Clearly it is not supposed that such an a^* belongs to A but of course (see section 2) the following inequality holds:

$$M \geqslant \max_{j=1,2} v(a^*, b_j) > 0.$$

By 5° the function $b(\boldsymbol{p}_z, 1)$ is a continuous mapping of the closed disc P into itself; thus by the fixed point theorem there exists such a $z(b)\in P$ that

$$b(\boldsymbol{p}_{z(b)}, 1) = z(b).$$

Since $\boldsymbol{p}_z(1) = z$, it is clear that

$$0 \leqslant m \leqslant \sup_{b\in B}\inf_{z\in P} v(a_z, b) \leqslant \sup_{b\in B} v(a_{z(b)}, b) = |\boldsymbol{p}_{z(b)}(1) - b(\boldsymbol{p}_{z(b)}, 1)| = 0$$

and $m = 0$.

Therefore $M > m$, q. e. d.

8. A game of pursuit and evasion of two points in the space R^n has a value since the optimal strategies (in the sense of section 3) are

$$\boldsymbol{p}'(t) = c_1 \frac{\boldsymbol{q}(t)-\boldsymbol{p}(t)}{\|\boldsymbol{q}(t)-\boldsymbol{p}(t)\|}, \quad \boldsymbol{q}'(t) = c_2 \frac{\boldsymbol{p}(t)-\boldsymbol{q}(t)}{\|\boldsymbol{p}(t)-\boldsymbol{q}(t)\|},$$

$$\boldsymbol{p}(0) = p_0, \quad \boldsymbol{q}(0) = q_0.$$

Therefore it is essential in our example that P and Q should not be whole plane. It is not so if the number of moving points of the two players is greater. Consider the game described briefly as follows. The pursuer $\mathfrak{A}$ has three moving points in the plane, situated at the moment 0 in the angles of an equilateral triangle with side 1. The evader $\mathfrak{B}$ has one point situated at the moment 0 at the central point of this triangle. $P = Q = R^2$. The velocity of all points is at most $1/\sqrt{3}$. Again, under some very general assumptions regarding the classes of strategies ($\mathfrak{A}$ must have some straight line strategies, the strategies of $\mathfrak{B}$ must be continuous on the set of straight line movements of $\mathfrak{A}$, and $\mathfrak{B}$ must have some appropriate finite set of strategies) this game has no value. The idea of the proof is the same as that of section 7.

References

[1] L. D. Berkowitz, W. H. Fleming and H. E. Scarf, *Contributions to the theory of games III*, Part V, Annals of Math. Studies 39 (1957), p. 393-435.

[2] J. C. C. McKinsey, *Introduction to the theory of games*, New York 1952.

[3] Jan Mycielski, S. Świerczkowski and A. Zięba, *On infinite games*, Bull. Acad. Pol. Sci., Cl. III, 3 (1955), p. 133-136.

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Analytic functions of polynomial growth

by

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In this paper we give simple proofs of some theorems (of type Paley-Wiener) on the representation of analytic functions in the form

$$\Phi(z) = \int_{-\infty}^{\infty} e^{-zt} f(t)\,dt,$$

where $f(t)$ is a distribution. Usual proofs are based on similar theorems of classical Analysis [1], [2], [3], [4]. This way is circuitous for distributions, for in this case it suffices to start from a class of very regular functions, which makes proofs easier and shorter, the tools of the theory of the Lebesgue integral being then superflous. This idea is followed in this paper.

The proof of the fundamental theorem is essentially similar to that in [5], but some further simplifications are introduced.

A function $\Phi(z)$ is said of *polynomial growth* in a set G if there exists a polynomial P such that

$$|\Phi(z)| < P(r) \text{ in } G \qquad (r = |z|).$$

The aim of this paper is to prove the following

THEOREM. *If a function $\Phi(z)$, analytic in* $\mathrm{Re}\, z > 0$, *is of polynomial growth, then it can be represented in the form*

$$\Phi(z) = \int_{-\infty}^{\infty} e^{-zt} f(t)\,dt \,(^1), \tag{1}$$

where $f(t)$ is a distribution, temperated for $t > 0$ and vanishing for $t < 0$.

Here, by a distribution *temperated* for $t > 0$ we understand every distribution which is a derivative of some order of a continuous function of polynomial growth for $t > 0$.

(1) The meaning of this integral may be understood in the sense of [6]. See also [7].

In section 1 we give the proof of the above theorem. In next sections we characterize the general class of functions of the form (1), and state, as simple corollaries, some theorems of Paley-Wiener type.

1. The function

$$\Psi(z)=\int_1^z \Phi(s)\,ds$$

is also of polynomial growth in $\operatorname{Re} z>0$, together with $\Phi(z)$. Moreover, it can be completed to a continuous function in the closed half-plane $\operatorname{Re} z\geqslant 0$. For properly chosen positive integer n, the function

$$\frac{\Psi(z)}{(1+z)^n}$$

is bounded in $\operatorname{Re} z\geqslant 0$. Let

$$\Omega(z)=\frac{\Psi(z)}{(1+z)^{n+2}};$$

we have

(2) $$\Omega(z)=\frac{1}{2\pi i}\int_{C_R}\Omega(s)\frac{ds}{s-z},$$

where the contour C_R ($R>|z|$) is composed of the semi-circle

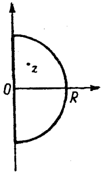

Fig. 1

(3) $$|s|=R,\quad \operatorname{Re} s\geqslant 0$$

and of the segment of the imaginary axis embraced by this semi-circle. On the other hand, given any real u, we have

(4) $$0=\frac{1}{2\pi i}\int_{C_R}\Omega(s)\frac{e^{-(s-z)u}-1}{s-z}\,ds,$$

Ar the integrand is analytic inside the contour C_R and continuous on it. odding (2) and (4) we get

$$\Omega(z)=\frac{1}{2\pi i}\int_{C_R}\Omega(s)e^{-(s-z)u}\frac{ds}{s-z}.$$

If $u\geqslant 0$, the part of that integral belonging to the semi-circle (3) tends to 0, as $R\to\infty$, and so we get

(5) $$\Omega(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Omega(iy)\frac{e^{-(iy-z)u}}{iy-z}\,dy.$$

But

$$\frac{e^{-(s-z)u}}{z-s}=\int_{-u}^{\infty}e^{(s-z)t}\,dt\qquad\text{for}\qquad \operatorname{Re} s<\operatorname{Re} z;$$

substituting this into (5) and interchanging the order of integration we get

$$\Omega(z)=\int_{-u}^{\infty}e^{-zt}\,dt\,\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{iyt}\,\Omega(iy)\,dy.$$

Since the formula holds for every $u\geqslant 0$ and its left side does not depend on u, the function

(6) $$g(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{iyt}\,\Omega(iy)\,dy$$

vanishes for $t<0$ and we can write

(7) $$\Omega(z)=\int_{-\infty}^{\infty}e^{-zt}g(t)\,dt.$$

Since

$$|\Omega(iy)\,dy|<\frac{M}{|1+iy|^2},$$

the integral (6) represents a continuous and bounded function in $-\infty<x<\infty$.

From (7) we get

$$(1+z)\,\Omega(z)=\int_{-\infty}^{\infty}e^{-zt}g_1(t)\,dt\ (^2),$$

where $g_1(t)=g(t)+g'(t)$, and $g'(t)$ is the distributional derivative of $g(t)$. Of course $g_1(t)$ is a distribution temperated for $t>0$ and vanishing for $t<0$. Similarly,

$$(1+z)^2\,\Omega(z)=\int_{-\infty}^{\infty}e^{-zt}g_2(t)\,dt,$$

where $g_2(t)=g_1(t)+g_1'(t)$. After n similar steps we get

$$\Psi(z)=\int_{-\infty}^{\infty}e^{-zt}g_n(t)\,dt,$$

(²) The properties of the Laplace integral of a distribution and their proofs are analogous to those of Fourier integral (see [6] and [7]). Remark that, in the complex domain, the difference between the two integrals is inessential.

where $g_n(t)$ is a distribution temperated for $t>0$ and vanishing for $t<0$. Hence, on differentiating, we obtain (1) with $f(t) = -tg_n(t)$.

2. An analytic function $\Phi(z)$ is said *temperated* in G if it is, in G, a derivative of some order of a function of polynomial growth.

Corollary 1. *A function $\Phi(z)$, analytic in the half-plane $\operatorname{Re} z>0$, is temperated in that half-plane, if and only if it is of the form* (1), *where $f(t)$ is a distribution temperated for $t>0$ and vanishing for $t<0$.*

In fact, if $f(t)$ is temperated for $t>0$ and vanishing for $t<0$, there is a continuous function $F(t)$ of polynomial growth for $t>0$ and vanishing for $t<0$ such that $f(t) = F^{(k)}(t)$. There is a positive integer n such that

$$\frac{F(t)}{(1+t)^n}$$

is bounded. Let

$$G(t) = \frac{F(t)}{(1+t)^{n+2}}.$$

The function

$$\Psi(z) = \int_{-\infty}^{\infty} e^{-zt} G(t)\,dt$$

is analytic and bounded in $\operatorname{Re} z>0$. The function

$$\Psi_1(z) = \Psi(z) - \Psi'(z) = \int_{-\infty}^{\infty} e^{-zt}(1+t)G(t)\,dt$$

is analytic and temperate in $\operatorname{Re} z>0$. After $n+2$ similar steps we obtain the function

$$\Psi_{n+2}(z) = \int_{-\infty}^{\infty} e^{-zt} F(t)\,dt,$$

which is analytic and temperated in $\operatorname{Re} z>0$. Hence

$$z^k \Psi_{n+2}(z) = \int_{-\infty}^{\infty} e^{-zt} F^{(k)}(t)\,dt,$$

which proves that the function (1) is analytic and temperated in $\operatorname{Re} z>0$.

Conversely, suppose now that $\Phi(z)$ is analytic and temperated in $\operatorname{Re} z>0$. Then it is a derivative of some order k of a function $\overline{\Phi}(z)$ of polynomial type in $\operatorname{Re} z>0$. By Theorem, we can write

$$\overline{\Phi}(z) = \int_{-\infty}^{\infty} e^{-zt} \bar{f}(t)\,dt,$$

where $\bar{f}(t)$ is a distribution temperated for $t>0$ and vanishing for $t<0$. Hence, on differentiating k times, we obtain (1) with $f(t) = (-t)^k \bar{f}(t)$.

Corollary 2. *The only entire functions temperated in the whole complex plane are polynomials.*

In fact, such a function can be represented, for $\operatorname{Re} z>0$, in the form (1). On the other hand, we can apply the same argument to the function $\overline{\Phi}(z) = \Phi(-z)$; thus we have on $\operatorname{Re} z>0$

$$\overline{\Phi}(z) = \int_{-\infty}^{\infty} e^{-zt} \bar{f}(t)\,dt,$$

where the distribution $\bar{f}(t)$ is temperated for $t>0$ and vanishing for $t<0$. Replacing t by $-t$, we find that

$$\bar{f}(t) = f(-t).$$

This implies that the distribution $f(t)$ vanishes also for $t<0$. Thus the support of $f(t)$ is reduced to the single point $t=0$ and, consequently, $f(t)$ is of the form

$$f(t) = a_0\delta(t) + \ldots + a_m\delta^{(m)}(t),$$

where a_i are numbers and $\delta(t)$ is the Dirac delta-distribution. Hence

$$\Phi(z) = a_0 + \ldots + a_m z^m.$$

3. The following two corollaries are concerned with functions $\Phi(z)$ of exponential type in $\operatorname{Re} z>0$.

Corollary 3. *The function $e^{-az}\Phi(z)$ (a real) is analytic and temperated for $\operatorname{Re} z>0$ if and only if $\Phi(z)$ is of the form* (1), *where $f(t)$ is a distribution temperated for $t>0$ and vanishing for $t<-a$.*

In fact, we have

$$e^{-az}\Phi(z) = \int_{-\infty}^{\infty} e^{-zt} f_a(t)\,dt,$$

where $f_a(t)$ is a distribution temperated for $t>0$ and vanishing for $t<0$. Hence

$$\Phi(z) = \int_{-\infty}^{\infty} e^{-zt} f_a(t+a)\,dt. \tag{8}$$

The distribution $f_a(t+a)$ is temperated for $t>0$ and vanishing for $t<-a$.

Corollary 4. *If, for every $a>0$, $e^{-az}\Phi(z)$ is a temperated analytic function in $\operatorname{Re} z>0$, then so is $\Phi(z)$.*

In fact, then (8) holds for every $a > 0$ and the distribution $f_a(t+a)$ vanishes for $t < -a$. But this distribution does not depend, in fact, on a, for it is determined uniquely by $\Phi(z)$. Thus it vanishes for $t < 0$.

4. In order to obtain theorems of Paley-Wiener type, we need also the following Lemma, whose proof is based on the theorem of Phragmén and Lindelöf and is entirely independent of the preceding considerations:

LEMMA. *If $\Phi(z)$ is analytic in $\mathrm{Re}\, z > 0$, tempered in the neighbourhood of the imaginary axis, and such that the product $e^{-a|z|}\Phi(z)$ (a real) is of polynomial growth in $\mathrm{Re}\, z > 0$, then the product $e^{-az}\Phi(z)$ is tempered in $\mathrm{Re}\, z > 0$.*

In fact, there exists a positive integer k and a function $\Psi(z)$ analytic in $\mathrm{Re}\, z > 0$, continuous and of polynomial growth on the imaginary axis, such that $\Psi^{(k)}(z) = \Phi(z)$ in $\mathrm{Re}\, z > 0$. Moreover, $e^{-a|z|}\Psi(z)$ is of polynomial growth. There exists a polynomial $P(z)$, whose all zeros are negative, such that

$$\left|\frac{\Phi(z)}{P(z)}\right| < 1 \text{ for imaginary } z,$$

and

$$\left|\frac{e^{-a|z|}\Psi(z)}{P(z)}\right| < 1 \quad \text{for} \quad \mathrm{Re}\, z \geqslant 0.$$

The function

$$\frac{e^{-az}\Psi(z)}{P(z)}$$

is of exponential type in $\mathrm{Re}\, z \geqslant 0$, and less than 1 on the imaginary axis and on the positive part of the real axis. By the Phragmén-Lindelöf theorem, it is less than 1 in the whole half-plane $\mathrm{Re}\, z \geqslant 0$. Thus $e^{-az}\Psi(z)$ is of polynomial growth in $\mathrm{Re}\, z > 0$. Hence it follows easily that $e^{-az}\Phi(z)$ is tempered in $\mathrm{Re}\, z \geqslant 0$.

On combining the preceding lemma with Corollary 3, we obtain

COROLLARY 5. *If $\Phi(z)$ satisfies conditions of Lemma, then it is of the form (1), where $f(t)$ is a distribution tempered for $t > 0$ and vanishing for $t < -a$.*

Our last corollary will be concerned with entire functions:

COROLLARY 6. *An entire function $\Phi(z)$ of exponential type is tempered on the imaginary axis if and only if it is of the form (1), where $f(t)$ is a distribution of bounded support.*

It is easy to verify that if $f(t)$ is of bounded support, then (1) is an entire function, tempered on the imaginary axis. Suppose, conversely, that $\Phi(z)$ is an entire function, tempered on the imaginary axis. Then

by Corollary 5, it can be represented in the form (1), where $f(t)$ is a distribution whose support is bounded from below. The same result can be stated for the function $\Phi(z)$. Reasoning as in the proof of Corollary 2, we find that the support of $f(t)$ is also bounded from above.

References

[1] Н. Н. Боголюбов О. С. Парасюк, *Об аналитическом продолжении обобщенных функций*, Докл. Ак. Наук СССР 109 (1956), p. 717-719.

[2] О. С. Парасюк, *Аналитическое продолжение обобщенных функций*, Revue de Math. Pures et Appliquées 1 (1956), № 3, p. 51-57.

[3] J. L. Lions, *Supports dans la transformation de Laplace*, Journal d'Analyse Mathématique 2 (1952-53), p. 369-380.

[4] — *Support dans la transformation de Laplace*, Journal d'Anal. Math. Israel, 2 (1952-53), p. 369-380.

[5] J. Mikusiński, *On the Paley-Wiener theorem*, Studia Mathematica 13 (1953), p. 287-295.

[6] R. Sikorski, *Integrals of distributions*, ibidem, in print.

[7] L. Schwartz, *Théorie des distributions*, II, Paris 1951.

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