STUDIA MATHEMATICA, T. XXII. (1963)

On some spaces of strongly summable sequences

by

W. ORLICZ (Poznań)

1. This paper contributes some further remarks to the problem treated in [4]. Notations of [4] will be used throughout; if $x = \{t_i\} \epsilon T$ and $y = \{s_i\} \epsilon T$, $x \geqslant y$ will mean $t_i \geqslant s_i$ for all i, and $|x|$ will stand for the sequence $\{|t_i|\}$. Let

$$\sigma_n^\varphi(x) = \frac{1}{n}\sum_{\nu=1}^{n} \varphi(|t_\nu|) \quad \text{for} \quad n = 1, 2, \ldots$$

Then we shall write

$$T_{\varphi_0} = \{x \epsilon T\colon \sigma_n^\varphi(x) \to 0 \text{ as } n \to \infty\},$$

$$T_\varphi^{a*} = \{x \epsilon T\colon \sigma_n^\varphi(\lambda x) \to 0 \text{ for a certain } \lambda > 0 \text{ as } n \to \infty\}.$$

It is easily seen that T_{φ_0} is a convex set and T_φ^{a*} is a linear set; we have $T_\varphi^a \subset T_\varphi^{a*}$, where

$$T_\varphi^a = \{x \epsilon T\colon \lambda x \epsilon T_{\varphi_0} \text{ for arbitrary } \lambda\},$$

as in [4].

1.1. We define a functional $\varrho_\varphi(x)$ in T_φ^{a*}, $\varrho_\varphi(x) = \sup_n \sigma_n^\varphi(x)$ for $x \epsilon T_{\varphi_0}$, and $\varrho_\varphi(x) = \infty$ for $x \epsilon T_\varphi^{a*} \setminus T_{\varphi_0}$. It is easily verified that ϱ_φ is a modular in T_φ^{a*} in the sense of [3] and satisfies the condition B.1, i. e.

$$\varrho_\varphi(\lambda x) \to 0 \text{ as } \lambda \to 0 \quad \text{for} \quad x \epsilon T_\varphi^{a*}.$$

For, assuming $x \epsilon T_\varphi^{a*}$, we have $\sigma_n^\varphi(\lambda_0 x) \to 0$ as $n \to \infty$ for a certain λ_0; hence there is an index m satisfying the condition

$$\frac{1}{n}\sum_{\nu=1}^{n} \varphi(\lambda_0 |t_\nu|) < \varepsilon \quad \text{for} \quad n \geqslant m.$$

Consequently,

$$\sigma_n^\varphi(\lambda x) \leqslant \sup_{n\leqslant m} \sigma_n^\varphi(\lambda x)+\varepsilon \quad \text{for} \quad 0<\lambda\leqslant\lambda_0.$$

By [3], an F-norm $\|\cdot\|_\varphi^a$ may be introduced in T_φ^{a*} by means of the formula

$$(*) \qquad \|x\|_\varphi^a = \inf\left\{\varepsilon>0:\ \varrho_\varphi\left(\frac{x}{\varepsilon}\right)\leqslant\varepsilon\right\}.$$

Let us remark that in [4] the norm $(*)$ was defined only in the space T_φ^a, that is in the subspace of all elements of T_φ^{a*} which satisfy the condition $\varrho_\varphi(\lambda x)<\infty$ for an arbitrary λ. In the terminology of the theory of modular spaces, T_φ^a means *the subspace of all finite elements of the space* T_φ^{a*}. If φ is an s-convex φ-function, then another norm may be defined in T_φ^{a*}, namely

$$(**) \qquad \|x\|_{s\varphi}^a = \inf\left\{\varepsilon>0:\ \varrho_\varphi\left(\frac{x}{\varepsilon^{1/s}}\right)\leqslant 1\right\}.$$

This norm is s-homogeneous and equivalent to the norm $(*)$. Let $\varphi(u)=u^\alpha$, $\alpha>0$. If we denote then the norm $(*)$ by $\|\cdot\|_\alpha^{0a}$ and the norm $(**)$ with $s=\alpha$, $s=1$ if $\alpha>1$, by $\|\cdot\|_\alpha^a$, we obtain $\|x\|_\alpha^{0a}=\left(\sup_n \sigma_n^\varphi(x)\right)^{1/(1+\alpha)}$, $\|x\|_\alpha^a=\sup_n\sigma_n^\varphi(x)$ for $0<\alpha\leqslant 1$, and $\|x\|_\alpha^a=\left(\sup_n\sigma_n^\varphi(x)\right)^{1/\alpha}$ when $\alpha\geqslant 1$.

Let us remark, besides, that the terms t_n of the sequence $x=\{t_i\}$ are linear functionals over T_φ^{a*} with respect to the norms $\|\cdot\|_\varphi^a$ and $\|\cdot\|_{s\varphi}^a$, and the relations (α) $\|x_n\|_\varphi^a\to 0$, (β) $\varrho_\varphi(\lambda x_n)\to 0$ as $n\to\infty$ for every $\lambda\geqslant 0$, are equivalent.

1.2. *The norm* $\|\cdot\|_\varphi^a$ *is complete in* T_φ^{a*} *and* T_φ^a *is a complete subspace of* T_φ^{a*}.

Let $x_n=\{t_i^n\}\in T_\varphi^{a*}$, let $\lambda>0$ be a given number and let $\|x_n-x_m\|_\varphi^a\to 0$ as $n,m\to\infty$. By the definition of $\|\cdot\|_\varphi^a$ there exists an increasing sequence of indices n_k possessing the following two properties:

$$(+) \qquad 2^{2k}(x_{n_{k+1}}-x_{n_k})\in T_{\varphi_0},$$

$$(++) \qquad \sup_n \sigma_n^\varphi\left(2^{2k}(x_{n_{k+1}}-x_{n_k})\right)\leqslant\frac{1}{k}\cdot\frac{1}{2^{2k}} \quad \text{for} \quad k=1,2,\ldots$$

Property $(++)$ gives

$$\sum_{k=1}^\infty \varphi\left(2^{2k}(t_i^{n_{k+1}}-t_i^{n_k})\right)<\infty \quad \text{for} \quad i=1,2,\ldots$$

Since $\varphi(\sum\alpha_\nu u_\nu)\leqslant\sum\varphi(u_\nu)$ for $\alpha_i>0$, $\sum\alpha_i=1$, $u_i\geqslant 0$ we have

$$\left({+\atop +}{}^{+}\right) \qquad \varphi\left(\lambda\sum_{\nu=p}^\infty |t_i^{n_{\nu+1}}-t_i^{n_\nu}|\right)\leqslant\sum_{\nu=p}^\infty\varphi\left(2^{2^\nu}|t_i^{n_{\nu+1}}-t_i^{n_\nu}|\right), \quad i=1,2,\ldots,$$

for all indices p for which $2^p>\lambda$. From $\left({+\atop +}{}^{+}\right)$ it follows that $t_i^{n_k}\to t_i$ for $i=1,2,\ldots$ and, if $l\geqslant k\geqslant p$, we have

$$\varphi(\lambda|t_i^{n_k}-t_i^{n_l}|)\leqslant\sum_{\nu=p}^\infty\varphi(2^{2^\nu}|t_i^{n_{\nu+1}}-t_i^{n_\nu}|),$$

whence

$$\varphi(\lambda|t_i^{n_k}-t_i|)\leqslant\sum_{\nu=p}^\infty\varphi(2^{2^\nu}|t_i^{n_{\nu+1}}-t_i^{n_\nu}|).$$

Let $x=\{t_i\}$; the last inequality implies

$$\sigma_n^\varphi\left(\lambda(x_{n_k}-x)\right)\leqslant\sum_{\nu=p}^\infty\sigma_n^\varphi\left(2^{2^\nu}(x_{n_{\nu+1}}-x_{n_\nu})\right)=\sigma_n<\infty \quad \text{for} \quad n=1,2,\ldots,$$

$$\left({++\atop ++}\right) \qquad \sup_n\sigma_n^\varphi\left(\lambda(x_{n_k}-x)\right)\leqslant\sum_{\nu=p}^\infty\sup_n\sigma_n^\varphi\left(2^{2^\nu}(x_{n_{\nu+1}}-x_{n_\nu})\right).$$

However, the sequences $\sigma_n^\varphi\left(2^{2k}(x_{n_{k+1}}-x_{n_k})\right)$ converge to 0; hence by $(++)$ we have $\sigma_n\to 0$, i. e. $\sigma_n^\varphi\left(\lambda(x_{n_k}-x)\right)\to 0$ as $n\to\infty$ for $k\geqslant p$. Now $(++)$ and $\left({++\atop ++}\right)$ imply

$$\sup_n\sigma_n^\varphi\left(\lambda(x_{n_k}-x)\right)\leqslant\sum_{k=p}^\infty\frac{1}{k}2^{-2k},$$

whence $\sup_n\sigma_n^\varphi\left(\lambda(x_{n_k}-x)\right)\to 0$ for every $\lambda>0$, i. e. $\|x_{n_k}-x\|_\varphi^a\to 0$. Since $\lambda(x_{n_k}-x)\in T_\varphi^{a*}$ and $x_{n_k}\in T_\varphi^{a*}$, we have $x\in T_\varphi^{a*}$. The inequality

$$\|x_m-x\|_\varphi^a\leqslant\|x_m-x_{n_k}\|_\varphi^a+\|x_{n_k}-x\|_\varphi^a$$

yields finally $\|x_m-x\|_\varphi^a\to 0$ as $m\to\infty$.

In order to prove the second part of the theorem let us note that if $x_n\in T_\varphi^a$ for $n=1,2,\ldots$, then the inequality

$$\sigma_n^\varphi(\tfrac{1}{2}\lambda x)\leqslant\sigma_n^\varphi\left(\lambda(x_m-x)\right)+\sigma_n^\varphi(\lambda x_m) \quad \text{for} \quad n=1,2,\ldots$$

implies $\frac{1}{2}\lambda x\in T_{\varphi_0}$ for an arbitrary λ, i. e. $x\in T_\varphi^a$.

2.1. *For arbitrary two* φ*-functions* φ, ψ *we have* $T_\varphi^{a*}\cap T_b=T_\psi^{a*}\cap T_b$.

We repeat here with a slight modification the proof given in [4]. Let $\eta>0$ and let $0\leqslant u\leqslant m\eta$ for a certain positive integer m. Then

$$(+) \qquad \psi(u)\leqslant\psi(\eta)+\frac{\psi(m\eta)}{\varphi(\eta)}\varphi(u).$$

Indeed, if $0\leqslant u\leqslant\eta$, then $\psi(u)\leqslant\psi(\eta)$, and if $(k-1)\eta<u\leqslant k\eta$, where k is an integer satisfying the inequalities $2\leqslant k\leqslant m$, then

$$\psi(u)\leqslant\psi(m\eta)\left[\varphi\left((k-1)\eta\right)\right]^{-1}\varphi(u)\leqslant\psi(m\eta)[\varphi(\eta)]^{-1}\varphi(u).$$

Now, if $x \in T_\varphi^{a*} \cap T_b$, then $\lambda_0 x \in T_{\varphi_0}$ for a certain $\lambda_0 > 0$, and we have $\lambda_0 |t_i| \leqslant m\eta$ for a sufficiently large m; thus (+) yields

$$\sigma_n^\psi(\lambda_0 x) \leqslant \psi(\eta) + \frac{\psi(m\eta)}{\varphi(\eta)} \sigma_n^\varphi(\lambda_0 x),$$

whence

$$(++) \qquad \limsup_n \sigma_n^\psi(\lambda_0 x) \leqslant \psi(\eta), \qquad \lim_{n\to\infty} \sigma_n^\varphi(\lambda_0 x) = 0.$$

2.11. *For an arbitrary φ-function φ we have $T_\varphi^{a*} \cap T_b = T_\varphi^a \cap T_b$.*

Let $\lambda_0 x \in T_{\varphi_0}$, $\lambda > 0$, $\psi(u) = \varphi(\lambda_0^{-1} u)$. It follows from the proof of 2.1 that 2.1 (++) holds, i. e. $\lim\limits_n \sigma_n^\varphi(\lambda x) = 0$, and consequently $x \in T_\varphi^a$. The inclusion $T_\varphi^a \cap T_b \subset T_\varphi^{a*} \cap T_b$ is obvious.

2.2. *If $\varphi \overset{l}{\prec} \psi$, then $T_\psi^{a*} \subset T_\varphi^{a*}$. The conditions $\|x_i\|_\psi^a \to 0$ as $i \to \infty$, $x_i \in T_\psi^{a*}$, imply $\|x_i\|_\varphi^a \to 0$ as $i \to \infty$.*

The inequality $\varphi(u) \leqslant b\psi(ku)$ holds for $u \geqslant u_0$, where k, b, u_0 are positive constants. Let $\lambda_0 x \in T_{\psi_0}$, $\lambda_0 > 0$. Since $\varphi(\lambda_0 k^{-1} t_i) \leqslant b\psi(\lambda_0 t_i)$ for indices i satisfying the inequality $\lambda_0 k^{-1} t_i \geqslant u_0$ holds, and since the sequence $x_i' = \{t_i'\}$ with terms $t_i' = t_i$ when $\lambda_0 k^{-1} t_i < u_0$ and $t_i' = 0$ for the remaining indices i belongs to $T_\psi^{a*} \cap T_b = T_\varphi^{a*} \cap T_b$, we have $x'' = x - x' \in T_\varphi^{a*}$ $x' \in T_\varphi^{a*}$, and consequently $x \in T_\varphi^{a*}$. A similar method of proof may be applied to the second part of the theorem; it follows also trivially from the closed graph theorem, for the terms of the sequence are linear functionals with respect to the norms $\|\cdot\|_\varphi^a$, $\|\cdot\|_\psi^a$.

By [4], 2.3, if convergence with respect to the norm $\|\cdot\|_\varphi^a$ implies convergence with respect to the norm $\|\cdot\|_\psi^a$ in T_f, then $\psi \overset{l}{\prec} \varphi$. Hence we obtain from 2.2 immediately:

2.3. *The following properties are equivalent*: (α) $\psi \overset{l}{\prec} \varphi$, (β) $T_\varphi^{a*} \supset T_\psi^{a*}$, (γ) $\|x_i\|_\varphi^a \to 0$ *implies* $\|x_i\|_\psi^a \to 0$ *for arbitrary* $x_i \in T_f$.

Corollary. (A) $T_\varphi^{a*} \subset T_\psi^{a*}$ *if and only if* $T_\varphi^a \subset T_\psi^a$.

(B) *The following properties are equivalent*: (α) $\varphi \overset{l}{\sim} \psi$, (β) $T_\varphi^{a*} = T_\psi^{a*}$, (γ) $T_\varphi^a = T_\psi^a$, (δ) *the norms* $\|\cdot\|_\varphi^a$, $\|\cdot\|_\psi^a$ *are equivalent in* T_f.

2.4. *The following conditions are equivalent*:

(α) *φ satisfies the condition* (Δ_2) *for large* u,

(β) $T_\varphi^{a*} = T_\varphi^a$,

(γ) *the space* $[T_\varphi^{a*}, \|\cdot\|_\varphi^a]$ *is separable.*

(α) ⇒ (β). If (α) holds, $\lambda > 1$, then there is a $c_\lambda > 1$ such that the inequality $\varphi(\lambda u) \leqslant c_\lambda \varphi(u)$ holds for $u \geqslant u_0(\lambda)$. Arguing as in 2.2 we may show by means of this inequality that $\sigma_n^\varphi(x) \to 0$ as $n \to \infty$ implies $\sigma_n^\psi(x) \to 0$ as $n \to \infty$, where $\psi(u) = \varphi(\lambda u)$.

(β) ⇒ (γ) follows from [4], Corollary to 1.3.

(γ) ⇒ (α). Let $\psi(u) = \varphi(2u)$ and let us suppose that (Δ_2) does not hold. By the arguments of [4] (2.3, the proof of (γ) ⇒ (α)) sequences of the form $x_i = v_i e_{n_i}$ may be chosen so that $n_i \neq n_j$ for $i \neq j$, $\|x_i\|_\psi^a \geqslant 1$, $\|x_i\|_\varphi^a < 2^{-i}$. Let $x_\eta = \eta_1 x_1 + \eta_2 x_2 + \ldots$, where $\eta = \{\eta_i\}$, $\eta_i = 0,1$; we have $x_\eta \in T_\varphi^{a*}$ for arbitrary η, for $\eta_1 \|x_1\|_\varphi^a + \eta_2 \|x_2\|_\varphi^a + \ldots \leqslant 1$, and T_φ^{a*} is complete with respect to the norm $\|\cdot\|_\varphi^a$. If $\eta' \neq \eta''$, then $|x_{\eta'} - x_{\eta''}| \geqslant x_m$ for a certain m, whence $\|2(x_{\eta'} - x_{\eta''})\|_\varphi^a \geqslant \|x_m\|_\psi^a \geqslant 1$, i. e. T_φ^{a*} is not separable, a contradiction.

2.5. *In order that the $\|\cdot\|_\varphi^a$-norm topology in T_φ^{a*} be locally s-convex it is necessary and sufficient that $\varphi \overset{l}{\sim} \chi$, where $\chi(u) = \psi(u^s)$, ψ is a convex function.*

Necessity. If the $\|\cdot\|_\varphi^a$-norm topology is locally s-convex in T_φ^{a*}, then this topology is also locally s-convex in T_φ^a; hence the necessity follows from [4], 2.5.

Sufficiency. If $\varphi \overset{l}{\sim} \chi$, then by 2.2 there exists an s-convex norm equivalent to $\|\cdot\|_\varphi^a$.

3. Besides the modular space T_φ^a, a space T_φ^b was considered in [4], 3 for convex φ-functions φ. The following question arises: assuming φ and ψ to be convex φ-functions, when does the identity $T_\varphi^a = T_\psi^b$ hold? The following theorem generalizes a result of [4]:

3.1. *If $T_\varphi^a = T_\psi^b$, then*

$$(*) \qquad \lim_{u\to\infty} \frac{\lg \varphi(u)}{\lg u} = r, \qquad (**) \qquad \lim_{v\to 0} \frac{\lg \psi(v)}{\lg v} = r,$$

where $1 \leqslant r < \infty$.

Let us write $\sigma_0 = \inf \sigma$, where the infimum is taken over all exponents $\sigma > 0$ satisfying the inequality $\limsup\limits_{u\to\infty} \varphi(u)/u^\sigma < \infty$, and $\sigma_0 = \infty$ if such exponents σ do not exist. Moreover, let us write $s_0 = \inf s$, where the infimum is taken over all exponents $s > 0$ such that $\liminf\limits_{v\to 0} \psi(v)/v^s > 0$, and $s_0 = \infty$ if such exponents do not exist. It is readily seen that $\limsup\limits_{u\to\infty} \lg \varphi(u)/\lg u \leqslant \sigma_0$. Let $T_\varphi^a = T_\psi^b$. By [4], 3.3,

$$(+) \qquad \psi(\delta uv) \leqslant \varphi(u)\psi(v) \quad \text{if} \quad \varphi(u)\psi(v) \leqslant \delta,\ \varphi(u) \geqslant 1.$$

Put $a = \delta u$, $c_a = \varphi(u)$; if $\varphi(u) > 1$, for $u \geqslant u_0$, $a > 1$, then $\psi(av) \leqslant c_a \psi(v)$ for $0 < v \leqslant v_0(u)$, i. e. ψ satisfies the condition (Δ_a) for small v. It may be easily verified that

$$(++) \qquad \frac{\psi(v_0)}{v_0^s} \leqslant c_a \frac{\psi(v)}{v^s} \quad \text{for} \quad 0 < v \leqslant v_0(u),$$

where $s = \lg c_a/\lg a$. By [4], 3.4, (3.15)

$$(\overset{+}{++}) \qquad \varphi(\bar{\delta}u)\psi\left(\frac{1}{u}\right) \leqslant \delta \quad \text{for} \quad u \geqslant \bar{u}.$$

The last inequality shows that the inequalities $\limsup_{u\to\infty}\varphi(u)/u^r < \infty$ and $\liminf_{v\to 0}\psi(v)/v^r > 0$ are equivalent; hence $\sigma_0 = s_0$. This equality holds also when one of the values σ_0, s_0 is equal to ∞.

By (++),

$$s_0 \leqslant \frac{\lg c_a}{\lg a} = \frac{\lg\varphi(u)}{\lg\delta+\lg u} \quad \text{for} \quad u \geqslant u_0,$$

whence $\liminf_{u\to\infty}\lg\varphi(u)/\lg u \geqslant s_0$, and since $\sigma_0 \geqslant \limsup_{u\to\infty}\lg\varphi(u)/\lg u$, (*) holds with $r = s_0 = \sigma_0$. From (+) and (++) it follows also that

$$\lim_{v\to 0}\frac{\lg\psi(v)}{\lg v} = r.$$

A function $\varphi(u)$ is called *regularly increasing with index* r_φ if $\varphi(u) = u^{r_\varphi}\varrho(u)$, where $\varrho(\lambda u)/\varrho(u) \to 1$ as $u\to\infty$, $\lambda > 0$ (cf. [1], [2]).

3.2. *If the relation* 3.1 (*) *holds and if* $\lg\varphi(e^u)$ *is a convex function for large* u, *then* $\varphi(u)$ *is a regularly increasing function* (*for large* u) *with index* $r_\varphi = r$.

The function $u\varphi'(u)/\varphi(u)$ is defined for $u\in B = (a,\infty)\setminus A$, where the set A is of measure 0, and $u\varphi'(u)/\varphi(u)\to g$ as $u\to\infty$, $u\in B$. Since

$$\frac{\lg\varphi(u)-\lg\varphi(a)}{\lg u-\lg a} = \frac{\int_a^u \frac{\varphi'(t)}{\varphi(t)}\,dt}{\int_a^u \frac{dt}{t}}$$

and the ratio of integrals on the right-hand side of the above equality tends to g, the value g is finite and $g = r$. Now it is sufficient to apply a known criterion (cf. e. g. [2]).

References

[1] J. Karamata, *Sur un mode de croissance régulière*, Bull. Soc. Math. France 61 (1933), p. 55-62.

[2] W. Matuszewska, *Regularly increasing functions in connection with the theory of $L^{*\varphi}$-spaces*, Studia Math. 21 (1962), p. 317-344.

[3] J. Musielak and W. Orlicz, *On modular spaces*, ibidem 18 (1959), p. 49-65.

[4] — *On modular spaces of strongly summable sequences*, ibidem 22 (1962), p. 127-146.

Reçu par la Rédaction le 20. 6. 1962

STUDIA MATHEMATICA, T. XXII. (1963)

Équations avec opérations algébriques

par

D. PRZEWORSKA-ROLEWICZ (Warszawa)

Sommaire

I. Introduction

Soit une équation intégrale singulière:

$$A(t)x(t) + \frac{B(t)}{\pi i}\int_L \frac{x(s)}{s-t}\,ds = x_0(t) \qquad (s, t\in L),$$

où L désigne un arc de Jordan, régulier et fermé, l'intégrale a le sens de la valeur principale de Cauchy, les fonctions complexes $A(t)$, $B(t)$, $x(t)$, $x_0(t)$ définies pour $t\in L$ satisfont à la condition de Hölder. Les méthodes classiques de Carleman, Vécoua et Muskhelichvili (voir la monographie [10]) utilisées dans l'étude de ces équations, ainsi que les considérations plus abstraites de Halilov [6], Tcherski [4] sont fondées en principe sur la propriété unique qu'une transformation intégrale singulière (pour les arcs fermés)

$$S(x) = \frac{1}{\pi i}\int_L \frac{x(s)}{s-t}\,ds$$