The determinant theory of generalized Fredholm operators

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In 1932 Leżański [3], [4] generalized the determinant theory over the linear equations with operators of Fredholm type in Banach spaces and Sikorski [8]-[14] developed and modified this theory by introducing the notion of determinant system. Independently of Leżański and Sikorski, similar theories were obtained by Ruston [6], [7] and Grothendieck [2]. However, their theories are more complicated and less general than Leżański’s theory.

So far the determinant theory has been applied only to Fredholm operators. The main purpose of this paper is a generalization of this theory over a larger class of operators, called generalized Fredholm operators (see p. 267). Formulse obtained for solutions are analogous to the formulae in the Fredholm case.

The paper consists of an algebrical and an analytical part. The former contains a discussion of properties of generalized Fredholm operators and determinant systems in arbitrary linear spaces. The analytical part concerns determinant systems for operators of the form $S + T$ in Banach spaces where $S$ is a fixed generalized Fredholm operator of order zero (see p. 292) and $T$ is any quasi-nuclear operator (see p. 267). The analytical part can be applied in the theory of singular integral equations.

The notation used here is not traditional. It was adopted from papers of Sikorski for convenience because it enables us to calculate in a simple and mechanical way.

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1. Operators. We shall consider two fixed linear spaces $E$ and $X$ over the same real or complex field $F$. The letters $x, y, z$ (with indices, if necessary) always denote elements of $X$, and $a, b, c$ — scalars of $F$. Every mapping into $F$ will be called a functional.
Following Sikorski [10] we suppose that $\mathcal{E}$ and $X$ are *conjugate*, i.e., there exists a bilinear functional on $\mathcal{E} \times X$ whose value at the point $(\xi, x)$ is denoted by $\xi x$ and which satisfies two additional conditions:

(a) if $\xi x = 0$ for every $\xi \in \mathcal{E}$, then $x = 0$;

(a') if $\xi x = 0$ for every $x \in X$, then $\xi = 0$.

Let $\mathfrak{A}$ be the class of all endomorphisms $A$ in $X$ such that the following condition is satisfied:

(b) for every fixed $\xi \in \mathcal{E}$ there exists an $\eta \in \mathcal{E}$ such that $\xi (Ax) = \eta x$ for every $x \in X$.

It is easy to see that every endomorphism $A \in \mathfrak{A}$ induces an adjoint one in $\mathcal{E}$ which will also be denoted by $A$, and whose value at the point $\xi$ will be denoted by $\xi A$. Namely $\xi A$ is the only element $\eta$ satisfying (b).

By definition of $\xi A$,

(b') for every fixed $x \in X$ there exists a $\eta \in \mathcal{E}$ such that $\xi (Ax) = \eta x$ for every $\xi \in \mathcal{E}$.

Endomorphisms $A \in \mathfrak{A}$ can be interpreted as bilinear functionals on $\mathcal{E} \times X$ as follows:

\begin{equation}
\xi Ax = \xi (Ax) = (\xi A)x.
\end{equation}

It is obvious that $\mathfrak{A}$ is a ring. Instead of $\xi x$, we can write $\xi Ix$, where $I$ is the identity operator.

For any $A \in \mathfrak{A}$ let us introduce the following notation:

\begin{align*}
Y(A) &= \{Ax: x \in X\}, \\
Z(A) &= \{x: Ax = 0, x \in X\}, \\
\mathcal{Y}(A) &= \{\xi A: \xi \in \mathcal{E}\}, \\
\mathcal{Z}(A) &= \{\xi: \xi A = 0, \xi \in \mathcal{E}\}, \\
\dim \mathcal{Z} &= \text{algebraic dimension of a subspace } Z \text{ of } X \text{ or } \mathcal{E}.
\end{align*}

For fixed $\xi_0$ and $x_0$, let $x_0, \xi_0$ denote the *one-dimensional operator* $K \in \mathfrak{A}$ defined by the formula (1)

\begin{equation}
Kx = x_0, \xi_0 x.
\end{equation}

Thus the value of its adjoint endomorphism at a point $\xi$ is

\begin{equation}
\xi K = \xi x_0, \xi_0.
\end{equation}

Any finite sum of one-dimensional operators

\begin{equation}
K = \sum_{i=1}^{n} x_i, \xi_i
\end{equation}

is called a *finitely dimensional operator* because the sets $Y(K)$ and $\mathcal{Y}(K)$ are at most $n$-dimensional.

Observe that if $K$ is a finitely dimensional operator, then so are $AK$ and $KA$ for every $A \in \mathfrak{A}$, and

\begin{align*}
AK &= \sum_{i=1}^{n} x_i, \xi_i, \\
KA &= \sum_{i=1}^{n} x_i, \xi_i A.
\end{align*}

In particular, if $K' = \sum_{i=1}^{n} x_i', \xi_i'$, then

\begin{equation}
KK' = \sum_{i=1}^{n} \sum_{j=1}^{n} (\xi_i x_i') \eta_j, \xi_j',
\end{equation}

An operator $B$ is said to be a quasi-inverse (*) of $A$, if

\begin{equation}
ABA = A, \quad BAB = B.
\end{equation}

Clearly, if $B$ is a quasi-inverse of $A$, then $A$ is a quasi-inverse of $B$.

2. Definition of the generalized Fredholm operator. A linear operator (bilinear functional) $A \in \mathfrak{A}$ is said to be a *generalized Fredholm operator* if:

\begin{enumerate}
\item [$\mathcal{Y}(A) \dim \mathcal{Z}(A) < \infty$, \dim \mathcal{Y}(A) < \infty;
\item [g] the equation $Ax = x_0$ has a solution $x$ if and only if $\xi x_0 = 0$
for every $\xi \in \mathcal{Y}(A)$;
\item [$\mathcal{Y}$] the equation $\xi A = x_0$ has a solution $\xi$ and if only if $\xi_0 x = 0$
for every $\xi \in \mathcal{Y}(A)$.
\end{enumerate}

The integers $r(A) = \min (\dim Z(A), \dim \mathcal{Y}(A))$ and $d(A) = \dim Z(A) - \dim \mathcal{Z}(A)$ will be called the *order* and the *defect* of $A$, respectively.

If $d(A) = 0$, then $A$ is said to be a *Fredholm operator*.

Let $x_0, \ldots, x_n$ and $\xi_1, \ldots, \xi_m$ be the bases of the subspaces $Z(A)$ and $\mathcal{Z}(A)$ respectively.

There exist linearly independent elements $\eta_1, \ldots, \eta_n$ such that (*)

\begin{equation}
\eta_i x_j = \delta_{ij} \quad \text{for } i, j = 1, \ldots, n
\end{equation}

and every element $x \in X$ is uniquely represented in the form

\begin{equation}
x = \eta + \sum_{i=1}^{n} a_i x_i, \quad \text{where } \eta \in \mathcal{Y}(A).
\end{equation}

Similarly there exist linearly independent elements $y_1, \ldots, y_m \in X$ such that

\begin{equation}
\xi_i y_j = \delta_{ij} \quad \text{for } i, j = 1, \ldots, m.
\end{equation}

(*) For the properties of this notion see Sikorski [10].

(*) As usual $\delta_{ij}$ means the Kronecker symbol.
and every element $x \in X$ is uniquely represented in the form

\[(4') \quad x = y' + a_1 y_1 + \ldots + a_n y_n, \quad \text{where} \quad y' \in Y(A).\]

If $X$ and $Y$ are finite-dimensional, then their dimensions are equal, and every endomorphism on $X$ is a Fredholm operator. Therefore, generalized Fredholm operators with a non-vanishing defect exist only in finite-dimensional spaces.

Now let $X$ and $Y = X^*$ be Banach spaces (1). Let us denote by $\xi \in X$ the value of a functional $\xi \in X^*$ at a point $x \in X$, and by $\mathcal{B}$ the class of all bounded endomorphisms defined on $X$.

Following Atkinson [1], one of the conditions (g) or (g') in the definition of the generalized Fredholm operator is a consequence of the remaining conditions, and therefore it can be omitted.

3. Properties of generalized Fredholm operators. Now we shall give some known properties of generalized Fredholm operators. The proof of some of them is adopted from the paper of Atkinson [1].

(i) If $A \in \mathcal{B}$ is a generalized Fredholm operator and $C \in \mathcal{B}$ has the inverse $C^{-1} \in \mathcal{B}$, then $CA$ and $AC$ are generalized Fredholm operators and $r(AC) = r(AC) = r(A)$, $d(AC) = d(AC) = d(A)$.

The proof is obvious.

(ii) Every generalized Fredholm operator $A \in \mathcal{B}$ has a quasi-inverse $B \in \mathcal{B}$. $B$ is also a generalized Fredholm operator and $r(B) = r(A)$, $d(B) = d(A)$.

To prove this statement we define two sets:

\[X = \{x: \eta \neq 0; \quad i = 1, \ldots, n; \quad x \in \mathcal{X}\}; \]

\[Y = \{x: \xi \neq 0; \quad i = 1, \ldots, m; \quad x \in \mathcal{X}\}.\]

where the elements $\eta_1, \ldots, \eta_n$ and $\xi_1, \ldots, \xi_m$ satisfy the conditions (3), (3'), and (4), (4'). It follows from the definition of $A$ that $Y = Y(A)$.

It is easy to see that the following equations hold:

\[X = \{x': \xi = x - \sum_{i=1}^{n} \eta_i x_i; \quad x \in \mathcal{X}\}; \]

\[Y = \{x': \xi = x - \sum_{i=1}^{m} \eta_i \xi_i; \quad x \in \mathcal{X}\}.\]

Let $A'$ denote the operator $A$ reduced to the subspace $X$, and for an $x \in X$, let

\[x' = x - \sum_{i=1}^{n} \eta_i x_i.\]

(1) $X^*$ means the space of all linear bounded functionals defined on $X$.

Obviously, $A' : X \to Y$, so it follows from this fact that $A'$ is a linear mapping of $X$ onto $Y$. But this mapping is not a one-to-one mapping, for otherwise, for each $x \in X$, there would exist two elements $x'$, $x'' \in X$ such that $x' \neq x''$ and $A' x' = A' x''$. Thus, $x'-x''$ is a linear combination of $\eta_1, \ldots, \eta_n$, and it belongs to $X$, which is impossible. Thus, there exists $A'^{-1}$ (the inverse of $A'$) which is a linear mapping of $X$ onto $Y$.

The mapping $B$, defined by the formula

\[Bx = A'^{-1} \left( x - \sum_{i=1}^{m} \eta_i \xi_i x_i \right),\]

is a linear mapping (1) of $X$ onto $X$. It is easily seen that $\eta_1, \ldots, \eta_n$ form a basis of the null space $Z(B)$ and that $\eta_i(Bx) = 0$ for each $x \in X$ and $i = 1, \ldots, n$.

The following formulæ hold:

\[(5) \quad AB = I - \sum_{i=1}^{n} \eta_i \xi_i; \]

\[(5') \quad BA = I - \sum_{i=1}^{m} \eta_i \xi_i.\]

Formula (5) follows immediately from the definition of the inverse. Formula (5') can be proved as follows:

\[BAX = A'^{-1} \left( AX - \sum_{i=1}^{n} \eta_i \xi_i A\right) = A'^{-1} (AX)\]

\[= A'^{-1} \left( x - \sum_{i=1}^{m} \eta_i \xi_i x_i \right) - \sum_{i=1}^{n} \eta_i \xi_i x_i.\]

Since $\eta_i(Bx) = 0$ for each $x \in X$ and $i = 1, \ldots, n$, we obtain by (5') and (5)

\[\xi(Bx) = \left( x - \sum_{i=1}^{m} \eta_i \xi_i x_i \right) = \eta x \quad \text{for each} \quad x \in X,\]

i.e., $B$ satisfies condition (1). Thus, it belongs to the class $\mathcal{B}$.

It is easy to verify that $B$ is a quasi-inverse of $A$ and that $B$ is also a generalized Fredholm operator such that $r(B) = r(A)$ and $d(B) = d(A)$. This completes the proof.

If $B \in \mathcal{B}$ is a fixed quasi-inverse of a generalized Fredholm operator $A \in \mathcal{B}$, then for every fixed basis $x_1, \ldots, x_n$, and $\xi_1, \ldots, \xi_m$ of $Z(A)$ and

(1) If $X$ is a Banach space and $A$ is bounded, then $B$ is also bounded.
There exist elements $\eta_1, \ldots, \eta_m$ and $y_1, \ldots, y_m$, uniquely determined, such that formulas \((5)', (5'')\), and \((3), (4)\) hold.

In particular, if $r(A) = 0$, $d(A) = d > 0$, then

\[ AB = I, \]
\[ BA = I - \sum_{i=1}^d a_i \cdot \eta_i. \]  

Observe that in case $r(A) = 0$ and $d(A) = d > 0$, the operator $A$ transforms the space $X$ onto $X$, and the adjoint operator $A$ transforms $\mathcal{E}$ into $\mathcal{E}$ in a one-to-one way.

Similarly, if $r(A) = 0$ and $d(A) < d$, then the operator $A$ transforms $X$ into $X$ in a one-to-one way, and the adjoint operator $A$ transforms $\mathcal{E}$ onto $\mathcal{E}$.

If $r(A) = 0$ and $d(A) = 0$, then the Fredholm operator $A$ has an inverse $A^{-1}$.

(iii) Let $z_1, \ldots, z_n$ and $\xi_1, \ldots, \xi_m$ be all linearly independent solutions of the equations $Ax = 0$ and $\xi A = 0$, respectively, and let $B \in \mathbb{K}$ be a quasi-inverse of the generalized Fredholm operator $A \in \mathbb{K}$.

Every solution of the equation

\[ Ax = x, \text{ where } x \text{ is orthogonal to } \xi_1, \ldots, \xi_m, \]

is of the form

\[ x = a_1 z_1 + \ldots + a_m z_m + B x; \]

\[ B x \text{ is the only solution of } (7) \text{ orthogonal to } \eta_1, \ldots, \eta_m. \]

Every solution of the equation

\[ \xi A = \xi, \text{ where } \xi \text{ is orthogonal to } z_1, \ldots, z_n, \]

is of the form

\[ \xi = a_1 \xi_1 + \ldots + a_m \xi_m + \xi B; \]

\[ \xi B \text{ is the only solution of } (8) \text{ orthogonal to } y_1, \ldots, y_m. \]

In fact, multiplying \((7)\) from the left side by $B$ and applying \((8)\), we obtain \((7')\). It is easy to prove (by leading to a contradiction) that $B x$ is the only solution of \((7)\) orthogonal to $\eta_1, \ldots, \eta_m$.

Similarly, multiplying \((8)\) from the right side by $B$ and applying \((5)\) we obtain \((8')\). Analogously we verify that $\xi B$ is the only solution of \((8)\) orthogonal to $y_1, \ldots, y_m$. This completes the proof.

(iv) $A \in \mathbb{K}$ is a generalized Fredholm operator if and only if there exist $B_1, B_1 \in \mathbb{K}$ and finitely dimensional operators $K_1, K_1 \in \mathbb{K}$ such that

\[ AB_1 = I - K_1, \]
\[ B_1 A = I - K_1. \]

Necessity is obvious. It is sufficient to take a quasi-inverse $B \in \mathbb{K}$ of $A$ instead of $B_1$ and $B_1$ and to apply \((5)\) and \((5')\).

Sufficiency. Let $B_1, B_2 \in \mathbb{K}$ be such that formulas \((9)\) and \((9')\) are satisfied. It follows from this that condition \((g)\) is satisfied.

To prove condition \((g)\) let us take an element $x \in \mathcal{X}$ such that $\xi x = 0$ for each $\xi \in \mathcal{E}$. Let $x_1, \ldots, x_m$ form a basis of $Y(K_1)$, and let $X_1$ be the set

\[ X_1 = \{ x \in Y(K_1) \text{ and } x \in Y(A) \}. \]

There exist linearly independent elements $\eta_1, \ldots, \eta_m$, such that

\[ K_1 = \sum_{i=1}^m x_i \cdot \eta_i. \]

Since $X_1$ is a linear subspace contained in $Y(K_1)$, the basis of $X_1$ can be denoted by $x_1, \ldots, x_n$, where $m \leq n$. By \((9), (10), (11)\), there exists an element $x \in X$ such that

\[ Ax = x - \sum_{i=1}^m x_i \cdot \eta_i. \]

The elements $\eta_i$ ($i = 1, \ldots, m$) satisfy the condition $\xi A = 0$. For otherwise there exists an element $\xi \in X$ such that $\xi X \neq 0$. Thus by \((9)\) the element $\sum_{i=1}^m x_i \cdot \eta_i$ belongs to $Y(A)$. But this is impossible because it contradicts the definition of $X_1$. Hence $Ax = x$ and the condition \((g)\) is satisfied. Proof of condition \((g')\) is analogous. This completes the proof.

Corollary. If $A_1, A_2 \in \mathbb{K}$ and products $A_1 A_2, A_2 A_1$ are generalized Fredholm operators, then $A_1$ and $A_2$ are also generalized Fredholm operators.

In fact, let $B_1$ and $B_2$ be quasi-inverses of $A_1 A_2$ and $A_2 A_1$, respectively. By \((5)\) and \((5')\) we obtain

\[ (B_1 A_1) A_2 = I - K_1 \quad \text{and} \quad A_2 (B_2 A_1) = I - K_2, \]

where $K_1$ and $K_2$ are finitely dimensional. Hence $A_1$ is a generalized Fredholm operator. The proof for $A_2$ is analogous.

It is not enough to suppose that one of the products is a generalized Fredholm operator, as the following example shows.

Let $X$ be the space of all convergent sequences $(a_n)$ and let $\mathcal{E}$ be the space $l^1$ of all sequences $(a_n)$ such that $\sum |a_n| < \infty$. We define the operators $A_1$ and $A_2$ on $X$ as follows:

\[ A_1 x = (a_1, a_2, \ldots) \quad \text{and} \quad A_2 x = (a_1, 0, a_2, 0, \ldots), \]

where $x = (a_1, a_2, \ldots)$. 

The operators $A_1$ and $A_2$ are quasi-inverses of each other, and hence are generalized Fredholm operators.
It is easy to verify that $A_1 A_2 = I$ but $d(A_1) = +\infty$ and $d(A_2) = -\infty$. Neither $A_1$ nor $A_2$ are generalized Fredholm operators.

(v) If $A_1, A_2 \in \mathcal{A}$ are generalized Fredholm operators, then $A_1 A_2$ is also a generalized Fredholm operator and $d(A_1 A_2) = d(A_1) + d(A_2)$.

The first part of the theorem follows from (5), (5'), and corollary. To prove the second part let us consider the equations: $A_1 A_2 x = 0, \xi A_2 = 0$.

Obviously, we can take for the basis of $Z(A_1 A_2)$ elements which belong either to the basis of $Z(A_1)$ or to the basis of the linear subspace $X_1 = \{x: x \in Z(A_1) \cap Z(A_2)\}$. So,

$$\text{dim} Z(A_1 A_2) = \text{dim} Z(A_1) + \text{dim} X_1.$$

Similarly, we can take for the basis of $Z(A_1 A_2)$ elements which belong either to the basis of $Z(A_1)$ or to the basis of the linear subspace $X_2 = \{x: x \in Z(A_1) \cap Z(A_2)\}$.

Thus:

$$\text{dim} Z(A_1 A_2) = \text{dim} Z(A_1) + \text{dim} X_1.$$

Let $Y_1, \ldots, Y_r$ and $\zeta_1, \ldots, \zeta_s$ denote the bases of $Z(A_1)$ and $Z(A_2)$, respectively. It follows from the definition of $X_1$ that each element $x \in X_1$ is of the form $x = \sum_{i=1}^{k} a_i Y_i$ and satisfies the condition

$$\sum_{i=1}^{k} a_i \zeta_i = 0 \quad \text{for} \quad j = 1, \ldots, s.$$

(12)

Analogously it follows from the definition of $X_2$ that each element $x \in X_2$ is of the form $x = \sum_{i=1}^{k} b_i \zeta_i$ and satisfies the condition

$$\sum_{i=1}^{k} b_i \zeta_i = 0 \quad \text{for} \quad j = 1, \ldots, n.$$

(12')

Let $r$ denote the rank of the matrix $(\zeta_i \zeta_j)_{i=1, \ldots, r, j=1, \ldots, s}$ of the system of equations (12). The rank of the analogous matrix of the system (12') is also $r$. Since $x \in X_1$ if and only if $\zeta_i x = 0$ ($i = 1, \ldots, r$) and similarly, $x \in X_2$ if and only if $\zeta_i x = 0$ ($i = 1, \ldots, s$), it is not difficult to deduce that $\text{dim} X_1 = n - r$ and $\text{dim} X_2 = k - r$. Hence $d(A_1 A_2) = d(A_1) + d(A_2)$.

This completes the proof.

(vi) If $A$ is a generalized Fredholm operator and $K$ is finitely dimensional, then $A + K$ is also a generalized Fredholm operator and

$$d(A + K) = d(A).$$

(13)

To prove that $A + K$ is a generalized Fredholm operator it is sufficient to multiply $A + K$ from the left and the right side by a quasi-inverse of $A$ and to apply (iv).

Formula (13) follows immediately from (v) and from the fact that every operator of the form $I + K S^{*}$, where $K$ is finitely dimensional, is a Fredholm operator; so we have

$$d(A + K) = d(B) = 0, \quad d(A) = d(B) = 0.$$

Hence, $d(A + K) = d(A)$.

(vii) If $A$ is a generalized Fredholm operator, then $A$ can be represented in the form $A = S + K$, where $S \in \mathcal{A}$ is a generalized Fredholm operator such that $r(S) = 0, d(S) = d(A)$, and $K \in \mathcal{A}$ is finitely dimensional.

It is sufficient to consider the case $d(A) = \delta > 0$. The proof in the case $d(A) < 0$ is analogous.

Let $x_1, \ldots, x_r$ and $\zeta_1, \ldots, \zeta_s$ be the bases of $Z(A)$ and $Z(A)$, respectively, and let us introduce the following notation:

$$L = \sum_{i=1}^{s} y_i {\cdot} \eta_i, \quad L = \sum_{i=1}^{r} \zeta_i {\cdot} \gamma_i,$$

where $\eta_i$ ($i = 1, \ldots, r$) and $y_i$ ($i = 1, \ldots, s$) have the same meaning as above.

(15)

$$K = -LL = -\sum_{i=1}^{s} y_i {\cdot} \eta_i, \quad K = -LL = -\sum_{i=1}^{r} \zeta_i {\cdot} \gamma_i.$$

It is easily seen that

(16) $AL = LA, \quad BL = LB = 0, \quad LLL = L, \quad LLL = L,$

where $B$ is a quasi-inverse of $A$.

Using (5), (5'), (15) and (16), we obtain the following formulae:

(17) $(A + B) (B + L) = I$,

(17') $(B + L) (A + L) = I - \sum_{i=1}^{s} x_i {\cdot} y_r {\cdot} x_i.$

Let us write, for brevity, $S = A + L$ and $U = B + L$. It follows from (17) and (17') that $S$ is a generalized Fredholm operator such that $r(S) = 0, d(S) = d(A)$, and that $U$ is a quasi-inverse of $S$. Further, by (16), we have

$$A = S (I + LL) = (I + LL) S.$$

Hence, $A = S + K$ where $K = SLL = LLC$.~

(10) For the properties of Fredholm operators, see Sieradski [10].
In the sequel $S$ will always denote a fixed generalized Fredholm operator such that $r(S) = 0$, $d(S) = d > 0$, and $U$ is a fixed quasi-inverse of $S$. The elements $s_1, \ldots, s_d \in X$ will always denote fixed linearly independent solutions of the equation $Su = 0$, and $a_1, \ldots, a_d$ — the solutions of $\xi U = 0$ such that $c_i a_j = a_j$ for $i, j = 1, \ldots, d$.

Suppose that $S + T$ is a generalized Fredholm operator. Observe that
\begin{equation}
    r(I + UT) = r(I + TU) = r(S + T).
\end{equation}

To prove this let us remark that $S + T = S(I + UT)$ and that $I + UT$ and $I + TU$ are Fredholm operators. Let us write $r(S + T) = r$. Then there exist linearly independent solutions of the equation $(S + T)x = 0$, say $x_1, \ldots, x_d$, which are all linearly independent solutions of the equation $(I + UT)x = 0$. The elements $Sx_1, \ldots, Sx_d$, are all linearly independent solutions of $(I + TU)x = 0$. This completes the proof.

(viii) Let $\tilde{B}$ denote an inverse of $I - UT$. Let $\tilde{e}_1, \ldots, \tilde{e}_d$ and $\tilde{e}_1, \ldots, \tilde{e}_d$ denote bases of $Z(I + UT)$ and $Z(I + TU)$, respectively.

Elements
\begin{equation}
    \tilde{e}_1, \ldots, \tilde{e}_d, B_1, \ldots, B_d
\end{equation}

are all linearly independent solutions of the equation $(S + T)x = 0$ and elements
\begin{equation}
    \tilde{e}_1 U, \ldots, \tilde{e}_d U
\end{equation}

are all linearly independent solutions of the equation $\xi (S + T) = 0$.

Let us write $e_i = \tilde{e}_i$ for $i = 1, \ldots, r$. It is easy to see that $e_1, \ldots, e_r$ are solutions of $(S + T)x = 0$ and that they are not solutions of $Sx = 0$. Obviously, there exist linearly independent solutions $x_{r+1}, \ldots, x_d$ of $(S + T)x = 0$ such that $x_j = (I + UT)x_j$ for $j = 1, \ldots, d$. Further, there exist $\eta_1, \ldots, \eta_d$ satisfying (4) and such that $B(1 + UT) = I - \sum_{i=1}^{d} \eta_i$ (see (5)). Hence, $B_i = B(1 + UK)x_{r+1} = x_j$ for $j = 1, \ldots, d$. Taking any element $\tilde{e}_j U$ of (19'), we obtain
\begin{equation}
    \tilde{e}_j U(S + K) = \tilde{e}_j \left( I - \sum_{i=1}^{d} s_i a_i + U K \right) = - \sum_{i=1}^{d} \tilde{e}_i s_i a_i \eta_i = 0
\end{equation}

because $\tilde{e}_j s_i = 0$ for $i = 1, \ldots, d$, and $j = 1, \ldots, r$. Since $\tilde{e}_i = -\tilde{e}_i U K$, it is easy to see that $\tilde{e}_j U = 0$ implies $\sum a_i \tilde{e}_j = 0$. It follows from this that the elements (19') are linearly independent. This completes the proof.

It is not yet known if in an arbitrary infinitely dimensional Banach space there exists a bounded generalized Fredholm operator $S_0$ such that $r(S_0) = 0$, $d(S_0) = d > 0$, and $U$ is a fixed quasi-inverse of $S_0$. The letters $s_1, \ldots, s_d \in X$ will always denote fixed linearly independent solutions of the equation $Su = 0$, and $a_1, \ldots, a_d$ — the solutions of $\xi U = 0$ such that $c_i a_j = a_j$ for $i, j = 1, \ldots, d$.

Suppose that $S + T$ is a generalized Fredholm operator. Observe that
\begin{equation}
    r(I + UT) = r(I + TU) = r(S + T).
\end{equation}

To prove this let us remark that $S + T = S(I + UT)$ and that $I + UT$ and $I + TU$ are Fredholm operators. Let us write $r(S + T) = r$. Then there exist linearly independent solutions of the equation $(S + T)x = 0$, say $x_1, \ldots, x_d$, which are all linearly independent solutions of the equation $(I + UT)x = 0$. The elements $Sx_1, \ldots, Sx_d$, are all linearly independent solutions of $(I + TU)x = 0$. This completes the proof.

(viii) Let $\tilde{B}$ denote an inverse of $I - UT$. Let $\tilde{e}_1, \ldots, \tilde{e}_d$ and $\tilde{e}_1, \ldots, \tilde{e}_d$ denote bases of $Z(I + UT)$ and $Z(I + TU)$, respectively.

Elements
\begin{equation}
    \tilde{e}_1, \ldots, \tilde{e}_d, B_1, \ldots, B_d
\end{equation}

are all linearly independent solutions of the equation $(S + T)x = 0$ and elements
\begin{equation}
    \tilde{e}_1 U, \ldots, \tilde{e}_d U
\end{equation}

are all linearly independent solutions of the equation $\xi (S + T) = 0$.

Let us write $e_i = \tilde{e}_i$ for $i = 1, \ldots, r$. It is easy to see that $e_1, \ldots, e_r$ are solutions of $(S + T)x = 0$ and that they are not solutions of $Sx = 0$. Obviously, there exist linearly independent solutions $x_{r+1}, \ldots, x_d$ of $(S + T)x = 0$ such that $x_j = (I + UT)x_j$ for $j = 1, \ldots, d$. Further, there exist $\eta_1, \ldots, \eta_d$ satisfying (4) and such that $B(1 + UT) = I - \sum_{i=1}^{d} \eta_i$ (see (5')). Hence, $B_i = B(1 + UK)x_{r+1} = x_j$ for $j = 1, \ldots, d$. Taking any element $\tilde{e}_j U$ of (19'), we obtain
\begin{equation}
    \tilde{e}_j U(S + K) = \tilde{e}_j \left( I - \sum_{i=1}^{d} s_i a_i + U K \right) = - \sum_{i=1}^{d} \tilde{e}_i s_i a_i \eta_i = 0
\end{equation}

because $\tilde{e}_j s_i = 0$ for $i = 1, \ldots, d$, and $j = 1, \ldots, r$. Since $\tilde{e}_i = -\tilde{e}_i U K$, it is easy to see that $\tilde{e}_j U = 0$ implies $\sum a_i \tilde{e}_j = 0$. It follows from this that the elements (19') are linearly independent. This completes the proof.

It is not yet known if in an arbitrary infinitely dimensional Banach space there exists a bounded generalized Fredholm operator $S_0$ such that $r(S_0) = 0$, $d(S_0) = d > 0$. This problem is equivalent to the following: Is every infinitely dimensional Banach space $X$ isomorphic to the Cartesian product $X \times R$, where $R$ is a straight line? If such an operator $S_0$ exists, then there exists also a generalized Fredholm operator $U$ such that $r(U) = 0$, $d(U) = -1$, and $S_0 U$ is a Fredholm operator. Then, repeatedly considering Atkinson’s results, we obtain that each bounded generalized Fredholm operator $A$ can be represented in the form
\begin{equation}
    A = O(S^0 \xi) + T
\end{equation}

provided that $d(A) > 0$, and in the form
\begin{equation}
    A = O(U^0 \xi) + T
\end{equation}

provided that $d(A) < 0$, where $T$ is a compact operator and $O^{-1}$ exists.

4. Examples of generalized Fredholm operators. Let us consider the space $X = c$ and $Z = I$ (see p. 271) and let $A$ be defined as follows:
\begin{equation}
    \xi = (a_1, a_2, \ldots), \quad A = (a_{11}, a_{12}, \ldots).
\end{equation}

It is easy to see that adjoint operator $A$ is of the form
\begin{equation}
    \xi = (a_{21}, a_{22}, \ldots), \quad A = (a_{11}, a_{12}, \ldots) \in \mathbb{N}.
\end{equation}

Obviously, the operator $A$ defined in such a way is a generalized Fredholm operator such that $r(A) = 0$ and $d(A) = -d$.

Now, let $X$ and $Z = X'$ be Banach spaces and let $\mathbb{N}$ be the ring of all bounded endomorphisms defined on $X$. As usual, $\xi$ is the value of a functional $\xi \in Z$ at a point $x \in X$.

Let $\mathfrak{T} \subset \mathbb{N}$ be an ideal consisting of operators $T \in \mathbb{N}$ such that $I + T$ are Fredholm operators. For instance, $\mathfrak{T}$ can be the ideal of all finitely dimensional operators or the ideal of all compact operators.

Now let us consider an operator $S \in \mathbb{N}$ such that $S^0 = I$ ($S$ is then said to be an isospectrum) and two operators $A, B \in \mathbb{N}$ such that the following conditions are satisfied:
\begin{equation}
    AB = BA, \quad (A + B)^{-1} \quad \text{and} \quad (A - B)^{-1} \quad \text{exist};
\end{equation}
\begin{equation}
    AS - SA \in \mathfrak{T}, \quad BS - SB \in \mathfrak{T}.
\end{equation}

Let $T \in \mathfrak{T}$. Then the operators
\begin{equation}
    A + BS + T, \quad A - BS + T, \quad A + SB + T, \quad A - SB + T,
\end{equation}

(1) See Przeworska-Rolewicz [6].
are generalized Fredholm operators such that their defects do not depend on \( T \) and satisfy the condition

\[
\delta(A + BS) = \delta(A + SB) = -\delta(A - SB) = -\delta(A - BS).
\]

To prove, e.g. that \( A + BS + T \) is a generalized Fredholm operator, it is sufficient to note that products \((A + BS + T)(A - SB)\) and \((A - SB)(A + BS + T)\) can be represented in the form \((I + T_1)(I - T_2)\) and \((I - T_2)(I + T_1)\), respectively, where \( T_1, T_2 \in \mathbb{T} \).

We can find examples of operators of this type, with a non-vanishing defect, in the theory of singular integral equations.

Let \( L \) be a closed rectifiable curve in the complex plane and \( X \) — the space of functions \( \varphi(t) \) defined on \( L \) and satisfying Hölder’s inequality on \( L \).

By Poincaré-Bertrand’s formula

\[
\frac{1}{(\pi i)^3} \int_\mathbb{R} \frac{\varphi(t)}{t - \tau} \, dt \, \frac{d\xi}{\xi} = \varphi(t) \, (\varphi(t) \ast X)
\]

(where the integral is taken in the sense of Cauchy’s main value) the linear operator \( S \) defined by the formula

\[
S\varphi(t) = \frac{1}{(\pi i)^3} \int_\mathbb{R} \frac{\varphi(t)}{t - \tau} \, dt \, \frac{d\xi}{\xi}
\]

is an involution.

Now, if the functions \( A(t), B(t) \in X \) satisfy the condition \( A^*(t)B(t) \neq 0 \), and if for \( A, B \) in (20) we substitute operators of multiplication by \( A(t) \) and \( B(t) \) respectively, then we obtain the theory of the singular integral equation

\[
A(t)\varphi(t) + B(t) \frac{\varphi(t)}{t - \tau} \, dt = f(t) \, (f(t) \ast X),
\]

where, in general, the operator \( A + BS \) has a non-vanishing defect.

5. Definition of the determinant system. Let \( E \) and \( X \) be two linear conjugate spaces and let \( \mathfrak{A} \) be the class of operators satisfying condition (b). Using the terminology of Sikorski [10] we shall understand by a determinant system (with a positive defect) for an operator \( A \in \mathfrak{A} \) every infinite sequence

\[
D_0, D_1, \ldots
\]

such that:

\((d)\) \( D_\eta \) is a \((2n + d)\)-linear functional on \( \mathbb{R}^{2n+d} \times \mathbb{X}^n \); the value of

\( D_\eta \) at the point \( (\xi_1, \ldots, \xi_{n+d}, \xi_1, \ldots, \xi_d) \) we denote by \( D_\eta(\xi_1, \ldots, \xi_{n+d}) \);

in particular, if \( n = 0 \), then \( D_\eta(\xi_1, \ldots, \xi_d) \) is a \( d \)-linear functional;

\((d)\) \( D_\eta \) is skew symmetric in \( \xi_1, \ldots, \xi_{n+d} \) and in \( x_1, \ldots, x_n \), i.e. for every permutation \( \sigma = (p_1, \ldots, p_n) \) of the integers \( 1, \ldots, n + d \) and for every permutation \( \tau = (q_1, \ldots, q_n) \) of the integers \( 1, \ldots, n \)

\[
D_\eta(\xi_{p_1}, \ldots, \xi_{p_n}, x_{q_1}, \ldots, x_{q_n}) = \text{sgn} \, \sigma \, \cdot \, D_\eta(\xi_1, \ldots, \xi_{n+d}),
\]

\[
D_\eta(\xi_{q_1}, \ldots, \xi_{q_n}, x_{p_1}, \ldots, x_{p_n}) = \text{sgn} \, \tau \, \cdot \, D_\eta(\xi_1, \ldots, \xi_{n+d}),
\]

respectively, where \( \text{sgn} \, \sigma = 1, \text{sgn} \, \tau = -1 \), if \( \sigma, \tau \) are even, and \( \text{sgn} \, \sigma = -1, \text{sgn} \, \tau = 1 \) if \( \sigma, \tau \) are odd;

\((d)\) if \( D_\eta(\xi_1, \ldots, \xi_{n+d}) \) is interpreted as a function of \( \xi_i \) only \( (1 \leq i \leq n + d) \), then there exists an element \( \zeta \in \mathbb{X} \) such that

\[
\xi_i x_i \quad \text{for every} \quad \xi \in \mathbb{X};
\]

\[(d)\] if \( D_\eta(\xi_1, \ldots, \xi_{n+d}) \) is interpreted as a function of \( x_j \) only \( (1 \leq j \leq n) \), then there exists an element \( \xi \in \mathbb{X} \) such that

\[
\xi x_j \quad \text{for every} \quad x_j \in \mathbb{X};
\]

\((d)\) there exists an integer \( r \geq 0 \) such that \( D_r \) does not vanish identically;

\((d)\) the following identities hold for \( n = 0, 1, \ldots \)

\[
(D) \quad D_{n+2}(\xi_1, \ldots, \xi_{n+d}) = \sum_{i=1}^{n+2} (-1)^i \xi_i x_i D_{n}(\xi_1, \ldots, \xi_{n+d}),
\]

\[
(D) \quad D_{n+1}(\xi_1, \ldots, \xi_{n+d}) = \sum_{i=1}^{n+d} (-1)^i \xi_i x_i D_{n}(\xi_1, \ldots, \xi_{n+d}),
\]

Analogously we define the determinant system with a negative defect. Then the number of \( \xi_i \) is larger than that of \( x_i \). The least integer \( r = r(D_\eta) \) such that \( D_r \) does not vanish identically, and the difference \( d(D_\eta) \) between the numbers of \( x_i \) and \( \xi_i \) in \( D_\eta \) is called the order and the defect of the determinant system (21) respectively.
If either $\xi_i = \xi_j$ or $a_i = a_j$ for $i \neq j$, then it follows from (d4) that
\[
D_n(\xi_1, \ldots, \xi_{n+d}) = 0,
\]
and it follows from (d4) that, for $n = 1, 2, \ldots, D_n(\xi_1, \ldots, \xi_{n+d})$, interpreted as a bilinear functional of variables $\xi_i$ and $a_i$ only, belongs to $\mathcal{U}$ (see (1)).

Analogously as in the Fredholm case (see Sikorski [14], p. 151) $D_n$ in (21) will be called the determinant of order $n$ of $A$.

The following simple remarks hold (compare with Sikorski [10]):

Remark 1. If $D_{\alpha}, D_1, \ldots$ is a determinant system for $A \in \mathfrak{U}$ and $\alpha \neq 0$, then
\[
\alpha D_{\alpha}, \alpha D_1, \ldots
\]
is also a determinant system of $A$, and
\[
D_1, \frac{1}{\alpha} D_1, \frac{1}{\alpha^2} D_1, \ldots
\]
is a determinant system for $\alpha A$.

Remark 2. If $D_{\alpha}, D_1, \ldots$ is a determinant system for $A \in \mathfrak{U}$, and $B \in \mathfrak{U}$ has the inverse $B^{-1} \in \mathfrak{U}$, then
\[
D_n(\xi B^{-1}, \ldots, \xi_{n+d} B^{-1})
\]
is a determinant system of $AB$, and
\[
D_n(B^{-1} \xi_1, \ldots, B^{-1} \xi_{n+d})
\]
is a determinant system for $BA$.

It follows from remark 1 that the determinant system for $A$, if it exists, is not uniquely determined by $A$.

(iii) If $S \in \mathcal{O}$ is a generalized Fredholm operator of order zero, $U \in \mathfrak{U}$ is a quasi-inverse of $S$ and $s_1, \ldots, s_d$ is a basis of the spaces of all solutions of the equation $S a = 0$, then the sequence $s_1, \xi_1, \ldots$ defined by the formula
\[
\theta_n(\xi_1, \ldots, \xi_d) =
\]
is a determinant system for $S$.

where $d = d(S) > 0$ and $p = n + d$ ($n = 0, 1, \ldots$), is a determinant system for $S$.

It is evident that conditions (d1), (d2), (d3), and (d4) are satisfied.

To prove the condition (d4) let us first observe that there exist points $a_1, \ldots, a_d$ such that the formulae
\[
SU = I, \quad US = I - \sum_{i=1}^{d} a_i \xi_i
\]
hold (see (6), (6')).

Then the condition (D4) follows from expanding the determinant
\[
\theta_n+1(\xi_S, \xi_1, \ldots, \xi_p) = \begin{vmatrix}
\xi_1 a_2 & \cdots & \xi_1 a_n & 0 & 0 & \cdots & 0 \\
\xi_2 a_3 & \cdots & \xi_2 a_n & \xi_1 a_1 & \xi_1 s_2 & \cdots & \xi_1 s_d \\
\xi_3 a_4 & \cdots & \xi_3 a_n & \xi_2 a_1 & \xi_2 a_2 & \cdots & \xi_2 s_d \\
\vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
\xi_p a_n & \cdots & \xi_p a_n & \xi_p a_1 & \xi_p a_2 & \cdots & \xi_p s_d
\end{vmatrix}
\]
(where $p = n + d$) in terms of its first row, and the condition (D4) follows from expanding the determinant
\[
\theta_{n+1}(\xi_1, \ldots, \xi_p) =
\]
in terms of its first column.

Similarly we can obtain the determinant system in the case $d = d(S) < 0$. For this purpose, let us set for $n = 0, 1, \ldots$
\[
\theta_n(\xi_1, \ldots, \xi_d) =
\]
where $s_1, \ldots, s_d$ is a basis of the space of all solutions of the equation $\xi S = 0$, and $U$ is a quasi-inverse of $S$. The sequence $\theta_0, \theta_1, \ldots$, just defined is a determinant system for $S$ with a negative defect.

In particular case, if $r(S) = 0$ and $d(S) = 0$, then $S$ has the only
quasi-inverse $S^{-1}$. It follows from (26) or (29) that the sequence $I_0, I_1, \ldots$, defined by the formula

$$
I_n = \begin{bmatrix}
\xi_1 S^{-1} x_1 & \cdots & \xi_n S^{-1} x_n \\
\vdots & \ddots & \vdots \\
\xi_1 S^{-1} x_n & \cdots & \xi_n S^{-1} x_n
\end{bmatrix} (n = 1, 2, \ldots),
$$

is a determinant system for $S$.

All theorems concerning operators and determinant systems with a negative defect are formulated in the same way as for operators and determinant systems with a positive defect. Therefore in the sequel, to avoid the duality of formulation, we shall only consider operators with a positive defect.

6. Fundamental theorems.

(x) If $A \in \mathbb{R}$ has a determinant system $(D_n)$, then $A$ is a generalized Fredholm operator such that $r(A) = r(D_n)$, $d(A) = d(D_n)$.

More exactly:

- If $r = r(D_n)$, $d = d(D_n) > 0$, and $\eta_1, \ldots, \eta_{r+d} \in \mathbb{R}$, $y_1, \ldots, y_r \in X$ are fixed elements such that

$$
D_n \begin{bmatrix}
\eta_1 \\
y_1 \\
\vdots \\
\eta_r \\
y_r
\end{bmatrix} \neq 0,
$$

then there exist elements $\xi_1, \ldots, \xi_r \in \mathbb{R}$ and $x_1, \ldots, x_{r+d} \in X$ such that

$$
\xi_j = \frac{D_n \begin{bmatrix}
\eta_j \\
y_1 \\
\vdots \\
\eta_r \\
y_r
\end{bmatrix}}{y_j} \quad \text{for every } j = 1, \ldots, r
$$

and

$$
\xi_j = \frac{D_n \begin{bmatrix}
\eta_j \\
y_1 \\
\vdots \\
\eta_r \\
y_r
\end{bmatrix}}{y_j} \quad \text{for every } j = r+1, \ldots, r+d
$$

The elements $\xi_1, \ldots, \xi_r$ are linearly independent and are solutions of the equation

$$
(29) \quad \xi A = 0.
$$

(*) This theorem is a slight generalization of a theorem of Sikorski [10].
(x!) If \( A \subset \mathcal{H} \) is a generalized Fredholm operator, then \( A \) has a determinant system \((D_n)\) and \( r(D_1) = r(A) \), \( d(D_n) = d(A) \). This system is determined by \( A \) uniquely up to a scalar factor \( \neq 0 \).

Suppose that \( r(A) > 0 \), \( d(A) = d \), and \( B \) is a quasi-inverse of \( A \). Let \( x_1, \ldots, x_{r+d} \), \( \xi_1, \ldots, \xi_r \) and \( y_1, \ldots, y_n, \eta_1, \ldots, \eta_r \) have the same meaning as in section 2.

Let us set

\[
\mathcal{D}_n = \begin{cases} 
0 & \text{for } n = 0, \ldots, r - 1, \\
\xi_1 x_1 \cdots \xi_r x_r & \text{for } n = r, \\
\xi_1 x_1 \cdots \xi_r x_r & \text{for } n = r + 1, \\
\vdots & \vdots \\
\xi_{r+d} x_1 \cdots \xi_{r+d} x_r & \text{for } n = r + r \end{cases}
\]

and for \( r = 1, 2, \ldots \)

\[
\mathcal{D}_{r+k} = \begin{cases} 
\xi_1 x_1 \cdots \xi_r x_r & \text{for } n = r, \\
\xi_1 x_1 \cdots \xi_r x_r & \text{for } n = r + 1, \\
\vdots & \vdots \\
\xi_{r+d} x_1 \cdots \xi_{r+d} x_r & \text{for } n = r + r \end{cases}
\]

where \( \sum \) is extended over all permutations \( p = (p_1, \ldots, p_{r+d}) \) and \( q = (q_1, \ldots, q_{r+d}) \) of the integers \( 1, \ldots, r + d + k \), and \( 1, \ldots, r + k \), respectively, such that

\[
p_1 < p_2 < \ldots < p_{r+d}, \quad q_1 < q_2 < \ldots < q_{r+d}.
\]

The sequence \( \mathcal{B}_0, \mathcal{B}_1, \ldots \), just defined, is a determinant system for \( A \). To prove this we have to verify that all conditions \((d_1), (d_4)\) in the definition of the determinant system are satisfied. Obviously \( \mathcal{D}_1 \), \( \mathcal{D}_2 \), \( \mathcal{D}_3 \), \( \mathcal{D}_4 \) satisfy conditions \((d_1), (d_4)\). Since

\[
\mathcal{D}_{r+1} = \begin{cases} 
\xi_1 x_1 \cdots \xi_r x_r & \text{for } n = r + 1, \\
\xi_1 x_1 \cdots \xi_r x_r & \text{for } n = r + 2, \\
\vdots & \vdots \\
\xi_{r+d} x_1 \cdots \xi_{r+d} x_r & \text{for } n = r + r \end{cases}
\]

\((d_4)\) is also satisfied.

\(^{(1)}\) It is easy to prove that \((2n + d)\)-linear functionals \( \mathcal{D}_n \) do not depend on the choice of \( B \).

It follows immediately from (32) that \((D_n)\) and \((D'_n)\) hold for \( n = 0, \ldots, n - 2 \). It is easy to see that they also hold for \( n = r - 1 \), because

\[
(D_{r-1} \xi_1, \ldots, \xi_r x_r) = 0
\]

if one of the points \( \xi_1, \ldots, \xi_r \) belongs to \( \mathcal{W}(A) \) or one of the points \( x_1, \ldots, x_r \) belongs to \( \mathcal{Y}(A) \).

The proof of condition \((d_4)\) for \( n > r \) is based on the formula

\[
\sum_{p} \text{sgn} \ p \sum_{q} \text{sgn} \ q \left( \begin{array}{c}
\xi_1 x_1 \cdots \xi_r x_r \\
\xi_{r+1} x_{r+1} \cdots \xi_{r+d} x_{r+d}
\end{array} \right) = \sum_{p} \text{sgn} \ p \sum_{q} \text{sgn} \ q
\]

where \( \sum \) and \( \sum \) are extended over all permutations \( p = (p_1, \ldots, p_{r+d}) \) and \( q = (q_1, \ldots, q_{r+d}) \) of the integers \( 1, \ldots, n + d \), such that

\[
p_1 < p_2 < \ldots < p_{r+d}, \quad q_1 < q_2 < \ldots < q_{r+d}.
\]

We obtain (see [10])

\[
\mathcal{D}_{r+k} = \begin{cases} 
\xi_1 x_1 \cdots \xi_r x_r & \text{for } n = r, \\
\xi_1 x_1 \cdots \xi_r x_r & \text{for } n = r + 1, \\
\vdots & \vdots \\
\xi_{r+d} x_1 \cdots \xi_{r+d} x_r & \text{for } n = r + r \end{cases}
\]
where \( p', q, p'' \) denote arbitrary permutations (of the integers 0, 1, \ldots, \nu + k\) of the form:
- \( p' = (0, p_2, \ldots, p_\nu, p_{\kappa+1}, \ldots, p_{\kappa+i+d}) \)
- \( p < p_2 < \ldots < p_\nu, \quad p_{\kappa+1} < \ldots < p_{\kappa+i+d} \)
- \( p' = (p_2, p_3, \ldots, p_\nu, 0, p_{\kappa+1}, \ldots, p_{\kappa+i+d}) \)
- \( p < p_2 < \ldots < p_\nu, \quad p_{\kappa+1} < \ldots < p_{\kappa+i+d} \)
- \( q = (q_1, q_2, \ldots, q_{\nu}, q_{\kappa+1}, \ldots, q_{\kappa+i+r}) \)
- \( g_1 < g_2 < \ldots < g_\nu, \quad g_{\kappa+1} < \ldots < g_{\kappa+i+r} \)

Hence, by (35), (36) and by the formula \( AB = I - \sum_{i=1}^\nu y_i \zeta_i \) (see (5)), we obtain

\[
\mathcal{D}_{\nu, k+d} \left( \xi_0, \zeta_0, \ldots, \xi_{\nu+k} \right) = \sum_{p'=1}^{\nu+k} \text{sgn} \ p' \text{sgn} \ q \cdot \mathcal{D}_{\nu, k+d} \left( \xi_{p'_0}, \zeta_{p'_0}, \ldots, \xi_{p'_{\nu+k}} \right) \times \mathcal{G} \left( \xi_{p'_0-1}, \ldots, \xi_{p'_{\nu+k+d}} ; \zeta_{p'_0-1}, \ldots, \zeta_{p'_{\nu+k+d}} \right)
\]

\[
= \sum_{p'=1}^{\nu+k} \text{sgn} \ p' \text{sgn} \ q \cdot \mathcal{D}_{\nu, k+d} \left( \xi_{p'_0}, \zeta_{p'_0}, \ldots, \xi_{p'_{\nu+k}} \right) \times \mathcal{G} \left( \xi_{p'_0-1}, \ldots, \xi_{p'_{\nu+k+d}} ; \zeta_{p'_0-1}, \ldots, \zeta_{p'_{\nu+k+d}} \right)
\]

where \( \sum \) is extended over all permutations \( p' = (p_1, \ldots, p_{\nu+k}) \) of the integers (1, \ldots, \nu + k) such that \( p_1 < p_2 < \ldots < p_{\nu+k} \) and all permutations \( q = (q_1, \ldots, q_{\nu+k}) \) of the integers 0, 1, \ldots, \nu + k + r such that \( q_1 < q_2 < \ldots < q_{\nu+k}, q_{\nu+k+1} < \ldots < q_{\nu+k+r} \). This proves (D). The proof of \( (D') \) is similar.

Now, we shall prove the last part of theorem (xi).

It has been proved by Sikorski (10) in the case \( d(A) = 0 \) that the determinant system for \( A \) is determined by \( A \) uniquely up to a scalar factor \( \neq 0 \). The proof in the case \( d(A) > 0 \) is analogous to that in above-mentioned paper [10]. Let \( D_0, D_1, \ldots \) be any determinant system for \( A \). We have to prove that there exists a scalar \( c \neq 0 \) such that

\[
(D_n) = c \mathcal{D}_n \quad \text{for} \quad n = 0, 1, 2, \ldots
\]

where \( \mathcal{D}_n \) is defined by (32), (33), and (34).

By theorem (x), we obtain \( r(D_0) = r(\mathcal{D}_n), \quad d(D_0) = d(\mathcal{D}_n) \), i.e.

\[
(D_n) = 0 \quad \text{for} \quad n = 0, 1, \ldots, \nu - 1.
\]
and $D_r \neq 0$. If one of the points $x_1, \ldots, x_r$ belongs to $Y(A)$ or one of the points $\xi_1, \ldots, \xi_r \in \mathcal{Q}(A)$, then it follows from (39) and (41) that

$$D_r(\xi_1, \ldots, \xi_r, \xi_{r+1}) = 0.$$

We know that the $(r+d)$-dimensional subspace $\mathcal{F}(B)$ is spanned by elements $\eta_1, \ldots, \eta_{r+d}$. Similarly, the $r$-dimensional subspace $Z(B)$ is spanned by elements $\eta_1, \ldots, \eta_r$. Each point $\xi \in \mathcal{F}(B)$ can be uniquely represented in the form $\xi = \xi^r + \xi^d$, where $\xi^r \in \mathcal{Q}(A)$ and $\xi^d \in \mathcal{F}(B)$. Analogously, each point $x \in X$ can be uniquely represented in the form $x = x^r + x^d$, where $x^r \in Y(A)$ and $x^d \in Z(B)$. By (39), we obtain

$$D_r(\xi_1, \ldots, \xi_{r+1}) = D_r(\xi^r_1, \ldots, \xi^r_r, \xi^d_{r+1})$$

for arbitrary $\xi_1, \ldots, \xi_{r+1} \in \mathcal{F}(B)$ and $x_1, \ldots, x_r \in X$. The same is true for $D_r$, i.e.,

$$D_r(\xi_1, \ldots, \xi_n) = D_r(\xi^r_1, \ldots, \xi^r_n).$$

Since two arbitrary $(2r+d)$-linear functionals defined on $Z(B)^{r+d} \times Z(B)$ are skew symmetric in $\xi^r_1, \ldots, \xi^r_{r+1}$, $\xi^d_{r+2}, \ldots, \xi^d_{r+d}$, and $\xi^d_{r+2}, \ldots, \xi^d_{r+d}$ differ only by a scalar factor, there exists a scalar $\epsilon \neq 0$ (since $D_r \neq 0$) such that

$$D_r(\xi^r_1, \ldots, \xi^r_{r+1}) = \xi^d_{r+2}, \ldots, \xi^d_{r+d}$$

for $\xi^r_1, \ldots, \xi^r_{r+1} \in \mathcal{F}(B)$ and $\xi^d_{r+2}, \ldots, \xi^d_{r+d} \in Z(B)$. It follows from this, (40), and (41), that

$$D_r(\xi_1, \ldots, \xi_n) = \xi^r_1, \ldots, \xi^r_n$$

for arbitrary $\xi_1, \ldots, \xi_n \in \mathcal{F}(B)$ and $x_1, \ldots, x_r \in X$. This and (38) proves that

$$\mathcal{Q}(A) = \{\xi^r_1, \ldots, \xi^r_n, \xi^d_{r+1}, \ldots, \xi^d_{r+d} : \eta_1, \ldots, \eta_{r+d} \in \mathcal{Q}(A)\}$$

is a determinant system for $S+T$ which does not depend on the choice of $U$.

By (18), $r(S+T) = r(I+UT)$. Let $\xi_{r+1}, \ldots, \xi_{r+n}$ be linearly independent solutions of the equations $\xi(I+UT) = 0$ and $(I+UT)\eta = 0$, respectively, and let $E$ be a quasi-inverse of $I+UT$.

It suffices to prove (xii) in the case where $(D_n)$ is defined by the formulae:

$$D_n(\xi_1, \ldots, \xi_r) = \xi_1, \ldots, \xi_r$$

for $n = 0, 1, \ldots, r-1$,

$$D_n(\xi_1, \ldots, \xi_r) = \xi_1, \ldots, \xi_r$$

for $n = 0, 1, \ldots, r-1$,

$$D_n(\xi_1, \ldots, \xi_r) = \xi_1, \ldots, \xi_r$$

for $n = 0, 1, \ldots, r-1$. 

If the sequence $\xi_1, \ldots, \xi_{n+1}$ contains one point belonging to $\mathcal{Q}(A)$, then it follows from the induction hypothesis and from the identity (D) for $D_{n+1}$ and (42), that (42) holds. If the sequence $\xi_1, \ldots, \xi_{n+1}$ contains only points $\eta_1, \ldots, \eta_{n+1}$, one of them appears at least twice. Hence, by the skew symmetry, both sides of (42) are equal to zero.

Theorem (xii) implies the following:

**Corollary.** Let $A \in \mathcal{A}$ be a generalized Fredholm operator such that $r(A) = 0$ and $d(A) = d > 0$. If $(D_n)$ is a determinant system for $A$, then

$$(43) \quad D_n(\xi_1, \ldots, \xi_n) = \xi_1, \ldots, \xi_n$$

for $n = 0, 1, \ldots$, where $B_{\pm} \in \mathcal{A}$ is a quasi-inverse of $A$ and $\sum$ is extended over all permutations $p = (p_0, \ldots, p_{n+d})$ of the integers $1, \ldots, n + d$ such that $p_1 < \ldots < p_n$,

$$(44) \quad D_n(\xi_1, \ldots, \xi_n) = \sum \xi_{p_0}, \ldots, \xi_{p_{n+1}}, \xi_{p_{n+2}}, \ldots, \xi_{p_{n+d}}$$

for $n = 0, 1, \ldots$. 

This completes the proof of Theorem (xii).
and for \( k = 1, 2, \ldots \)

\[
D_{k+1} \left( \xi_1, \ldots, \xi_{k+1} \right) = \sum_{p, q} \text{sgn } p \text{ sgn } q \begin{vmatrix} \xi_{p_1} \xi_{q_1} & \ldots & \xi_{p_1} \xi_{q_2} \\ \xi_{p_2} \xi_{q_1} & \ldots & \xi_{p_2} \xi_{q_2} \\ \vdots & \ddots & \vdots \\ \xi_{p_{k+1}} \xi_{q_1} & \ldots & \xi_{p_{k+1}} \xi_{q_{k+1}} \end{vmatrix} \cdot D_k \left( U_{p_1}, \ldots, U_{q_{k+1}} \right)
\]

where \( \sum \) is extended over all permutations \( p = (p_1, \ldots, p_{k+1}) \) and \( q = (q_1, \ldots, q_{k+1}) \) of the integers \( 1, \ldots, r+k \) such that

\[
p_1 < p_2 < \ldots < p_{k+1}, \quad p_{k+1} < p_{k+1} < \cdots < p_{k+1}, \\
q_1 < q_2 < \ldots < q_{k+1}, \quad q_{k+1} < q_{k+1} < \cdots < q_{k+1}.
\]

It follows immediately from (44) that \( D_k \) satisfies conditions (d_1), (d_2), (d_3). If \( n+d = r \), then it follows from (46) that

\[
D_k \left( \xi_1, \ldots, \xi_{k+1} \right) = \frac{1}{2^k} \prod_{j=1}^{k+1} \xi_j \left( U_{p_j}, \ldots, U_{p_{k+1}} \right) = 0,
\]

because all the solutions \( \xi_j \) of the equation \( \xi (I+UT) = 0 \) are orthogonal to all the solutions \( \xi_j \) of the equation \( \xi (S+T) = 0 \). If \( n < r \) but \( n+d > r \), then

\[
D_k \left( \xi_1, \ldots, \xi_{k+1} \right) = 0,
\]

since each term (k) of the sum (47) is equal to zero.

If \( n = r \), then by (46) and (47) we obtain

\[
D_k \left( \xi_1, \ldots, \xi_{k+1} \right) = \sum_{p, q} \text{sgn } p \text{ sgn } q \begin{vmatrix} \xi_{p_1} \xi_{q_1} & \ldots & \xi_{p_1} \xi_{q_2} \\ \xi_{p_2} \xi_{q_1} & \ldots & \xi_{p_2} \xi_{q_2} \\ \vdots & \ddots & \vdots \\ \xi_{p_{k+1}} \xi_{q_1} & \ldots & \xi_{p_{k+1}} \xi_{q_{k+1}} \end{vmatrix} \cdot D_k \left( U_{p_1}, \ldots, U_{q_{k+1}} \right)
\]

where \( \sum \) is extended over all permutations \( p \) satisfying (47'). Applying

(49) Each term of that sum contains as a factor a determinant with at least one zero column \( \xi_j \), where \( j = 1, \ldots, r \), and \( j \) is fixed.
\[ D_{a+1} \begin{pmatrix} x_1, x_2, \ldots, x_n, \xi_0, \xi_1, \ldots, \xi_{n-1}, \xi_n \end{pmatrix} \begin{pmatrix} x_1, y_1, \ldots, y_r \end{pmatrix} \]

is the only solution of the equation \( \xi A = \xi_0 \) orthogonal to \( y_1, \ldots, y_r \).

Analogously, if \( \xi_0 \) is orthogonal to \( z_1, \ldots, \zeta_r \), then the element \( x \) such that

\[ \xi x = \frac{D_{a+1} \begin{pmatrix} x_1, x_2, \ldots, x_n, \xi_0, \xi_1, \ldots, \xi_{n-1}, \xi_n \end{pmatrix} \begin{pmatrix} x_1, y_1, \ldots, y_r \end{pmatrix}}{D_{a+1} \begin{pmatrix} \xi_1, \xi_2, \ldots, \xi_n \end{pmatrix} \begin{pmatrix} y_1, y_2, \ldots, y_r \end{pmatrix}} \]

for every \( \xi \in \mathbb{E} \)

is the only solution of the equation \( Ax = x_0 \) orthogonal to \( \eta_1, \ldots, \eta_r, \omega_1, \ldots, \omega_l \), where \( \omega_1, \ldots, \omega_l \) are linearly independent solutions of the equation \( \xi U = 0 \).

Theorem (xiii) follows immediately from (x) and (xiii).

7. Analytic formulae for determinant systems. So far we have dealt with algebraic properties of determinant systems. If we know a determinant system for a generalized Fredholm operator \( A \), then we can solve the equations (see (2))

\[ \xi A = \xi_0, \quad Ax = x_0. \]

Formulæ (32), (33) and (34) defining a determinant system have no practical value from the point of view of solving equations because they are obtained by means of quasi-inverse \( B \) of \( A \) and of the solutions of the equations \( \xi A = 0, \quad Ax = 0 \). However, for a large class of generalized Fredholm operators, it is possible to give an analytic formula for determinant systems in Banach spaces. This will be done in this and the next sections.

From now on, let \( \mathbb{E} \) and \( X \) be conjugate (in the sense explained above) Banach spaces and let \( \xi x \) be a bilinear functional such that

\[ ||\xi|| = \sup_{||x|| = 1} ||\xi x||, \quad ||\xi|| = \sup_{||x|| = 1} ||\xi x|| \quad \text{for every} \quad \xi \in \mathbb{E}, \quad x \in X. \]

Suppose that \( A \in \mathbb{E} \) is bounded. It is easy to verify the equations of the norms:

\[ \sup_{||x|| = 1} ||Ax|| = \sup_{||x|| = 1} ||Ax|| = \sup_{||x|| = 1} ||Ax|| = ||A||. \]
Let us denote by \( \mathcal{M} \) the class of all linear bounded functionals \( \mathcal{F} \) on \( \mathbb{A} \) such that operators (bilinear functionals) \( T_{\mathcal{F}} \) defined by the formula \(^{(1)}\)

\[
\xi T_{\mathcal{F}} x = \mathcal{F}(\xi, x)
\]

belong to \( A \).

Obviously \( T_{\mathcal{F}} \) is also bounded, since \( \|T_{\mathcal{F}}\| \leq \|\mathcal{F}\| \).

Following Sikorski [12], elements \( \mathcal{F} \in \mathcal{M} \) will be called quasi-nuclei \(^{(2)}\).

If for an operator \( T \) there exists a quasi-nucleus \( \mathcal{F} \) such that \( T = T_{\mathcal{F}} \), then \( T \) is said to be quasi-nuclear and \( \mathcal{F} \) is said to be a quasi-nucleus of \( T \).

If \( \xi_1, \ldots, \xi_n \) and \( x_1, \ldots, x_n \) are fixed, then the quasi-nuclear \( \mathcal{F} \) defined by the formula

\[
\mathcal{F}(A) = \sum_{i=1}^{n} \xi_i \langle A \xi_i \rangle \quad \text{for every} \quad A \in \mathcal{A},
\]

will be called \textit{finitely dimensional}, and it will be denoted by

\[
\mathcal{F} = \sum_{i=1}^{n} \xi_i \otimes \xi_i.
\]

Evidently the finitely dimensional operator \( \sum_{i=1}^{n} \xi_i \otimes \xi_i \) is determined by the quasi-nucleus \( \mathcal{F} \).

In the sequel, if a quasi-nucleus \( \mathcal{F} \) is fixed, then, for brevity, instead of \( T_{\mathcal{F}} \) we shall write \( T \).

If \( \mathcal{F} \) is a quasi-nuclear and \( A \in \mathcal{M} \), then, following Leznaki [3], we shall write \( \mathcal{F}_{\mathcal{M}}(B, A) \) instead of \( \mathcal{F}(A) \).

According to this notation, formula (49) can be written in the form

\[
\xi A x = \mathcal{F}_{\mathcal{M}}(\eta A x : \xi) \quad (T = T_{\mathcal{F}}),
\]

or more generally

\[
\xi A x = \mathcal{F}_{\mathcal{M}}(\eta A x : \xi) \quad \text{where} \quad A \in A \mathcal{M}.
\]

Notice also that

\[
\|\mathcal{F}_{\mathcal{M}}(\eta A x)\| \leq \|\mathcal{F}\| \|\|A\|\| \quad \text{for every} \quad A \in \mathcal{M}.
\]

Let us consider the following expression for \( m \leq n \):

\[
\mathcal{F} \left( \xi_{1,1}, \ldots, \xi_{1,n} ; \xi_{2,1}, \ldots, \xi_{2,n} \right)
\]

\[
= \mathcal{F}_{\xi_{1,1}} \cdots \mathcal{F}_{\xi_{m,1}} \left( \xi_{1,1} U_{\xi_{1,1}} \cdots \xi_{1,m} U_{\xi_{1,m}} ; \xi_{2,1} \xi_{1,1} + \cdots + \xi_{2,m} \xi_{2,m} \right).
\]

Similarly as in paper [3], it can be proved that expression (53) is a \((n-m)+\)linear functional of the variables \( \xi_{m+1,1}, \ldots, \xi_{n+1,1}, \xi_{m+1,2}, \ldots, \xi_{n+1,n} \), whose value does not depend on the ordering of \( \mathcal{F}_{\xi_{m,1}} \). Clearly it satisfies condition (43) in the definition of the determinant system.

(xiv) Let \( S \) be a generalized Fredholm operator of order zero, let \( U \in \mathcal{M} \) be a quasi-inverse of \( S \), let \( x_1, \ldots, x_n \) be linearly independent solutions of \( S x = 0 \), and let \( \theta_{i,1}, \theta_{i,2}, \ldots \) be the determinant system for \( S \) defined by the formula (26). For any \( \mathcal{F} \in \mathcal{M} \), let

\[
D_n(\mathcal{F}) = \sum_{m=0}^{n} D_{n,m}(\mathcal{F}),
\]

where \( D_{n,m}(\mathcal{F}) \)

\[
= \frac{1}{m!} \mathcal{F}_{\eta_{1,1}} \cdots \mathcal{F}_{\eta_{n,1}} \theta_{n,1} \left( \eta_{1,1}, \eta_{1,2}, \eta_{1,3}, \ldots, \eta_{1,n} \right)
\]

\[
\left( \eta_{1,1}, \eta_{1,2}, \eta_{1,3}, \ldots, \eta_{1,n} \right)
\]

or for \( n = 0, 1, \ldots \)

Then the sequence \( D_n(\mathcal{F}), D_{n+1}(\mathcal{F}), \ldots \) is a determinant system for \( S+T_{\mathcal{F}} \) and, by fixed \( x_1, \ldots, x_n \), it does not depend on \( U \).

The proof is analogous to that of Leznaki [3].

To show that series (54) for \( n = 0, 1, \ldots \) are well defined we have first to prove that the series of norms of \((2n+d)\)-linear functionals \( D_{n,m}(\mathcal{F}) \) are convergent.

By Hadamard's inequality and by (52), we obtain

\[
\|D_{n,m}(\mathcal{F})\| \leq \sup_{m \leq n} \frac{\|D_{n,m}(\mathcal{F})\|}{\|\mathcal{F}_{\mathcal{M}}(\eta_{1,1}, \ldots, \eta_{n,1})\|}
\]

\[
= \frac{1}{m!} \sup_{m \leq n} \left( \mathcal{F}_{\eta_{1,1}} \cdots \mathcal{F}_{\eta_{n,1}} \theta_{n,1} \left( \eta_{1,1}, \eta_{1,2}, \eta_{1,3}, \ldots, \eta_{1,n} \right) \right)
\]

\[
\leq \left\| \mathcal{F} \right\| \left( \sum_{m \leq n} \sup_{m \leq n} \theta_{n,1} \left( \eta_{1,1}, \eta_{1,2}, \eta_{1,3}, \ldots, \eta_{1,n} \right) \right)
\]

\[
\leq \left\| \mathcal{F} \right\| \left( \sum_{m \leq n} \sup_{m \leq n} \theta_{n,1} \left( \eta_{1,1}, \eta_{1,2}, \eta_{1,3}, \ldots, \eta_{1,n} \right) \right)
\]

\[
= \left\| \mathcal{F} \right\| \left( \sum_{m \leq n} \sup_{m \leq n} \theta_{n,1} \left( \eta_{1,1}, \eta_{1,2}, \eta_{1,3}, \ldots, \eta_{1,n} \right) \right)
\]

\[
\leq \left\| \mathcal{F} \right\| \left( \sum_{m \leq n} \sup_{m \leq n} \theta_{n,1} \left( \eta_{1,1}, \eta_{1,2}, \eta_{1,3}, \ldots, \eta_{1,n} \right) \right)
\]

\[
\leq \left\| \mathcal{F} \right\| \left( \sum_{m \leq n} \sup_{m \leq n} \theta_{n,1} \left( \eta_{1,1}, \eta_{1,2}, \eta_{1,3}, \ldots, \eta_{1,n} \right) \right)
\]

\[
\leq \left\| \mathcal{F} \right\| \left( \sum_{m \leq n} \sup_{m \leq n} \theta_{n,1} \left( \eta_{1,1}, \eta_{1,2}, \eta_{1,3}, \ldots, \eta_{1,n} \right) \right)
\]

\[
\leq \left\| \mathcal{F} \right\| \left( \sum_{m \leq n} \sup_{m \leq n} \theta_{n,1} \left( \eta_{1,1}, \eta_{1,2}, \eta_{1,3}, \ldots, \eta_{1,n} \right) \right)
\]
Thus we have proved

\[ \|D_n\|_m \leq \frac{\|\mathcal{F}\|_m}{m!} (n + d + m)^{n-d} \|\mathcal{F}\|_m \|\mathcal{I}\|_m \|\mathcal{F}\|_m \|\mathcal{I}\|_m \cdots |x_d|. \]

It follows from this immediately that

\[ \sum_{n=0}^{\infty} \|D_n\|_m < \infty \quad \text{for} \quad n = 0, 1, \ldots \]

Clearly, multilinear functionals \( D_n(\mathcal{F}) \) \((n = 0, 1, \ldots)\) satisfy conditions \((d_1),(d_2),(d_3)\).

To prove condition \((d_4)\) let us consider the series

\[ D_n(\lambda \mathcal{F})(\alpha_1, \ldots, \alpha_d) = \sum_{n=0}^{\infty} \lambda^n D_n(\mathcal{F})(\alpha_1, \ldots, \alpha_d), \]

where \(\alpha_1, \ldots, \alpha_d\) are solutions of \(\xi U = 0\) such that \(\alpha_i \alpha_j = \delta_{ij}\) for \(i, j = 1, \ldots, d\). Series (57) is convergent for every complex \(\lambda\) and it is a holomorphic function of the variable \(\lambda\). It can also be verified that the following identity holds:

\[ \frac{d}{d\lambda} D_n(\lambda \mathcal{F})(\alpha_1, \ldots, \alpha_d) = \mathcal{F}(\alpha_1, \ldots, \mathcal{F}_{\alpha_1}, \ldots, \mathcal{F}_{\alpha_d}) D_n(\lambda \mathcal{F})(\epsilon_1, \ldots, \epsilon_n, \alpha_1, \ldots, \alpha_d), \]

where \(\mathcal{F}(\alpha_1, \ldots, \mathcal{F}_{\alpha_1}, \ldots, \mathcal{F}_{\alpha_d}) = 1\), the holomorphic function (57) is not identically equal to zero. Thus by the well-known property of holomorphic functions, there exists an integer \(\gamma \geq 0\) such that

\[ \left[ \frac{d^n}{d\lambda^n} D_n(\lambda \mathcal{F})(\alpha_1, \ldots, \alpha_d) \right]_{\lambda = 0} \neq 0. \]

Hence it follows from (58) that \(D_n(\mathcal{F}) = 0\).

Expanding the determinant \(\theta_{m+1}(\eta_1, \ldots, \eta_m, \xi_1, \xi_2, \ldots, \xi_{n+1})\) in terms of its \((m+1)\)-st column, applying the formula \(SU = I - \sum_{i=1}^m \eta_i \alpha_i\) and basic properties of determinants, we obtain

\[ \theta_{m+1}(\eta_1, \ldots, \eta_m, \xi_1, \xi_2, \ldots, \xi_{n+1}) = \sum_{\gamma, \nu=1}^{m+1} (-1)^{\nu+1} \eta_\gamma \theta_{m+1}(\eta_1, \ldots, \eta_{\gamma-1}, \eta_{\gamma+1}, \ldots, \eta_m, \xi_1, \ldots, \xi_{n+1}, \xi_{n+2}) + \sum_{\gamma, \nu=1}^{m} (-1)^{\nu+1} \xi_\nu \theta_{m+1}(\eta_1, \ldots, \eta_m, \xi_1, \ldots, \xi_{n-1}, \xi_{n+1}, \xi_{n+2}). \]

Applying the operator \(1/m! \mathcal{F}_{\eta_m} \cdots \mathcal{F}_{\eta_1} \mathcal{F}_{\xi_m} \cdots \mathcal{F}_{\xi_1}\) to both sides of (59) and performing the calculation, we obtain according to (55)

\[ D_{n+1, m}(\mathcal{F})(\xi_1, \xi_2, \ldots, \xi_{n+1}) \]

\[ = -D_{n+1, m-1}(\mathcal{F})(\xi_1, \xi_2, \ldots, \xi_{n+1}) + \sum_{\alpha=1}^{n} (-1)^{\alpha} \xi_\alpha \theta_{n+1}(\mathcal{F})(\xi_1, \xi_2, \ldots, \xi_{n+1}, \xi_{n+2}), \]

where \(D_{n+1, m-1}(\mathcal{F}) = 0\) for \(m = 0\).

Adding the terms of both sides of (60) (from \(m = 0\) up to \(\infty\)), we obtain

\[ D_{n+1}(\mathcal{F})(\xi_1, \xi_2, \ldots, \xi_{n+1}) = \]

\[ = -D_{n+1}(\mathcal{F})(\xi_1, \xi_2, \ldots, \xi_{n+1}) + \sum_{\alpha=1}^{n} (-1)^{\alpha} \xi_\alpha \theta_{n+1}(\mathcal{F})(\xi_1, \xi_2, \ldots, \xi_{n+1}, \xi_{n+2}). \]

Hence the condition \((d_4)\) is satisfied.

The proof of condition \((d_4)\) is analogous.

Expanding the determinant \(\theta_{m+1}(\eta_1, \ldots, \eta_m, \xi_1, \xi_2, \ldots, \xi_{n+1})\) in terms of its \((m+1)\)-st column, applying the formula \(US = I - \sum_{i=1}^m \eta_i \alpha_i\) and basic properties of determinants, we obtain

\[ \theta_{m+1}(\eta_1, \ldots, \eta_m, \xi_1, \xi_2, \ldots, \xi_{n+1}) = \sum_{\gamma, \nu=1}^{m+1} (-1)^{\nu+1} \eta_\gamma \theta_{m+1}(\eta_1, \ldots, \eta_{\gamma-1}, \eta_{\gamma+1}, \ldots, \eta_m, \xi_1, \ldots, \xi_{n+1}, \xi_{n+2}) + \sum_{\gamma, \nu=1}^{m} (-1)^{\nu+1} \xi_\nu \theta_{m+1}(\eta_1, \ldots, \eta_m, \xi_1, \ldots, \xi_{n-1}, \xi_{n+1}, \xi_{n+2}). \]

Applying the operator \(1/m! \mathcal{F}_{\eta_m} \cdots \mathcal{F}_{\eta_1} \mathcal{F}_{\xi_m} \cdots \mathcal{F}_{\xi_1}\) to both sides of (61) and calculating, we obtain
and \(\sum\) is extended over all sequences of non-negative integers \(i_1, \ldots, i_n\), such that \(i_1 + i_2 + \cdots + i_n = m\).

There is a formula for the determinant system (64) for \(S + T\), analogous to (63). It can be obtained from the determinant system for \(I + UT\) by means of theorem (xii).

For this purpose we introduce the following notation.

If \(\mathcal{F}\) and \(\mathcal{G}\) are \(\mathcal{H}\) or \(\mathcal{C}\), then \(\mathcal{F}\) and \(\mathcal{G}\) denote the functionals defined by the formulae (18):

\[\mathcal{F}(A) = \mathcal{G}(CA),\quad \mathcal{G}(A) = \mathcal{G}(AC)\quad \text{for all} A \in \mathcal{H}.\]

It is obvious that \(\mathcal{C}\) and \(\mathcal{G}\) are quasi-nuclear.

Observe that \(\mathcal{C}\) and \(\mathcal{G}\) determine the quasi-nuclear operators \(CT_p\) and \(T_pG\), respectively.

Let \(U\) be a quasi-inverse of \(S\). If a quasi-nuclear \(\mathcal{F}\) determines the quasi-nuclear operator \(T\), then \(\mathcal{U}\) determines the operator \(UT_p\) and the determinant system for \(I + UT\):

\[\bar{D}_n(U\mathcal{F}) = \sum_{m=0}^{n} \bar{D}_{n,m}(U\mathcal{F})\quad (n = 0, 1, \ldots),\]

where

\[\bar{D}_{n,m}(U\mathcal{F}) = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} = \begin{pmatrix} \eta_1 Y_1 \cdots \eta_1 Y_m \\ \eta_1 Y_1 \cdots \eta_1 Y_m \\ \vdots \\ \vdots \\ \eta_1 Y_1 \cdots \eta_1 Y_m \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{pmatrix} \quad (m = 0, 1, \ldots),\]

\(\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} = \begin{pmatrix} \eta_1 Y_1 \cdots \eta_1 Y_m \\ \eta_1 Y_1 \cdots \eta_1 Y_m \\ \vdots \\ \vdots \\ \eta_1 Y_1 \cdots \eta_1 Y_m \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{pmatrix} \quad (m = 0, 1, \ldots),\]

Similarly, \(\mathcal{F}U\) determines the determinant system for \(I + TU\):

\[\bar{D}_n(\mathcal{F}U) = \sum_{m=0}^{n} \bar{D}_{n,m}(\mathcal{F}U)\quad (n = 0, 1, \ldots),\]

(18) See Sikorski [14].
where

\[
(66') \quad D_{n,m}(U) \left( \xi_1, \ldots, \xi_n \middle| \eta_1, \ldots, \eta_m \right) = \frac{1}{m!} \mathcal{F} \left( U \eta_1 \cdots U \eta_m \right)
\]

\[
= \frac{1}{m!} \mathcal{F} \left( U \eta_1 \cdots U \eta_m \right)
\]

\[
= \frac{1}{m!} \mathcal{F} \left( U \eta_1 \cdots U \eta_m \right)
\]

\[
\eta_1 \xi_1 \cdots \eta_n \xi_n \quad \eta_1 \xi_1 \cdots \eta_n \xi_n
\]

\[
\eta_1 U \eta_1 \cdots \eta_n U \eta_n \quad \eta_1 U \eta_1 \cdots \eta_n U \eta_n
\]

\[
\xi_1 \xi_1 \cdots \xi_n \xi_n \quad \xi_1 \xi_1 \cdots \xi_n \xi_n
\]

\[
\xi_1 \eta_1 \cdots \xi_n \eta_n \quad \xi_1 \eta_1 \cdots \xi_n \eta_n
\]

\[
\xi_1 \eta_1 \cdots \xi_n \eta_n
\]

\[
\eta_1 \xi_1 \cdots \eta_n \xi_n
\]

\[
\eta_1 U \eta_1 \cdots \eta_n U \eta_n
\]

\[
\xi_1 \xi_1 \cdots \xi_n \xi_n
\]

\[
\xi_1 \eta_1 \cdots \xi_n \eta_n
\]

\[
\xi_1 \eta_1 \cdots \xi_n \eta_n
\]

Evidently, formula (63) for determinant systems (65) is of the same form. The operator \( T \) is replaced by \( UT \) and the numbers \( s_m \) are equal

\[
(67) \quad s_m = U \mathcal{F} \left( UT^{m-1}U \right) (m = 1, 2, \ldots).
\]

If we replace \( z_i \) by \( Uz_i \) for \( i = 1, \ldots, n \) and \( a_{i,j} \) by \( a_{i,j} \) for \( j = 1, \ldots, d \) in (65'), then we obtain the determinant system (54) for \( S+T \) (see (xii)). Hence by (63), (64), we obtain the following theorem:

(xv) The determinant system (54) for \( S+T \) satisfies the identities

\[
D_n(\mathcal{F}) \left( \xi_1, \ldots, \xi_n; a_{1,1}, \ldots, a_{n,d} \right) = D_n(\mathcal{F}) \left( \xi_1, \ldots, \xi_n; a_{1,1}, \ldots, a_{n,d} \right) (n = 0, 1, \ldots),
\]

\[
D_n(\mathcal{F}) \left( \xi_1, \ldots, \xi_n; a_{1,1}, \ldots, a_{n,d} \right) = D_n(\mathcal{F}) \left( \xi_1, \ldots, \xi_n; a_{1,1}, \ldots, a_{n,d} \right) (n = 0, 1, \ldots),
\]

\[
D_n(\mathcal{F}) \left( \xi_1, \ldots, \xi_n; a_{1,1}, \ldots, a_{n,d} \right) = D_n(\mathcal{F}) \left( \xi_1, \ldots, \xi_n; a_{1,1}, \ldots, a_{n,d} \right) (n = 0, 1, \ldots),
\]

where \( a_{m} \) are defined by (67) and \( T_m^a \) is the \((2n+d)\)-linear functional

\[
T_m^a \left( \xi_1, \ldots, \xi_n; a_{1,1}, \ldots, a_{n,d} \right) = \sum a_m \cdot \text{det}(a_{i,j}),
\]

where for \( k, \ell = 1, \ldots, n+d \)

\[
a_{k, \ell} = \begin{cases} \xi_k U \eta_{\ell} & \text{for } \ell \leq n, \\ \xi_k U \eta_{\ell-n} & \text{for } \ell > n. \end{cases}
\]

\[
b_{k, \ell} = \begin{cases} \xi_k U \eta_{\ell} & \text{for } \ell \leq n, \\ \xi_k U \eta_{\ell-n} & \text{for } \ell > n. \end{cases}
\]

and \( \sum \) is extended over all finite sequences of non-negative integers \( i_1, \ldots, i_n, j \), such that \( i_1 + \cdots + j = m \).

(xvi) The following connections between the determinant system (\( D_n(\mathcal{F}) \)) for \( S+T \) (see (54)) and the determinant systems (\( D_n(\mathcal{F}) \)), \( (D_n(\mathcal{F}) U) \) hold (see (65), (66)):

\[
D_n(\mathcal{F}) \left( \xi_1, \ldots, \xi_n; a_{1,1}, \ldots, a_{n,d} \right) = D_n(\mathcal{F}) \left( \xi_1, \ldots, \xi_n; a_{1,1}, \ldots, a_{n,d} \right) (n = 0, 1, \ldots),
\]

\[
D_n(\mathcal{F}) \left( \xi_1, \ldots, \xi_n; a_{1,1}, \ldots, a_{n,d} \right) = D_n(\mathcal{F}) \left( \xi_1, \ldots, \xi_n; a_{1,1}, \ldots, a_{n,d} \right) (n = 0, 1, \ldots),
\]

where \( a_{m} \), \( a_{m} \) are linearly independent solutions of the equation \( \xi U = 0 \) such that \( a_m \) \( a_m \) for \( i, j = 1, \ldots, d \).

These formulae can be proved by applying the identities \( SU = I \), \( US = I - \sum \xi_1 U \eta_1 \), to their left sides.

9. Determinant systems in spaces of sequences. Now we shall consider the spaces of sequences \( E \) and \( X \) of infinite sequences

\[
\xi = (\xi_1, \xi_2, \ldots) \in E, \quad x = (x_1, x_2, \ldots) \in X.
\]

We suppose:

1) bilinear functional \( \xi x \) is the ordinary scalar product of the sequences \( \xi \), \( x \), i.e.,

\[
\xi x = \sum_1 \xi_1 x_1
\]

2) the series \( \xi x \) is absolutely convergent for each \( \xi \in E \) and \( x \in X \),

3) the sequences

\[
\xi_1 = (1, 0, 0, \ldots),
\]

\[
x_1 = (0, 1, 0, \ldots),
\]

\[
\ldots,
\]

form a basis in \( E \) and in \( X \).
Each operator $A \in \mathfrak{A}$ is uniquely represented by an infinite square matrix $a = (a_{ij})$. Thus we can identify operators $A$ with the corresponding matrices $a$, writing $A = a = (a_{ij})$.

In particular, the unit operator is represented by the matrix $\delta = (\delta_{ij})$.

By a matrix quasi-nuclear we understand any quasi-nuclear operator $T$ of the form

$$\mathcal{F}(A) = \sum_{i,j=1}^{\infty} \tau_{ij} a_{ij}, \quad (\tau_{ij}) \in \mathbb{R},$$

where $\tau = (\tau_{ij})$ is an infinite square matrix. In particular,

$$\mathcal{F}(x; \tau) = \sum_{i,j=1}^{\infty} \tau_{ij} \tau_i \chi$$

for $\tau = (\tau_1, \tau_2, \ldots), \quad x = (x_1, x_2, \ldots)$.

The class of all matrix quasi-nuclear will be denoted by $\mathfrak{M}_\mathbb{R}$. If $\mathcal{F} \times \mathfrak{M}_\mathbb{R}$, then the corresponding quasi-nuclear operator will be denoted by $T$.

It is easily seen that both correspondences

$$T \rightarrow \tau, \quad \mathcal{F} \rightarrow \tau$$

are one-to-one correspondences. Consequently we can identify $T$ with $\mathcal{F}$:

$$T = \tau = \mathcal{F}.$$  

However, this identification is not extended over the norms. In general, the norm of a matrix quasi-nuclear $T = \tau$ is not equal to the norm of a quasi-nuclear operator $T = \tau$.

Let us consider the operator $S$ defined by the formula

$$Sx = (a_{d+1}^d, a_{d+2}^d, \ldots, a_{d}^d), \quad \div \mathfrak{X} \text{ for } d \in \mathfrak{X}.$$

Clearly, $S$ is a generalized Fredholm operator such that $r(S) = 0$, $d(S) = d > 0$.

The operator $U$ defined by the formula

$$Ux = (0, \ldots, 0, \mathfrak{X}, \ldots), \quad \div \mathfrak{X} \text{ for } t \mathfrak{X} \text{ such that } Sx = 0$$

is a quasi-inverse of $S$. The elements $a_{d+1}^d, \ldots, a_{d}^d$ form bases of solutions of the equations $Sx = 0$ and $t \mathfrak{X} = 0$. It is easy to see that

$$S = (\delta_{d+1}^d | \tau) \quad U = (\delta_{d+1}^d | \tau).$$

Let $S$ be fixed and let $\mathfrak{M}_\mathbb{R}$ denote the class of matrices (operators) $a \in \mathfrak{A}$ such that

$$a = S + T = s + \tau = (\delta_{d+1}^d | \tau),$$

where $\tau \in \mathfrak{M}_\mathbb{R}$.

Every matrix $a \in \mathfrak{A}$ has exactly one determinant system (54), provided the solutions $s_1, \ldots, s_n$ of the equation $Sx = 0$ are fixed. This follows from the identification of the matrix $\tau$ as a quasi-nuclear operator $T$ with the quasi-nuclear $\mathcal{F} \times \mathfrak{M}_\mathbb{R}$ and from the fact that the determinant system does not depend on $U$. Consequently we can denote the determinant system by $D_a(a), D_a(a), \ldots$ for $a \in \mathfrak{A}$.

It follows from theorem (xii) that for every $a \in \mathfrak{A}$ we can obtain the determinant system $(D(a))$ from $(D(a))$.

For this purpose let us denote by $\mathfrak{M}_\mathbb{R}$ the class of matrices (operators) $b$ such that

$$b = \delta + \tau = (\delta_{d+1}^d + \tau), \quad \tau = (\tau_{ij}) \in \mathfrak{M}_\mathbb{R}.$$

For every matrix $b \in \mathfrak{M}_\mathbb{R}$, the determinant $D_a(b)$ will be denoted by $D(b)$ or by

$$\begin{pmatrix} \beta_1 & \beta_2 & \cdots \\ \beta_1 & \beta_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

If we replace $j$-th column in the matrix $b$ by terms of a sequence $x \in \mathfrak{X}$, then the matrix $\mathfrak{X}_X \mathfrak{X}$ obtained in such a way belongs also to $\mathfrak{M}_\mathbb{R}$. Analogously, any matrix $b$ obtained from $b$ by replacing the $i$-th row by terms of a sequence $\mathfrak{X} \in \mathfrak{X}$ belongs also to $\mathfrak{M}_\mathbb{R}$.

Moreover, the determinants $D(b), D(b)$ are linear functionals of the variables $x$ and $\mathfrak{X}$, respectively.

Observe also that every column in $\beta$ is an element of $\mathbb{X}$ and every row in $\beta$ is an element of $\mathfrak{X}$.

Let $\beta \in \mathfrak{A}$ and let $t_1, \ldots, t_n$ and $j_1, \ldots, j_n$ be two finite sequences of positive integers. To quote several properties of determinant systems for $a \in \mathfrak{A}$, we define, following Sikorski [14], for every $b \in \mathfrak{A}$, the numbers $\beta(t_1, \ldots, t_n)$ as follows: if either in the sequence $i_1, \ldots, i_n$ or in $j_1, \ldots, j_n$, two of the integers are equal, then $\beta(t_1, \ldots, t_n) = 0$; otherwise $\beta(t_1, \ldots, t_n)$ is the determinant of the matrix belonging to $\mathfrak{M}_\mathbb{R}$ and obtained from the matrix $b$ by replacing the $i_1$-th column in $\beta$ by the columns $b(t_1, \ldots, t_n)$, respectively, and the $j_1$-th column in $\beta$ by the rows $b(t_1, \ldots, t_n)$, respectively, and the $j_1$-th column in $\beta$ by the rows $b(t_1, \ldots, t_n)$, respectively.

The following theorem of Sikorski [14] holds:

(14) See Sikorski [14].
For any $\beta \in \mathcal{A}$, the following formulae hold:

\[
\beta \left( \begin{array}{c} e_1, \ldots, e_n \\ f_1, \ldots, f_m \end{array} \right) = D_n(\beta) \left( \begin{array}{c} e_1, \ldots, e_n \\ f_1, \ldots, f_m \end{array} \right),
\]

\[
D_n(\beta) \left( \begin{array}{c} e_1, \ldots, e_n \\ f_1, \ldots, f_m \end{array} \right) = \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} \prod_{k=1}^{n} \alpha_{e_k, f_{i_k} f_{j_k}} \beta_{f_{i_1}, \ldots, f_{i_n}},
\]

where $\alpha = (\alpha_{e_k, f_{i_k} f_{j_k}})$ for $k = 1, \ldots, n$.

\[
\beta \left( \begin{array}{c} e_1, \ldots, e_n \\ f_1, \ldots, f_m \end{array} \right) = \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} \prod_{k=1}^{n} \alpha_{e_k, f_{i_k} f_{j_k}} \beta_{f_{i_1}, \ldots, f_{i_n}},
\]

where

\[
\beta_{e_k, f_{i_k} f_{j_k}} = \delta_{e_k, f_{i_k} f_{j_k}},
\]

Now, let us consider the matrix $\alpha = \delta + wz \mathcal{A}$, where $w$ is a quasi-inverse of $\alpha$.

\[
\beta = \delta + wz = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
\end{bmatrix},
\]

We introduce the following notation for every $\alpha \in \mathcal{A}$:

\[
\alpha \left( \begin{array}{c} i_1, \ldots, i_m \\ j_1, \ldots, j_n \end{array} \right) = \beta \left( \begin{array}{c} i_1, \ldots, i_m \\ j_1+a, \ldots, j_n+a \end{array} \right)
\]

where $\alpha = \delta + wz$, $\beta = \delta + wz$.

Applying theorem (xii) to the theorem of Sikorski, in view of $U(e_1) = \epsilon_1 \delta$, we obtain the following formulae for every $\alpha \in \mathcal{A}$:

\[
\alpha \left( \begin{array}{c} i_1, \ldots, i_m \\ j_1, \ldots, j_n \end{array} \right) = D_n(\alpha) \left( \begin{array}{c} e_1, \ldots, e_n \\ f_1, \ldots, f_m \end{array} \right) (n = 0, 1, \ldots).
\]

Hence, by (72),

\[
D_n(\alpha) \left( \begin{array}{c} i_1, \ldots, i_m \\ j_1, \ldots, j_n \end{array} \right) = \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} \prod_{k=1}^{n} \alpha_{e_k, f_{i_k} f_{j_k}} \beta_{f_{i_1}, \ldots, f_{i_n}},
\]

where $\alpha = (\alpha_{e_k, f_{i_k} f_{j_k}})$ for $k = 1, \ldots, n$ and $l = 1, \ldots, n + d$. The numbers (76) will be said to be the coordinates (14) of the $n$-subdeterminant $D_n(\alpha)$.

There are formulae for the coordinates of $D_n(\alpha)$ which follow immediately from (73), (74), and (76):

\[
\alpha \left( \begin{array}{c} i_1, \ldots, i_m \\ j_1, \ldots, j_n \end{array} \right) = \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} \alpha_{e_k, f_{i_k} f_{j_k}} \beta_{f_{i_1}, \ldots, f_{i_n}},
\]

where

\[
\alpha_{e_k, f_{i_k} f_{j_k}} = \beta_{f_{i_1}, \ldots, f_{i_n}},
\]

(14) See Sikorski [14].
$D_a(x_1, x_2, \ldots, x_d)$, where $A = a \in \mathfrak{H}$, we obtain the following formula:

$$D_a(x_1, x_2, \ldots, x_d) = \sum_{i_1, \ldots, i_d=1}^n a_{i_1} a_{i_2} \cdots a_{i_d} \frac{d}{dx} \left( x_1^{i_1}, x_2^{i_2}, \ldots, x_d^{i_d} \right).$$

It is easy to establish by (81) that the value of $D_a(x)$ at the point $(x_1, \ldots, x_d)$ is the determinant of the matrix

$$\begin{pmatrix}
\varphi_{1,1} & \varphi_{1,2} & \cdots & \varphi_{1,d} \\
\varphi_{2,1} & \varphi_{2,2} & \cdots & \varphi_{2,d} \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_{d,1} & \varphi_{d,2} & \cdots & \varphi_{d,d}
\end{pmatrix}
$$

($x_k = \varphi_{k,1}, \varphi_{k,2}, \ldots$ for $k = 1, 2, \ldots, d$)

belonging to $\mathfrak{H}$.

Thus we can write $D_a(x_1, \ldots, x_d)$ in the form

$$D_a(x_1, \ldots, x_d) = \sum_{i_1, \ldots, i_d=1}^n a_{i_1} a_{i_2} \cdots a_{i_d} \varphi_{1,i_1} \cdots \varphi_{d,i_d}.$$

(82)

or, for brevity,

$$D_a(x_1, \ldots, x_d) = D_a(\sum_{i_1=1}^d a_i \cdot x_i).$$

(83)

Using the same method as in proof of (82), we obtain

$$D_a(x_1, \ldots, x_d) = \sum_{i_1, \ldots, i_d=1}^n a_{i_1} a_{i_2} \cdots a_{i_d} \varphi_{1,i_1} \cdots \varphi_{d,i_d}.$$

(84)

where $x = (x_1, x_2, \ldots), \xi = (\xi_1, \xi_2, \ldots)$.

Let $\alpha \in \mathfrak{H}$ be of order zero and let $\xi_1, \ldots, \xi_d \in \mathfrak{H}$ be such that $D_a(\xi_1, \ldots, \xi_d) \neq 0$. It follows from theorem (x) and from (82), (84), that for every $x_0 = (x_0, x_1, x_2, \ldots) \in \mathfrak{X}$ the system of linear equations

$$\sum_{i=1}^d a_{i,j} x_i = w_i \quad (i = 1, 2, \ldots)$$

has a solution of the form

$$x = \sum_{i=1}^d c_i x_i + \xi,$$

(86)

where $c_i$ are arbitrary constants, $x_i = (x_{i,1}, x_{i,2}, \ldots)$ are linearly independent solutions of a homogeneous system, determined by the formula

$$\begin{pmatrix}
\varphi_{1,1} & \varphi_{1,2} & \cdots & \varphi_{1,i} & \cdots & \varphi_{1,d} \\
\varphi_{2,1} & \varphi_{2,2} & \cdots & \varphi_{2,i} & \cdots & \varphi_{2,d} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\varphi_{d,1} & \varphi_{d,2} & \cdots & \varphi_{d,i} & \cdots & \varphi_{d,d}
\end{pmatrix}$$

(87)

and $\xi = (x_1, x_2, \ldots)$ is the only solution of the system (85), determined by the formula

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and orthogonal to \(\xi_1, \ldots, \xi_d\).

If \(\xi = (\varphi_1, \varphi_2, \ldots) \in E\) is orthogonal to \(x_1, \ldots, x_d\), then there exists the only solution \(x = (\varphi_1, \varphi_2, \ldots)\) of the adjoint system

\[
\sum_{j=1}^{d} \varphi_j \psi_{i,j} = \psi_i \quad (i = 1, 2, \ldots)
\]

given by the formula

\[
\psi_i = \frac{1}{D_3(\xi_1, \ldots, \xi_d)}
\]

References


