0 \in \mathrm{III}_1\sigma(A)$, then $\mu(A) > 0$, and we see from (5.14) that $r > 0$. This shows us in another way that 0 is then an interior point of $\mathrm{III}_1\sigma(A)$. One might suppose that (5.14) would be useful for obtaining an estimate of the III_1-radius of 0. Actually, however, we can get nothing better this way than what we already know from Theorem 3.3, namely that the III_1-radius of 0 in $\sigma(A)$ is not less than $\mu(A)$. For, $0 < \mu(A) \leqslant M(J_\Theta)$, and from this it is easy to see that

$$M(J_\Theta) - \sqrt{\{M(J_\Theta)\}^2 - \{\mu(A)\}^2} \leqslant \mu(A).$$

It is of some interest to see a non-trivial example in which we can calculate $M[2rJ_\Theta - A^*A]$. Let A be the operator in l^2 defined as follows:

$$Au_k = u_k + au_{k+1},$$

where $a > 1$ and u_k is the k^{th} basis vector,

$$u_k = (0, \ldots, 0, 1, 0, \ldots) \quad (1 \text{ in } k^{th} \text{ position}).$$

We readily find that A^*A and J_Θ are represented by the following infinite matrices:

$$A^*A: \quad \begin{array}{|cccccc} 1+a^2 & a & 0 & & \ldots & \\ a & 1+a^2 & a & 0 & & \\ 0 & a & 1+a^2 & a & 0 & \ldots \\ \ldots & & & & & \\ \ldots & & & & & \end{array}$$

$$J_\Theta: \quad \begin{array}{|ccccc} \cos\Theta & \frac{a}{2}e^{i\Theta} & 0 & \ldots & \\ \frac{a}{2}e^{-i\Theta} & \cos\Theta & \frac{a}{2}e^{i\Theta} & 0 & \\ 0 & \frac{a}{2}e^{-i\Theta} & \cos\Theta & \frac{a}{2}e^{i\Theta} & \ldots \\ \ldots & & & & \\ \ldots & & & & \end{array}$$

For our purposes we need the following facts. If B is the operator defined by $Bu_1 = u_2$, $Bu_k = u_{k-1} + u_{k+1}$ if $k \geqslant 2$, then $\sigma(B) = [-2, 2]$.

If $|c| = 1$, and if C is the operator defined by $Cu_k = c^{k-1}u_k$, then $C^{-1}BC$ is defined by

$$C^{-1}BCu_1 = \bar{c}u_2, \qquad C^{-1}BCu_k = cu_{k-1}+\bar{c}u_{k+1}$$

if $k \geqslant 2$, and $\sigma(C^{-1}BC) = \sigma(B)$. The facts about B are classical, and well known. See [3], for instance.

It is now easy to see that

$$(5.15) \qquad 2rJ_\Theta - A^*A = [2r\cos\Theta-(1+a^2)]I + a|re^{i\Theta}-1|C^{-1}BC,$$

where

$$c = \frac{re^{i\Theta}-1}{|re^{i\Theta}-1|}.$$

This is the situation if $re^{i\Theta} \neq 1$. We are not concerned with the contrary case, for we can easily see that the point $\lambda = 1$ belongs to $\mathrm{III}_1\sigma(A)$, and hence it cannot be in $\partial\sigma(A)$. From (5.15) and the fact that $\sigma(B) = [-2, 2]$ we now see that $\sigma(2rJ_\Theta - A^*A)$ consists of all real values of λ such that

$$\frac{|\lambda-[2r\cos\Theta-(1+a^2)]|}{a|re^{i\Theta}-1|} \leqslant 2.$$

In particular $M[2rJ_\Theta - A^*A]$ is the largest such value of λ, so that

$$M[2rJ_\Theta - A^*A] = 2r\cos\Theta-(1+a^2)+2a\sqrt{1+r^2-2r\cos\Theta}.$$

Equation (5.9) now takes the form

$$r^2 = 2r\cos\Theta-(1+a^2)+2a\sqrt{1+r^2-2r\cos\Theta},$$

which can be rewritten as

$$(\sqrt{1+r^2-2r\cos\Theta}-a)^2 = 0, \qquad \text{or} \qquad 1+r^2-2r\cos\Theta = a^2.$$

This is the equation in polar form, of the circle consisting of all λ such that $|\lambda-1| = a$. The set $\partial\sigma(A)$ lies on this circumference, by Theorem 5.1. Since the center of the circle is in $\sigma(A)$, the entire circumference is $\partial\sigma(A)$.

Actually, it is easier to determine $\sigma(A)$ in another way. We can write $A = I+aT$, where T is defined by $Tu_k = u_{k+1}$. Since $\lambda I - A = a\{(\lambda-1)/a - T\}$, and since it is known (and easily proved) that $\sigma(T)$ is the set $\{\lambda: |\lambda| \leqslant 1\}$, we see that $\sigma(A)$ is the set $\{\lambda: |\lambda-1| \leqslant a\}$. (For the facts about T see p. 266-267 in Taylor [6]). Since $A^*A = (1+a^2)I+aB$, it is readily apparent that $\sigma(A^*A) = [(a-1)^2, (a+1)^2]$. It then follows from (5.5) that $\mu(A) = a-1$. Since $I-A = -aT$, we readily compute

$\mu(I-A) = a\mu(T) = a$, whence the III_1-radius of 1 in $\sigma(A)$ is at least a. It cannot exceed a, of course; in this case all interior points of $\sigma(A)$ are in $\mathrm{III}_1\sigma(A)$.

We shall conclude this section with some theorems about seminormal operators.

THEOREM 5.4. *Let A be a positive seminormal operator. (That is, if $H = AA^*-A^*A$, then $m(H) \geqslant 0$). Then the sets $\mathrm{III}_1\sigma(A)$ and $\mathrm{III}_2\sigma(A)$ are empty (i. e., in another terminology, the residual spectrum of A is empty).*

Proof. Observe that

$$(\lambda I-A)(\lambda I-A)^*-(\lambda I-A)^*(\lambda I-A) = H.$$

Hence, $\lambda I - A$ is a positive seminormal operator if A is. It therefore suffices to prove that, if $\overline{R(A)} \neq X$, then A has no inverse. Now, if $\overline{R(A)} \neq X$, there is some $x \neq 0$ such that $A^*x = 0$. Then, by hypothesis,

$$0 \leqslant (Hx, x) = (AA^*x-A^*Ax, x) = -(A^*Ax, x) = -\|Ax\|^2 \leqslant 0.$$

Thus $Ax = 0$, and the proof is complete.

THEOREM 5.5. *If A is a negative seminormal operator, the sets $I_3\sigma(A)$ and $\mathrm{II}_3\sigma(A)$ are empty.*

Proof. The hypothesis is that $((AA^*-A^*A)x, x) \leqslant 0$ for all x. This implies that A^* is a positive seminormal operator. Now (by the state diagram)

$$\mathrm{I}_3\sigma(A) = \{\lambda: \bar{\lambda} \in \mathrm{III}_1\sigma(A^*)\} \quad \text{and} \quad \mathrm{II}_3\sigma(A) = \{\lambda: \bar{\lambda} \in \mathrm{III}_2\sigma(A^*)\}.$$

The conclusion now follows from Theorem 5.4.

6. Examples. The examples we consider are all obtained as particular cases of the following general example. Let X be the space l. Let $\{a_n\}$ be a sequence of complex numbers such that

$$(6.1) \qquad c = \inf|a_n| > 0, \qquad C = \sup|a_n| < +\infty.$$

Let A be the bounded linear operator defined by

$$(6.2) \qquad A(\xi_1, \xi_2, \ldots) = (0, a_1\xi_1, a_2\xi_2, \ldots).$$

If we think of A as defined by an infinite matrix, the matrix is

$$\begin{array}{|llll} 0 & 0 & 0 & \ldots \\ a_1 & 0 & 0 & \ldots \\ 0 & a_2 & 0 & \ldots \\ 0 & 0 & a_3 & \ldots \\ \ldots & & & \\ \ldots & & & \end{array}$$

The elements in the first subdiagonal are $a_1, a_2, \ldots$ and all other elements are 0. For A^2 the corresponding matrix has elements $a_1 a_2$, $a_2 a_3$, $a_3 a_4, \ldots$ in the second subdiagonal, and all other elements are 0. Likewise, the elements in the n^{th} subdiagonal of the matrix for A^n are

$$a_1 a_2 \ldots a_n, \quad a_2 a_3 \ldots a_{n+1}, \quad a_3 a_4 \ldots a_{n+2}, \quad \ldots$$

and all other elements are 0. From these calculations we can see that

$$\|A^n\| = \sup_{p\geqslant 1} |a_p a_{p+1} \ldots a_{p+n-1}| \tag{6.3}$$

and

$$\mu(A^n) = \inf_{p\geqslant 1} |a_p a_{p+1} \ldots a_{p+n-1}|.$$

To determine $\sigma(A)$ we observe in the first place that $0 \in \mathrm{III}_1\sigma(A)$, because $\mu(A) = c > 0$ and $\overline{R(A)} \neq X$. We shall see that $\sigma(A)$ is a disk of radius $r_\sigma(A)$ with center at 0, where the spectral radius $r_\sigma(A)$ is to be calculated with the aid of (6.3). If $\lambda \neq 0$ and $y = (\lambda I - A)x$, we can solve for x explicitly. The inverse $(\lambda I - A)^{-1}$ exists; the matrix which determines it is found to be

$$\left|\begin{array}{ccccc} 1/\lambda & 0 & 0 & 0 & \ldots \\ a_1/\lambda^2 & 1/\lambda & 0 & 0 & \ldots \\ a_1 a_2/\lambda^3 & a_2/\lambda^2 & 1/\lambda & 0 & \ldots \\ a_1 a_2 a_3/\lambda^4 & a_2 a_3/\lambda^3 & a_3/\lambda^2 & 1/\lambda & 0 \\ \ldots & & & & \end{array}\right.$$

Thus, λ will be in $\varrho(A)$ if and only if this matrix determines a bounded operator defined on all of X. If we write

$$r_k(\lambda) = \frac{1}{|\lambda|} + \sum_{n=k}^{\infty} \frac{|a_k a_{k+1} \ldots a_n|}{|\lambda|^{n+2-k}}, \tag{6.5}$$

the condition that λ be in $\varrho(A)$ is that the series for $r_1(\lambda), r_2(\lambda), \ldots$ all be convergent and that

$$\|(\lambda I - A)^{-1}\| = \sup_k |r_k(\lambda)| \tag{6.6}$$

be finite. Observe that if the series for $r_1(\lambda)$ is convergent, the series for $r_k(\lambda)$ converges for every value of k. Since $r_k(\lambda)$ depends only on $|\lambda|$, and decreases as $|\lambda|$ increases, it is clear that $\varrho(A)$ is the exterior of some disk centered at 0.

We shall attempt to classify the points of $\sigma(A)$. To do this it is helpful to study the conjugate operator A', which acts in l^∞. It is represented by the transpose of the matrix which represents A. It is not difficult to see that λ is an eigenvalue of A' if and only if

$$\left(1, \frac{\lambda}{a_1}, \frac{\lambda^2}{a_1 a_2}, \frac{\lambda^3}{a_1 a_2 a_3}, \ldots\right)$$

is a vector in l^∞, i. e. if and only if

$$\sup_n \left|\frac{\lambda^n}{a_1 \ldots a_n}\right| < \infty. \tag{6.7}$$

For this $|\lambda| \leqslant \liminf_{n\to\infty} |a_1 \ldots a_n|^{1/n}$ is necessary and $|\lambda| < \liminf_{n\to\infty} |a_1 \ldots a_n|^{1/n}$ is sufficient.

From now on we impose various special conditions on the a_n's.

Case I. Let us suppose that $|a_{n+1}| \leqslant a_n|$ for each n. Then $|a_n| \to c$ as $n \to \infty$. In this case we see from (6.3) that

$$\|A^n\|^{1/n} = |a_1 \ldots a_n|^{1/n} \to c.$$

Also, $\mu(A^n) = c^n$. Thus we see that $r_\sigma(A) = c$, and from this and Theorem 3.8 we see that the III_1-radius of 0 in $\sigma(A)$ is c. To classify the points λ for which $|\lambda| = c$, we observe that

$$\frac{c^n}{|a_1 \ldots a_n|} \leqslant 1;$$

by (6.7), therefore λ is an eigenvalue of A' if $|\lambda| = c$. From the state diagram (Taylor [6], p. 237) and the fact that $(\lambda I - A)^{-1}$ exists but is not continuous (see Theorem 3.2), we conclude that $\lambda \in \mathrm{III}_2\sigma(A)$ when $|\lambda| = c$.

Case II. Suppose now that $|a_n| \leqslant |a_{n+1}|$ for each n. Then $|a_n| \to C$. In this case $\|A^n\| = C^n$ and $\mu(A^n) = |a_1 \ldots a_n|$. We see that $r_\sigma(A) = C$. Now $[\mu(A^n)]^{1/n} \to C$, so that $\sup_n [\mu(A^n)]^{1/n} \geqslant C$. On the other hand, $|\lambda| < \sup[\mu(A^n)]^{1/n}$ implies $\lambda \in \mathrm{III}_1\sigma(A)$. Consequently, since $|\lambda| > C$ implies $\lambda \in \varrho(A)$, we conclude that $\sup[\mu(A^n)]^{1/n} = C$. To classify the points for which $|\lambda| = C$, we observe that they are either in $\mathrm{II}_2\sigma(A)$ or in $\mathrm{III}_2\sigma(A)$, because $(\lambda I - A)^{-1}$ exists but is not continuous. We can distinguish between these two cases by looking to see if λ an eigenvalue of A'.

We shall arrange a special case within Case II, with A depending on a real parameter t. Suppose that $1/2 \leqslant t \leqslant 1$, and define

$$a_{k+1} = \frac{k+2t}{k+2}, \quad k = 0, 1, 2, \ldots$$

Then $t = a_1 \leqslant a_2 \leqslant \ldots, a_n \to 1 = C$. If $2t = 1+h$, we can write

$$a_1 \ldots a_n = (1+h)\left(1+\frac{h}{2}\right)\ldots\left(1+\frac{h}{n}\right)\frac{1}{n+1},$$

and so

$$\log\frac{1}{a_1\ldots a_n} = \log(n+1) - \sum_{k=1}^{n}\log\left(1+\frac{h}{k}\right),$$

$$\log\frac{1}{a_1\ldots a_n} \geqslant \log(n+1) - \sum_{k=1}^{n}\frac{h}{k} \geqslant (1-h)\log(n+1).$$

If $1/2 \leqslant t < 1$, we have $0 \leqslant h < 1$, and we see from the foregoing that

$$\frac{1}{a_1\ldots a_n} \to +\infty.$$

Referring back to (6.7), we see that this means that points λ for which $|\lambda| = C = 1$ are in $\mathrm{II}_2\sigma(A)$, because they are not eigenvalues of A'. On the other hand, if $t = 1$ we have $a_1\ldots a_n = 1$, and in this case the points for which $|\lambda| = 1$ are in $\mathrm{III}_2\sigma(A)$, because they are eigenvalues of A'. If we denote A by $A(t)$ to show its dependence on t, it is interesting to observe that

$$\|A(1)-A(t)\| = \sup_n\left|1-\frac{n-1+2t}{n+1}\right| = 1-t.$$

Thus we see that an arbitrarily small perturbation of the operator $A(1)$ can shift the classification of the points for which $|\lambda| = 1$ from $\mathrm{III}_2\sigma(A)$ to $\mathrm{II}_2\sigma(A)$.

Case III. This case is more complicated. Let a, b be fixed real numbers such that $0 < a < b$, and let the sequence $\{a_n\}$ be constructed so that each a_n is either a or b, be rule being that

$$a_n = a \quad \text{if} \quad k^2-k < n \leqslant k^2, \quad k = 1, 2, \ldots$$

and $a_n = b$ for other values of n. The succesion of terms in the sequence $\{a_n\}$ is then one a, one b, two a's, two b's, three a's, three b's, and so on. It is clear after a little inspection that $\|A^n\| = b^n$ and $\mu(A^n) = a^n$. Thus, $\sigma(A)$ consists of all λ such that $|\lambda| \leqslant b$. We shall show that

$$\lambda \in \mathrm{III}_1\sigma(A) \quad \text{if} \quad |\lambda| < a,$$

$$\lambda \in \mathrm{III}_2\sigma(A) \quad \text{if} \quad a \leqslant |\lambda| < \sqrt{ab},$$

and

$$\lambda \in \mathrm{II}_2(A) \quad \text{if} \quad \sqrt{ab} \leqslant |\lambda| \leqslant b.$$

Let $c_n = a_1\ldots a_n$. The first twelve terms of the sequence $\{c_n\}$ are

$$a,\ ab,\ a^2b,\ a^3b,\ a^3b^2,\ a^3b^3,\ a^4b^3,\ a^5b^3,\ a^6b^3,\ a^6b^4,\ a^6b^5,\ a^6b^6.$$

Note that $c_n = a^pb^q$, where p and q depend on n and $p+q = n$. The rule is that

$$p = \frac{k(k+1)}{2} \quad \text{if} \quad k^2 \leqslant n \leqslant k(k+1),$$

$$q = \frac{k(k+1)}{2} \quad \text{if} \quad k(k+1) < n < (k+1)^2.$$

Since $[\mu(A^n)]^{1/n} = a$, we know that the III_1-radius of 0 in $\sigma(A)$ is not less than a. Let us use (6.7) to test for eigenvalues of A'. We see that

$$\frac{\lambda^n}{a_1\ldots a_n} = \frac{\lambda^n}{a^pb^q}.$$

If $|\lambda| \leqslant a$, the absolute value of this ratio is not larger than $(a/b)^q$, which is less than 1. Hence such λ's are eigenvalues of A'. We consider the situation when $a < |\lambda|$. Let us write $|\lambda|/a = \alpha$, $|\lambda|/b = \beta$. Then $0 < \beta < 1 < \alpha$. The sequence $\lambda^n/a_1\ldots a_n$ is now

$$\alpha,\ \alpha\beta,\ \alpha^2\beta,\ \alpha^3\beta,\ \alpha^3\beta^2,\ \alpha^4\beta^3,\ \ldots.$$

That is, it is just like the sequence $\{a_1\ldots a_n\}$, except that a and b have been replaced by α and β. Since $\beta < 1$ and $\alpha > 1$, we see that, in looking for

$$\sup_n\left|\frac{\lambda^n}{a_1\ldots a_n}\right|,$$

we can confine attention to the values $n = k^2, k = 1, 2, 3, \ldots$ This means we let

$$y_k = \alpha^{k(k+1)/2}\beta^{k(k-1)/2}$$

and look to see if the y_k's are bounded. Now

$$y_{k+1}/y_k = (\alpha\beta)^k\alpha.$$

Since $\alpha > 1$, we see that $y_n \to +\infty$ if $\alpha\beta \geqslant 1$ and $y_n \to 0$ if $\alpha\beta < 1$. The meaning of $\alpha\beta < 1$ is $|\lambda| < \sqrt{ab}$. Thus, λ is an eigenvalue of A' if and only if $|\lambda| < \sqrt{ab}$. This information enables us to assert that $\lambda \in \mathrm{II}_2\sigma(A)$ if $\sqrt{ab} \leqslant |\lambda| \leqslant b$. The only other possibilities (since λ is not an eigenvalue of A) are $\mathrm{III}_1\sigma(A)$ and $\mathrm{III}_2\sigma(A)$; but these are ruled out because, by the state diagram, they would involve the impossible classifications $\mathrm{I}_3\sigma(A')$ and $\mathrm{II}_3\sigma(A')$ or $\mathrm{III}_3\sigma(A')$.

We are now left to decide between $\mathrm{III}_1\sigma(A)$ and $\mathrm{III}_2\sigma(A)$ for λ when $a \leqslant |\lambda| < \sqrt{ab}$. We shall show that the classification must be $\mathrm{III}_2\sigma(A)$ because $R(\lambda I - A') \neq l^\infty$ for these values of λ, so that λ cannot be classified $\mathrm{I}_3\sigma(A')$.

Writing $x = \{\xi_n\}$ and $y = \{\eta_n\}$ for vectors in l^∞, we see that $(\lambda I - A')x = y$ means

$$\eta_k = \lambda\xi_k - a_k\xi_{k+1}, \qquad k = 1, 2, \ldots$$

If we solve for $\xi_2, \xi_3, \ldots$ we find

$$\xi_{n+1} = \frac{\lambda^n \xi_1}{a_1 \ldots a_n} - \sum_{k=1}^{n} \frac{\lambda^{n-k}}{a_k \ldots a_n}\eta_k.$$

Since (6.7) holds when $a \leqslant |\lambda| < \sqrt{ab}$, we see that the condition for $R(\lambda I - A') = l^\infty$ is that the vector $(0, \xi_2, \xi_3 \ldots)$ defined by

$$\xi_{n+1} = \sum_{k=1}^{n} \frac{\lambda^{n-k}}{a_k \ldots a_n}\eta_k, \qquad n = 1, 2, \ldots$$

shall always be in l^∞ if the vector $(\eta_1, \eta_2, \ldots)$ is in l^∞. The necessary and sufficient condition that this be so is that

(6.8) $$\sup_{n \geqslant 1} \sum_{k=1}^{n} \frac{|\lambda|^{n-k}}{|a_k \ldots a_n|} < +\infty.$$

If we write

$$C_n = \frac{|\lambda|^n}{|a_1 \ldots a_n|},$$

the sum in (6.8) takes the form

$$\frac{C_n}{|\lambda|}\left[1 + \frac{1}{C_1} + \ldots + \frac{1}{C_{n-1}}\right].$$

We shall show that

(6.9) $$\sup_n C_n\left[\frac{1}{C_1} + \ldots + \frac{1}{C_{n-1}}\right] = +\infty,$$

thus demonstrating that $R(\lambda I - A') \neq l^\infty$ under the stated restrictions on λ. If we set $|\lambda|/a = \alpha$ and $|\lambda|/b = \beta$ as before, we can express C_n in the form $\alpha^p\beta^q$, where $p + q = n$. We shall be interested in the case $n = k^2$, and we shall want to know the value of C_n/C_j for j such that $k^2 - k < j \leqslant k^2 - 1$. We know that

$$C_{k^2} = \alpha^{k(k+1)/2}\beta^{k(k-1)/2}, \qquad C_j = \alpha^{[2j-k(k-1)]/2}\beta^{k(k-1)/2}, \qquad k^2 - k < j \leqslant k^2 - 1.$$

Thus, since $\alpha \geqslant 1$, we have

$$\sum_{j=k^2-k+1}^{k^2-1} \left(\frac{C_k^2}{C_j}\right) = \sum_{j=1}^{k-1} \alpha^j \geqslant k - 1.$$

This shows that

$$C_n\left[\frac{1}{C_1} + \ldots + \frac{1}{C_{n-1}}\right] > k - 1$$

if $n = k^2$, and hence (6.9) is proved.

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