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The determinant theory of generalized Fredholm operators

by

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In 1952 Leżański [3], [4] generalized the determinant theory over the linear equations with operators of Fredholm type in Banach spaces and Sikorski [8]-[14] developed and modified this theory by introducing the notion of determinant system. Independently of Leżański and Sikorski, similar theories were obtained by Ruston [6], [7] and Grothendieck [2]. However, their theories are more complicated and less general than Leżański's theory.

So far the determinant theory has been applied only to Fredholm operators. The main purpose of this paper is a generalization of this theory over a larger class of operators, called generalized Fredholm operators (see p. 267). Formulae obtained for solutions are analogous to the formulae in the Fredholm case.

The paper consists of an algebraical and an analytical part. The former contains a discussion of properties of generalized Fredholm operators and determinant systems in arbitrary linear spaces. The analytical part concerns determinant systems for operators of the form $S+T$ in Banach spaces where S is a fixed generalized Fredholm operator of order zero (see p. 292) and T is any quasi-nuclear operator (see p. 267). The analytical part can be applied in the theory of singular integral equations.

The notation used here is not traditional. It was adopted from papers of Sikorski for convience because it enables us to calculate in a simple and mechanical way.

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1. Operators. We shall consider two fixed linear spaces Ξ and X over the same real or complex field $\mathfrak{F}$. The letters ξ, η, ζ (with indices, if necessary) always denote elements of Ξ, the letters x, y, z denote elements of X, and a, b, c — scalars of $\mathfrak{F}$. Every mapping into $\mathfrak{F}$ will be called a *functional*.

Following Sikorski [10] we suppose that Ξ and X are *conjugate*, i. e. there exists a bilinear functional on $\Xi \times X$ whose value at the point (ξ, x) is denoted by ξx and which satisfies two additional conditions:

(a) if $\xi x = 0$ for every $\xi \in \Xi$, then $x = 0$;

(a′) if $\xi x = 0$ for every $x \in X$, then $\xi = 0$.

Let $\mathfrak{A}$ be the class of all endomorphisms A in X such that the following condition is satisfied:

(b) for every fixed $\xi \in \Xi$ there exists an $\eta \in \Xi$ such that $\xi(Ax) = \eta x$ for every $x \in X$.

It is easy to see that every endomorphism $A \in \mathfrak{A}$ induces an adjoint one in Ξ which will also be denoted by A, and whose value at the point ξ will be denoted by ξA. Namely ξA is the only element η satisfying (b).

By definition of ξA,

(b′) for every fixed $x \in X$ there exists a $y \in X$ such that $(\xi A)x = \xi y$ for every $\xi \in \Xi$.

Endomorphisms $A \in \mathfrak{A}$ can be interpreted as bilinear functionals on $\Xi \times X$ as follows:

$$\xi A x = \xi(Ax) = (\xi A)x. \tag{1}$$

It is obvious that $\mathfrak{A}$ is a ring. Instead of ξx, we can write $\xi I x$, where I is the identity operator.

If $\xi x = 0$, then ξ, x are said to be *orthogonal*.

For any $A \in \mathfrak{A}$ let us introduce the following notation:

$$Y(A) = \{Ax \colon x \in X\}, \qquad Z(A) = \{x \colon Ax = 0,\ x \in X\},$$

$$\mathscr{Y}(A) = \{\xi A \colon \xi \in \Xi\}, \qquad \mathscr{Z}(A) = \{\xi \colon \xi A = 0,\ \xi \in \Xi\},$$

$\dim Z$ = the algebraic dimension of a subspace Z of X or Ξ.

For fixed ξ_0 and x_0, let $x_0 \cdot \xi_0$ denote the *one-dimensional* operator $K \in \mathfrak{A}$ defined by the formula [1]

$$Kx = x_0 \cdot \xi_0 x.$$

Thus the value of its adjoint endomorphism at a point ξ is

$$\xi K = \xi x_0 \cdot \xi_0.$$

Any finite sum of one-dimensional operators

$$K = \sum_{i=1}^{n} x_i \cdot \xi_i$$

(1) $x_0 \cdot \xi_0 x$ means the product of the element x_0 by the scalar $\xi_0 x$.

is called a *finitely dimensional operator* because the sets $Y(K)$ and $\mathscr{Y}(K)$ are at most n-dimensional.

Observe that if K is a finitely dimensional operator, then so are AK and KA for every $A \in \mathfrak{A}$, and

$$AK = \sum_{i=1}^{n} Ax_i \cdot \xi_i,$$

$$KA = \sum_{i=1}^{n} x_i \cdot \xi_i A.$$

In particular, if $K' = \sum_{i=1}^{n'} x_i' \cdot \xi_i'$, then

$$KK' = \sum_{i=1}^{n} \sum_{j=1}^{n'} (\xi_i x_j') x_i \cdot \xi_j'. \tag{2}$$

An operator B is said to be a *quasi-inverse* (2) of A, if

$$ABA = A, \qquad BAB = B.$$

Clearly, if B is a quasi-inverse of A, then A is a quasi-inverse of B.

2. Definition of the generalized Fredholm operator. A linear operator (bilinear functional) $A \in \mathfrak{A}$ is said to be a *generalized Fredholm operator* if:

(g_0) $\dim Z(A) < \infty$, $\dim \mathscr{Z}(A) < \infty$;

(g) the equation $Ax = x_0$ has a solution x if and only if $\xi x_0 = 0$ for every $\xi \in \mathscr{Z}(A)$;

(g′) the equation $\xi A = \xi_0$ has a solution ξ if and only if $\xi_0 x = 0$ for every $x \in Z(A)$.

The integers $r(A) = \min(\dim Z(A), \dim \mathscr{Z}(A))$ and $d(A) = \dim Z(A) - \dim \mathscr{Z}(A)$ will be called the *order* and the *defect* of A, respectively.

If $d(A) = 0$, then A is said to be a *Fredholm operator*.

Let $z_1, \ldots, z_n$ and $\zeta_1, \ldots, \zeta_m$ be the bases of the subspaces $Z(A)$ and $\mathscr{Z}(A)$ respectively.

There exist linearly independent elements $\eta_1, \ldots, \eta_n$ such that (3)

$$\eta_i z_j = \delta_{ij} \quad \text{for} \quad i, j = 1, \ldots, n \tag{3}$$

and every element $\xi \in \Xi$ is uniquely represented in the form

$$\xi = \eta' + a_1\eta_1 + \ldots + a_n\eta_n, \quad \text{where} \quad \eta' \in \mathscr{Y}(A). \tag{3′}$$

Similarly there exist linearly independent elements $y_1, \ldots, y_m \in X$ such that

$$\zeta_i y_j = \delta_{ij} \quad \text{for} \quad i, j = 1, \ldots, m, \tag{4}$$

(2) For the properties of this notion see Sikorski [10].

(3) As usual δ_{ij} means the Kronecker symbol.

and every element $x \in X$ is uniquely represented in the form

(4') $$x = y' + a_1 y_1 + \ldots + a_m y_m, \quad \text{where} \quad y' \in Y(A).$$

If Ξ and X are conjugate and finitely dimensional, then their dimensions are equal and every endomorphism in X is a Fredholm operator. Therefore generalized Fredholm operators with a non-vanishing defect can exist only in infinitely dimensional spaces.

Now let X and $\Xi = X^*$ be Banach spaces ([4]). Let us denote by ξx the value of a functional $\xi \in \Xi$ at a point $x \in X$, and by $\mathfrak{A}$ — the class of all bounded endomorphisms defined on X.

Following Atkinson [1], one of the conditions (g) or (g') in the definition of the generalized Fredholm operator is a consequence of the remaining conditions, and therefore it can be omitted.

3. Properties of generalized Fredholm operators. Now we shall give some known properties of generalized Fredholm operators. The proof of some of them is adopted from the paper of Atkinson [1].

(i) *If $A \in \mathfrak{A}$ is a generalized Fredholm operator and $C \in \mathfrak{A}$ has the inverse $C^{-1} \in \mathfrak{A}$, then CA and AC are generalized Fredholm operators and $r(CA) = r(AC) = r(A)$, $d(CA) = d(AC) = d(A)$.*

The proof is obvious.

(ii) *Every generalized Fredholm operator $A \in \mathfrak{A}$ has a quasi-inverse $B \in \mathfrak{A}$. B is also a generalized Fredholm operator and $r(B) = r(A)$, $d(B) = -d(A)$.*

To prove this statement we define two sets:

$$X_0 = \{x\colon \eta_i x = 0;\ i = 1, \ldots, n;\ x \in X\},$$
$$Y_0 = \{x\colon \zeta_i x = 0;\ i = 1, \ldots, m;\ x \in X\},$$

where the elements $\eta_1, \ldots, \eta_n$ and $\zeta_1, \ldots, \zeta_m$ satisfy the conditions (3), (3'), and (4), (4'). It follows from the definition of A that $Y_0 = Y(A)$. It is easy to see that the following equations hold:

$$X_0 = \Big\{x'\colon x' = x - \sum_{i=1}^{n} z_i \cdot \eta_i x;\ x \in X\Big\},$$
$$Y_0 = \Big\{x'\colon x' = x - \sum_{i=1}^{m} y_i \cdot \zeta_i x;\ x \in X\Big\}.$$

Let A' denote the operator A reduced to the subspace X_0, and for an $x_0 \in Y_0$ let

$$x' = x_0 - \sum_{i=1}^{n} z_i \cdot \eta_i x_0 \in X_0.$$

([4]) X^* means the space of all linear bounded functionals defined on X.

Obviously, $A'x' = x_0$, so it follows from this immediately that A' is a linear mapping of X_0 onto Y_0. But this mapping is a one-to-one mapping, for otherwise for an element $x_0 \in Y_0$ there would exist two elements x', $x'' \in X_0$ such that $x' \neq x''$ and $A'x' = A'x'' = x_0$. Thus, $x' - x''$ is a linear combination of $z_1, \ldots, z_n$, and it belongs to X_0, which is impossible. Thus, there exists A'^{-1} (the inverse of A') which is a linear mapping of X_0 onto Y_0.

The mapping B, defined by the formula

$$Bx = A'^{-1}\Big(x - \sum_{i=1}^{m} y_i \cdot \zeta_i x\Big),$$

is a linear mapping ([5]) of X onto X_0. It is easily seen that $y_1, \ldots, y_m$ form a basis of the null space $Z(B)$ and that $\eta_i(Bx) = 0$ for each $x \in X$ and $i = 1, \ldots, n$.

The following formulae hold:

(5) $$AB = I - \sum_{i=1}^{m} y_i \cdot \zeta_i,$$

(5') $$BA = I - \sum_{i=1}^{n} z_i \cdot \eta_i.$$

Formula (5) follows immediately from the definition of the inverse. Formula (5') can be proved as follows:

$$BAx = A'^{-1}\Big(Ax - \sum_{i=1}^{m} y_i \cdot \zeta_i Ax\Big) = A'^{-1}(Ax)$$
$$= A'^{-1}A\Big(x - \sum_{i=1}^{n} z_i \cdot \eta_i x\Big) = x - \sum_{i=1}^{n} z_i \cdot \eta_i x.$$

Since $\eta_i(Bx) = 0$ for every $x \in X$ and $i = 1, \ldots, n$, we obtain by (3') and (5)

$$\xi(Bx) = \Big(\eta' + \sum_{i=1}^{n} a_i \eta_i\Big) Bx = \Big(\xi - \sum_{i=1}^{m} \bar{\xi}_i y_i \cdot \zeta_i\Big) x = \eta x \quad \text{for every} \quad x \in X,$$

i. e. B satisfies condition (b). Thus, it belongs to the class $\mathfrak{A}$.

It is easy to verify that B is a quasi-inverse of A and that B is a generalized Fredholm operator such that $r(B) = r(A)$ and $d(B) = -d(A)$. This completes the proof.

If $B \in \mathfrak{A}$ is a fixed quasi-inverse of a generalized Fredholm operator $A \in \mathfrak{A}$, then for every fixed basis $z_1, \ldots, z_n$, and $\zeta_1, \ldots, \zeta_m$, of $Z(A)$ and

([5]) If X is a Banach space and A is bounded, then B is also bounded.

$\mathscr{X}(A)$ respectively, there exist elements $\eta_1, \ldots, \eta_n$, and $y_1, \ldots, y_m$, uniquely determined, such that formulae (5), (5′), and (3), (4) hold.

In particular, if $r(A) = 0$, $d(A) = d > 0$, then

(6) $$AB = I,$$

(6′) $$BA = I - \sum_{i=1}^{d} z_i \cdot \eta_i.$$

Observe that in case $r(A) = 0$ and $d(A) = d > 0$, the operator A transforms the space X onto X, and the adjoint operator A transforms Ξ into Ξ in a one-to-one way.

Similarly, if $r(A) = 0$ and $d(A) < 0$, then the operator A transforms X into X in a one-to-one way, and the adjoint operator A transforms Ξ onto Ξ.

If $r(A) = 0$ and $d(A) = 0$, then the Fredholm operator A has an inverse A^{-1}.

(iii) *Let $z_1, \ldots, z_n$ and $\zeta_1, \ldots, \zeta_m$ be all linearly independent solutions of the equations $Ax = 0$ and $\xi A = 0$, respectively, and let $B \in \mathfrak{A}$ be a quasi-inverse of the generalized Fredholm operator $A \in \mathfrak{A}$.*

Every solution of the equation

(7) $$Ax = x_0, \quad \textit{where } x_0 \textit{ is orthogonal to } \zeta_1, \ldots, \zeta_m,$$

is of the form

(7′) $$x = a_1 z_1 + \ldots + a_n z_n + Bx_0;$$

Bx_0 is the only solution of (7) orthogonal to $\eta_1, \ldots, \eta_n$.

Every solution of the equation

(8) $$\xi A = \xi_0, \quad \textit{where } \xi_0 \textit{ is orthogonal to } z_1, \ldots, z_n,$$

is of the form

(8′) $$\xi = a_1 \zeta_1 + \ldots + a_m \zeta_m + \xi_0 B;$$

$\xi_0 B$ is the only solution of (8) orthogonal to $y_1, \ldots, y_m$.

In fact, multiplying (7) from the left side by B and applying (5′), we obtain (7′). It is easy to prove (by leading to a contradiction) that Bx_0 is the only solution of (7) orthogonal to $\eta_1, \ldots, \eta_n$.

Similarly, multiplying (8) from the right side by B and applying (5) we obtain (8′). Analogously we verify that $\xi_0 B$ is the only solution of (8) orthogonal to $y_1, \ldots, y_m$. This completes the proof.

(iv) *$A \in \mathfrak{A}$ is a generalized Fredholm operator if and only if there exist $B_1, B_2 \in \mathfrak{A}$ and finitely dimensional operators $K_1, K_2 \in \mathfrak{A}$ such that*

(9) $$AB_1 = I - K_1,$$

(9′) $$B_2 A = I - K_2.$$

Necessity is obvious. It is sufficient to take a quasi-inverse $B \in \mathfrak{A}$ of A instead of B_1 and B_2 and to apply (5) and (5′).

Sufficiency. Let $B_1, B_2 \in \mathfrak{A}$ be such that formulae (9) and (9′) are satisfied. It follows from this that condition (g_0) is satisfied.

To prove condition (g) let us take an element $x_0 \in X$ such that $\xi x_0 = 0$ for each $\xi \in \mathscr{X}(A)$. Let $x_1, \ldots, x_n$ form a basis of $Y(K_1)$, and let X_1 be the set

(10) $$X_1 = \{x \colon x \in Y(K_1) \text{ and } x \notin Y(A)\}.$$

There exist linearly independent elements $\bar{\eta}_1, \ldots, \bar{\eta}_n$, such that

(11) $$K_1 = \sum_{i=1}^{n} x_i \cdot \bar{\eta}_i.$$

Since X_1 is a linear subspace contained in $Y(K_1)$, the basis of X_1 can be denoted by $x_1, \ldots, x_m$, where $m \leqslant n$. By (9), (10), and (11), there exists an element $x \in X$ such that

$$Ax = x_0 - \sum_{i=1}^{m} x_i \cdot \bar{\eta}_i x_0.$$

The elements $\bar{\eta}_i$ $(i = 1, \ldots, m)$ satisfy the condition $\bar{\eta}_i A = 0$. For otherwise there exists an element $\bar{x} \in X$ such that $A\bar{x} \neq 0$. Thus by (9) the element $\sum_{i=1}^{m} x_i \cdot \bar{\eta}_i A \bar{x}$ belongs to $Y(A)$. But this is impossible because it contradicts the definition of X_1. Hence $Ax = x_0$ and the condition (g) is satisfied. Proof of condition (g′) is analogous. This completes the proof.

Corollary. *If $A_1, A_2 \in \mathfrak{A}$ and products $A_1 A_2$, $A_2 A_1$ are generalized Fredholm operators, then A_1 and A_2 are also generalized Fredholm operators.*

In fact, let B_1 and B_2 be quasi-inverses of $A_1 A_2$ and $A_2 A_1$, respectively. By (5) and (5′) we obtain

$$(B_1 A_1) A_2 = I - K_1 \quad \text{and} \quad A_2 (A_1 B_2) = I - K_2,$$

where K_1 and K_2 are finitely dimensional. Hence A_2 is a generalized Fredholm operator. The proof for A_1 is analogous.

It is not enough to suppose that one of the products is a generalized Fredholm operator, as the following example shows.

Let X be the space $\boldsymbol{c}$ of all convergent sequences $\{a_n\}$ and let Ξ be the space $\boldsymbol{l}$ of all sequences $\{a_n\}$ such that $\sum_{i=1}^{\infty} |a_n| < \infty$. We define the operators A_1 and A_2 on X as follows:

$$A_1 x = (a_1 a_3, \ldots) \quad \text{and} \quad A_2 x = (a_1, 0, a_2, 0, \ldots),$$

where $x = (a_1, a_2, \ldots)$.

It is easy to verify that $A_1A_2 = I$ but $d(A_1) = +\infty$ and $d(A_2) = -\infty$. Neither A_1 nor A_2 are generalized Fredholm operators.

(v) *If $A_1, A_2 \in \mathfrak{A}$ are generalized Fredholm operators, then A_1A_2 is also a generalized Fredholm operator and $d(A_1A_2) = d(A_1)+d(A_2)$.*

The first part of the theorem follows from (5), (5′), and corollary.

To prove the second part let us consider the equations: $A_1A_2x = 0$, $\xi A_1A_2 = 0$.

Obviously, we can take for the basis of $Z(A_1A_2)$ elements which belong either to the basis of $Z(A_2)$ or to the basis of the linear subspace $X_1 = \{x\colon xY(A_2)\cap Z(A_1)\}$. So,

$$\dim Z(A_1A_2) = \dim Z(A_2)+\dim X_1.$$

Similarly, we can take for the basis of $\mathscr{Z}(A_1A_2)$ elements which belong either to the basis of $\mathscr{Z}(A_1)$ or to the basis of the linear subspace $\Xi_1 = \{\xi\colon \mathscr{Y}(A_1)\cap\mathscr{Z}(A_2)\}$.

Thus:

$$\dim\mathscr{Z}(A_1A_2) = \dim\mathscr{Z}(A_1)+\dim\Xi_1.$$

Let $z_1, \ldots, z_n$ and $\bar\zeta_1, \ldots, \bar\zeta_k$ denote the bases of $Z(A_1)$ and $\mathscr{Z}(A_2)$, respectively. It follows from the definition of X_1 that each element $x \in X_1$ is of the form $x = \sum_{i=1}^{n} a_i z_i$ and satisfies the condition

$$\sum_{i=1}^{n} a_i\bar\zeta_j z_i = 0 \quad \text{for} \quad j = 1, \ldots, k. \tag{12}$$

Analogously it follows from the definition of Ξ_1 that each element $\xi\in\Xi_1$ is of the form $\xi = \sum_{i=1}^{k} b_i\bar\zeta_i$ and satisfies the condition

$$\sum_{i=1}^{k} b_i\bar\zeta_i z_j = 0 \quad \text{for} \quad j = 1, \ldots, n. \tag{12′}$$

Let r denote the rank of the matrix $(\bar\zeta_j z_i)_{\substack{j=1,\ldots,k\\ i=1,\ldots,n}}$ of the system of equations (12). The rank of the analogous matrix of the system (12′) is also r. Since $z_i \in X_1$ if and only if $\bar\zeta_j z_i = 0$ $(j = 1, \ldots, k)$ and similarly, $\bar\zeta_i \in \Xi_1$ if and only if $\bar\zeta_i z_j = 0$ $(j = 1, \ldots, n)$, it is not difficult to deduce that $\dim X_1 = n-r$ and $\dim\Xi = k-r$. Hence $d(A_1A_2) = d(A_1)+d(A_2)$. This completes the proof.

(vi) *If A is a generalized Fredholm operator and K is finitely dimensional, then $A+K$ is also a generalized Fredholm operator and*

$$d(A+K) = d(A). \tag{13}$$

To prove that $A+K$ is a generalized Fredholm operator it is sufficient to multiply $A+K$ from the left and the right side by a quasi-inverse of A and to apply (iv).

Formula (13) follows immediately from (v) and from the fact that every operator of the form $I+K_0$ (6), where K_0 is finitely dimensional, is a Fredholm operator; so we have

$$d(A+K)+d(B) = 0, \qquad d(A)+d(B) = 0.$$

Hence, $d(A+K) = d(A)$.

(vii) *If A is a generalized Fredholm operator, then A can be represented in the form $A = S+K$, where $S\in\mathfrak{A}$ is a generalized Fredholm operator such that $r(S) = 0$, $d(S) = d(A)$, and $K\in\mathfrak{A}$ is finitely dimensional.*

It is sufficient to consider the case $d(A) = d > 0$. The proof in the case $d(A) < 0$ is analogous.

Let $z_1, \ldots, z_{r+d}$ and $\zeta_1, \ldots, \zeta_r$ be the bases of $Z(A)$ and $\mathscr{Z}(A)$, respectively, and let us introduce the following notation:

$$L = \sum_{i=1}^{r} y_i\cdot\eta_i, \qquad \mathit{Ł} = \sum_{i=1}^{r} z_i\cdot\zeta_i, \tag{14}$$

where η_i $(i = 1, \ldots, r+d)$ and y_i $(i = 1, \ldots, r)$ have the same meaning as above;

$$K_1 = -L\mathit{Ł} = -\sum_{i=1}^{r} y_i\cdot\zeta_i, \qquad K_2 = -\mathit{Ł}L = -\sum_{i=1}^{r} z_i\cdot\eta_i. \tag{15}$$

It is easily seen that

$$A\mathit{Ł} = \mathit{Ł}A, \quad BL = LB = 0, \quad L\mathit{Ł}L = L, \quad \mathit{Ł}L\mathit{Ł} = \mathit{Ł}, \tag{16}$$

where B is a quasi-inverse of A.

Using (5), (5′), (15) and (16), we obtain the following formulae:

$$(A+L)(B+\mathit{Ł}) = I, \tag{17}$$

$$(B+\mathit{Ł})(A+L) = I-\sum_{j=1}^{d} z_{r+j}\cdot\eta_{r+j}. \tag{17′}$$

Let us write, for brevity, $S = A+L$ and $U = B+\mathit{Ł}$. It follows from (17) and (17′) that S is a generalized Fredholm operator such that $r(S) = 0$, $d(S) = d(A)$, and that U is a quasi-inverse of S. Further, by (16), we have

$$A = S(I+\mathit{Ł}L) = (I+L\mathit{Ł})S.$$

Hence, $A = S+K$ where $K = S\mathit{Ł}L = L\mathit{Ł}S$.

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(6) For the properties of Fredholm operators, see Sikorski [10].

In the sequel S will always denote a fixed generalized Fredholm operator such that $r(S) = 0$, $d(S) = d > 0$, and U is a fixed quasi-inverse of S. The letters $s_1, \ldots, s_d \in X$ will always denote fixed linearly independent solutions of the equation $Sx = 0$, and $\omega_1, \ldots, \omega_d$ — the solutions of $\xi U = 0$ such that $\omega_i s_j = \delta_{i,j}$ for $i, j = 1, \ldots, d$.

Suppose that $S+T$ is a generalized Fredholm operator. Observe that

(18) $$r(I+UT) = r(I+TU) = r(S+T).$$

To prove this let us remark that $S+T = S(I+UT)$ and that $I+UT$ and $I+TU$ are Fredholm operators. Let us write $r(S+T) = r$. Then there exist linearly independent solutions of the equation $(S+T)x = 0$, say $z_1, \ldots, z_r$, which are all linearly independent solutions of the equation $(I+UT)x = 0$. The elements $Sz_1, \ldots, Sz_r$ are all linearly independent solutions of $(I+TU)x = 0$. This completes the proof.

(viii) *Let $\overline{B}$ denote a quasi-inverse of $I+UT$. Let $\bar{z}_1, \ldots, \bar{z}_r$ and $\bar{\zeta}_1, \ldots, \bar{\zeta}_r$ denote bases of $Z(I+UT)$ and $\mathscr{Z}(I+UT)$, respectively.*

Elements

(19) $$\bar{z}_1, \ldots, \bar{z}_r, \overline{B}s_1, \ldots, \overline{B}s_d$$

are all linearly independent solutions of the equation $(S+T)x = 0$ and elements

(19′) $$\bar{\zeta}_1 U, \ldots, \bar{\zeta}_r U$$

are all linearly independent solutions of the equation $\xi(S+T) = 0$.

Let us write $z_i = \bar{z}_i$ for $i = 1, \ldots, r$. It is easy to see that $z_1, \ldots, z_r$ are solutions of $(S+T)x = 0$ and that they are not solutions of $Sx = 0$. Obviously, there exist linearly independent solutions $z_{r+1}, \ldots, z_{r+d}$ of $(S+T)x = 0$ such that $s_j = (I+UT)z_{r+j}$ for $j = 1, \ldots, d$. Further, there exist $\eta_1, \ldots, \eta_{r+d}$ satisfying (4) and such that $B(I+UT) = I - \sum_{i=1}^{r} z_i \cdot \eta_i$ (see (5′)). Hence, $\overline{B}s_j = \overline{B}(I+UK)z_{r+j} = z_{r+j}$ for $j = 1, \ldots, d$. Taking any element $\bar{\zeta}_j U$ of (19′), we obtain

$$\bar{\zeta}_j U(S+K) = \bar{\zeta}_j\Big(I - \sum_{i=1}^{d} s_i \cdot \omega_i + UK\Big) = -\sum_{i=1}^{d} \bar{\zeta}_j s_i \cdot \omega_i = 0$$

because $\bar{\zeta}_j s_i = 0$ ($i = 1, \ldots, d$, and $j = 1, \ldots, r$). Since $\bar{\zeta}_i = -\bar{\zeta}_i UK$, it is easy to see that $\sum_{i=1}^{r} a_i \bar{\zeta}_i U = 0$ implies $\sum_{i=1}^{r} a_i \bar{\zeta}_i = 0$. It follows from this that the elements (19′) are linearly independent. This completes the proof.

It is not yet known if in an arbitrary infinitely dimensional Banach space there exists a bounded generalized Fredholm operator S_0 such that $r(S_0) = 0$, $d(S_0) = 1$. This problem is equivalent to the following. Is every infinitely dimensional Banach space X isomorphic to the Cartesian product $X \times R$, where R is a straight line? If such an operator S_0 exists, then there exists also a generalized Fredholm operator U_0 such that $r(U_0) = 0$, $d(U_0) = -1$, and $S_0 U_0 = I$. Then, repeating completely considerations of Atkinson [1], we obtain that each bounded generalized Fredholm operator A can be represented in the form

$$A = C(S_0^{d(A)} + T)$$

provided that $d(A) > 0$, and in the form

$$A = C(U_0^{-d(A)} + T)$$

provided that $d(A) < 0$, where T is a compact operator and C^{-1} exists.

4. Examples of generalized Fredholm operators. Let us consider the space $X = \boldsymbol{c}$ and $\Xi = \boldsymbol{l}$ (see p. 271) and let A be defined as follows:

$$\text{if } x = (a_1, a_2, \ldots) \in X, \quad \text{then} \quad Ax = (a_{d+1}, a_{d+2}, \ldots).$$

It is easy to see that adjoint operator A is of the form

$$\xi A = (0, \ldots, 0, a_1, a_2, \ldots), \quad \text{where} \quad \xi = (a_1, a_2, \ldots) \in \Xi.$$

Obviously, the operator A defined in such a way is a generalized Fredholm operator such that $r(A) = 0$ and $d(A) = d$.

Now, let X and $\Xi = X^*$ be Banach spaces and let $\mathfrak{R}$ be the ring of all bounded endomorphisms defined on X. As usual, ξx is the value of a functional $\xi \in \Xi$ at a point $x \in X$.

Let $\mathfrak{J} \subset \mathfrak{R}$ be an ideal consisting of operators $T \in \mathfrak{R}$ such that $I+T$ are Fredholm operators. For instance, $\mathfrak{J}$ can be the ideal of all finitely dimensional operators or the ideal of all compact operators.

Now let us consider an operator $S \in \mathfrak{R}$ such that $S^2 = I$ (S is then said to be an *involution*) and two operators $A, B \in \mathfrak{R}$ such that the following conditions are satisfied:

$$AB = BA, \quad (A+B)^{-1} \quad \text{and} \quad (A-B)^{-1} \text{ exist};$$

$$AS - SA \in \mathfrak{J}, \quad BS - SB \in \mathfrak{J}.$$

Let $T \in \mathfrak{J}$. Then the operators [7]

(20) $$A+BS+T, \quad A-SB+T, \quad A+SB+T, \quad A-BS+T,$$

[7] See Przeworska-Rolewicz [5].

are generalized Fredholm operators such that their defects do not depend on T and satisfy the condition

$$d(A+BS) = d(A+SB) = -d(A-SB) = -d(A-BS).$$

To prove, e. g. that $A+BS+T$ is a generalized Fredholm operator, it is sufficient to note that products $(A+BS+T)(A-SB)$ and $(A-SB)(A+BS+T)$ can be represented in the form $(I+T_1)(A^2-B^2)$ and $(A^2-B^2)(I+T_2)$, respectively, where $T_1, T_2 \in \mathfrak{I}$.

We can find examples of operators of this type, with a non-vanishing defect, in the theory of singular integral equations.

Let L be a closed rectifiable curve in the complex plane and X — the space of functions $\varphi(t)$ defined on L and satisfying Hölder's inequality on L.

By Poincaré-Bertrand's formula

$$\frac{1}{(\pi i)^2}\int_L \frac{d\tau}{\tau-t}\int_L \frac{\varphi(\xi)}{\xi-\tau}\,d\xi = \varphi(t) \qquad (\varphi(t)\in X)$$

(where the integral is taken in the sence of Cauchy's main value) the linear operator S defined by the formula

$$S\varphi(t) = \frac{1}{\pi i}\int_L \frac{\varphi(\tau)}{\tau-t}\,d\tau$$

is an involution.

Now, if the functions $A(t), B(t)\in X$ satisfy the condition $A^2(t)-B^2(t) \neq 0$, and if for A, B, in (20) we substitute operators of multiplication by $A(t)$ and $B(t)$ respectively, then we obtain the theory of the singular integral equation

$$A(t)\varphi(t)+\frac{B(t)}{\pi i}\int_L \frac{\varphi(\tau)}{\tau-t}\,d\tau = f(t) \qquad (f(t)\in X),$$

where, in general, the operator $A+BS$ has a non-vanishing defect.

5. Definition of the determinant system. Let Ξ and X be two linear conjugate spaces and let $\mathfrak{A}$ be the class of operators satisfying condition (b). Using the terminology of Sikorski [10] we shall understand by a *determinant system (with a positive defect) for an operator* $A\in\mathfrak{A}$ every infinite sequence

(21) $$D_0, D_1, \ldots$$

such that:

(d_1) D_n is a $(2n+d)$-linear functional on $\Xi^{n+d}\times X^n$; the value of D_n at the point $(\xi_1,\ldots,\xi_{n+d},x_1,\ldots,x_n)$ we denote by $D_n\begin{pmatrix}\xi_1,\ldots,\xi_{n+d}\\ x_1,\ldots,x_n\end{pmatrix}$; in particular, if $n=0$, then $D_0(\xi_1,\ldots,\xi_d)$ is a d-linear functional;

(d_2) $D_n\begin{pmatrix}\xi_1,\ldots,\xi_{n+d}\\ x_1,\ldots,x_n\end{pmatrix}$ is skew symmetric in $\xi_1,\ldots,\xi_{n+d}$ and in $x_1,\ldots,x_n$, i. e. for every permutation $\mathfrak{p}=(p_1,\ldots,p_{n+d})$ of the integers $1,\ldots,n+d$ and for every permutation $\mathfrak{q}=(q_1,\ldots,q_n)$ of the integers $1,\ldots,n$

$$D_n\begin{pmatrix}\xi_{p_1},\ldots,\xi_{p_{n+d}}\\ x_1,\ldots,x_n\end{pmatrix} = \operatorname{sgn}\mathfrak{p}\cdot D_n\begin{pmatrix}\xi_1,\ldots,\xi_{n+d}\\ x_1,\ldots,x_n\end{pmatrix},$$

$$D_n\begin{pmatrix}\xi_1,\ldots,\xi_{n+d}\\ x_{q_1},\ldots,x_{q_n}\end{pmatrix} = \operatorname{sgn}\mathfrak{q}\cdot D_n\begin{pmatrix}\xi_1,\ldots,\xi_{n+d}\\ x_1,\ldots,x_n\end{pmatrix},$$

respectively, where $\operatorname{sgn}\mathfrak{p}=1$, $\operatorname{sgn}\mathfrak{q}=1$, if $\mathfrak{p},\mathfrak{q}$ are even, and $\operatorname{sgn}\mathfrak{p}=-1$, $\operatorname{sgn}\mathfrak{q}=-1$ if $\mathfrak{p},\mathfrak{q}$ are odd;

(d_3) if $D_n\begin{pmatrix}\xi_1,\ldots,\xi_{n+d}\\ x_1,\ldots,x_n\end{pmatrix}$ is interpreted as a function of ξ_i only $(1\leqslant i\leqslant n+d)$, then there exists an element $z_i\in X$ such that

$$D_n\begin{pmatrix}\xi_1,\ldots,\xi_{n+d}\\ x_1,\ldots,x_n\end{pmatrix} = \xi_i z_i \quad \text{for every} \quad \xi_i\in\Xi;$$

if $D_n\begin{pmatrix}\xi_1,\ldots,\xi_{n+d}\\ x_1,\ldots,x_n\end{pmatrix}$ is interpreted as a function of x_j only $(1\leqslant j\leqslant n)$, then there exists an element $\zeta_j\in\Xi$ such that

$$D_n\begin{pmatrix}\xi_1,\ldots,\xi_{n+d}\\ x_1,\ldots,x_n\end{pmatrix} = \zeta_j x_j \quad \text{for every} \quad x_j\in X;$$

(d_4) there exists an integer $r\geqslant 0$ such that D_r does not vanish identically;

(d_5) the following identities hold for $n=0,1,\ldots$:

$$(\mathrm{D}_n)\quad D_{n+1}\begin{pmatrix}\xi_0 A,\xi_1,\ldots,\xi_{n+d}\\ x_0,x_1,\ldots,x_n\end{pmatrix} = \sum_{i=0}^{n}(-1)^i\xi_0 x_i\cdot D_n\begin{pmatrix}\xi_1,\ldots\ldots\ldots,\xi_{n+d}\\ x_0,\ldots,x_{i-1},x_{i+1},\ldots,x_n\end{pmatrix},$$

$$(\mathrm{D}_n')\quad D_{n+1}\begin{pmatrix}\xi_0,\xi_1,\ldots,\xi_{n+d}\\ Ax_0,x_1,\ldots,x_n\end{pmatrix} = \sum_{i=0}^{n+d}(-1)^i\xi_i x_0\cdot D_n\begin{pmatrix}\xi_0,\ldots,\xi_{i-1},\xi_{i+1},\ldots,\xi_{n+d}\\ x_1,\ldots\ldots\ldots,x_n\end{pmatrix}.$$

Analogously we define the determinant system with a negative defect. Then the number of x_i is larger than that of ξ_i. The least integer $r=r\{D_n\}$ such that D_r does not vanish identically, and the difference $d\{D_n\}$ between the numbers of x_i and ξ_i in D_n is called the *order* and the *defect* of the determinant system (21) respectively.

If either $\xi_i = \xi_j$ or $x_i = x_j$ for $i \neq j$, then it follows from (d$_2$) that

$$D_n\begin{pmatrix}\xi_1, \ldots, \xi_{n+d}\\ x_1, \ldots, x_n\end{pmatrix} = 0,$$

and it follows from (d$_3$) that, for $n=1,2,\ldots$, $D_n\begin{pmatrix}\xi_1, \ldots, \xi_{n+d}\\ x_1, \ldots, x_n\end{pmatrix}$, interpreted as a bilinear functional of variables ξ_i and x_j only, belongs to $\mathfrak{A}$ (see (1)).

Analogously as in the Fredholm case (see Sikorski [14], p. 151) D_0 in (21) will be called the *determinant* of A, and D_n for $n > 0$ will be called the *subdeterminant of order* n of A.

The following simple remarks hold (compare with Sikorski [10]):

Remark 1. If $D_0, D_1, \ldots$ is a determinant system for $A \in \mathfrak{A}$ and $c \neq 0$, then

(23) $$cD_0, cD_1, \ldots$$

is also a determinant system for A, and

(24) $$D_0, \frac{1}{c}D_1, \frac{1}{c^2}D_2, \ldots$$

is a determinant system for cA.

Remark 2. If $D_0, D_1, \ldots$ is a determinant system for $A \in \mathfrak{A}$, and $B \in \mathfrak{A}$ has the inverse $B^{-1} \in \mathfrak{A}$, then

(25) $$D_n\begin{pmatrix}\xi_1 B^{-1}, \ldots, \xi_{n+d}B^{-1}\\ x_1, \ldots\ldots, x_n\end{pmatrix} \quad (n = 0, 1, 2, \ldots)$$

is a determinant system for AB, and

(25′) $$D_n\begin{pmatrix}\xi_1, \ldots\ldots, \xi_{n+d}\\ B^{-1}x_1, \ldots, B^{-1}x_n\end{pmatrix} \quad (n = 0, 1, 2, \ldots)$$

is a determinant system for BA.

It follows from remark 1 that the determinant system for A, if it exists, is not uniquely determined by A.

(ix) *If $S \in \mathfrak{A}$ is a generalized Fredholm operator of order zero, $U \in \mathfrak{A}$ is a quasi-inverse of S and $s_1, \ldots, s_d$ is a basis of the spaces of all solutions of the equation $Sx = 0$, then the sequence $\theta_0, \theta_1, \ldots$, defined by the formula*

(26) $$\theta_n\begin{pmatrix}\xi_1, \ldots, \xi_p\\ x_1, \ldots, x_n\end{pmatrix} = \begin{vmatrix}\xi_1 Ux_1 & \ldots & \xi_1 Ux_n & \xi_1 s_1 & \ldots & \xi_1 s_d\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \xi_p Ux_1 & \ldots & \xi_p Ux_n & \xi_p s_1 & \ldots & \xi_p s_d\end{vmatrix},$$

where $d = d(S) > 0$ and $p = n+d$ $(n = 0, 1, \ldots)$, is a determinant system for S.

It is evident that conditions (d$_1$), (d$_2$), (d$_3$), and (d$_4$) are satisfied. To prove the condition (d$_5$) let us first observe that there exist points $\omega_1, \ldots, \omega_d \in \Xi$, such that the formulae

$$SU = I, \quad US = I - \sum_{i=1}^{d} \omega_i \cdot s_i$$

hold (see (6), (6′)).

Then the condition (D$_n$) follows from expanding the determinant

$$\theta_{n+1}\begin{pmatrix}\xi_0 S, \xi_1, \ldots, \xi_p\\ x_0, x_1, \ldots, x_n\end{pmatrix} = \begin{vmatrix}\xi_0 x_0 & \ldots & \xi_0 x_n & 0 & 0 & \ldots & 0\\ \xi_1 Ux_0 & \ldots & \xi_1 Ux_n & \xi_1 s_1 & \xi_1 s_2 & \ldots & \xi_1 s_d\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \xi_p Ux_0 & \ldots & \xi_p Ux_n & \xi_p s_1 & \xi_p s_2 & \ldots & \xi_p s_d\end{vmatrix}$$

(where $p = n+d$) in terms of its first row, and the condition (D$'_n$) follows from expanding the determinant

$$\theta_{n+1}\begin{pmatrix}\xi_0, \ldots, \xi_p\\ Sx_0, \ldots, x_n\end{pmatrix} = \begin{vmatrix}\xi_0 x_0 & \xi_0 Ux_1 & \ldots & \xi_0 Ux_n & \xi_0 s_1 & \ldots & \xi_0 s_d\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \xi_p x_0 & \xi_p Ux_1 & \ldots & \xi_p Ux_n & \xi_p s_1 & \ldots & \xi_p s_d\end{vmatrix}$$

in terms of its first column.

Similarly we can obtain the determinant system in the case $d = d(S) < 0$. For this purpose, let us set for $n = 0, 1, \ldots$

(26′) $$\theta_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_{n-d}\end{pmatrix} = \begin{vmatrix}\xi_1 Ux_1 & \ldots & \xi_1 Ux_{n-d}\\ \cdot & \cdot & \cdot\\ \xi_n Ux_1 & \ldots & \xi_n Ux_{n-d}\\ s_1 x_1 & \ldots & s_1 x_{n-d}\\ \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\\ s_{-d} x_1 & \ldots & s_{-d} x_{n-d}\end{vmatrix},$$

where $s_1, \ldots, s_{-d}$ is a basis of the space of all solutions of the equation $\xi S = 0$, and U is a quasi-inverse of S. The sequence $\theta_0, \theta_1, \ldots$ just defined is a determinant system for S with a negative defect.

In particular case, if $r(S) = 0$ and $d(S) = 0$, then S has the only quasi-inverse S^{-1}. It follows from (26) or (26′) that the sequence $I_0, I_1, \ldots,$ defined by the formula

$$(27)\qquad I_0 = 1, \quad I_n\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix} = \begin{vmatrix} \xi_1 S^{-1}x_1 & \ldots & \xi_1 S^{-1}x_n \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \xi_n S^{-1}x_1 & \ldots & \xi_n S^{-1}x_n \end{vmatrix} \quad (n = 1, 2, \ldots),$$

is a determinant system for S.

All theorems concerning operators and determinant systems with a negative defect are formulated in the same way as for operators and determinant systems with a positive defect. Therefore in the sequel, to avoid the duality of formulation, we shall only consider operators with a positive defect.

6. Fundamental theorems.

(x) [8] *If $A \in \mathfrak{A}$ has a determinant system $\{D_n\}$, then A is a generalized Fredholm operator such that $r(A) = r\{D_n\}$, $d(A) = d\{D_n\}$.*

More exactly:

If $r = r\{D_n\}$, $d = d\{D_n\} > 0$, and $\eta_1, \ldots, \eta_{r+d} \in \Xi$, $y_1, \ldots, y_r \in X$ are fixed elements such that

$$D_r\begin{pmatrix}\eta_1, \ldots, \eta_{r+d} \\ y_1, \ldots, y_r\end{pmatrix} \neq 0,$$

then there exist elements $\zeta_1, \ldots, \zeta_r \in \Xi$ and $z_1, \ldots, z_{r+d} \in X$ such that

$$(28)\qquad \zeta_i x = \frac{D_r\begin{pmatrix}\eta_1, \ldots\ldots\ldots\ldots, \eta_{r+d} \\ y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_r\end{pmatrix}}{D_r\begin{pmatrix}\eta_1, \ldots, \eta_{r+d} \\ y_1, \ldots, y_r\end{pmatrix}} \quad \text{for every} \quad x \in X,$$

and

$$(28')\qquad \xi z_i = \frac{D_r\begin{pmatrix}\eta_1, \ldots, \eta_{i-1}, \xi, \eta_{i+1}, \ldots\ldots, \eta_{r+d} \\ y_1, \ldots\ldots\ldots\ldots, y_r\end{pmatrix}}{D_r\begin{pmatrix}\eta_1, \ldots, \eta_{r+d} \\ y_1, \ldots, y_r\end{pmatrix}} \quad \text{for every} \quad \xi \in \Xi.$$

The elements $\zeta_1, \ldots, \zeta_r$ are linearly independent and are solutions of the equation

$$(29)\qquad \xi A = 0.$$

[8] This theorem is a slight generalization of a theorem of Sikorski [10].

The elements $z_1, \ldots, z_{r+d}$, are linearly independent and are solutions of the equation

$$(29')\qquad Ax = 0.$$

Conversely, every solution ξ of (29) is a linear combination of $\zeta_1, \ldots, \zeta_r$, and every solution x of (29′) is a linear combination of $z_1, \ldots, z_{r+d}$.

The bilinear functional defined by the formula

$$(30)\qquad \xi Bx = \frac{D_{r+1}\begin{pmatrix}\xi, \eta_1, \ldots, \eta_{r+d} \\ x, y_1, \ldots, y_r\end{pmatrix}}{D_r\begin{pmatrix}\eta_1, \ldots, \eta_{r+d} \\ y_1, \ldots, y_r\end{pmatrix}}$$

determines a quasi-inverse $B \in \mathfrak{A}$ of A.

The equation

$$(31)\qquad \xi A = \xi_0$$

has a solution ξ if and only if $\xi_0 z_i$ for $i = 1, \ldots, r+d$. Then the general form of the solution ξ is

$$\xi = c_1\zeta_1 + \ldots + c_r\zeta_r + \xi_0 B,$$

and $\xi_0 B$ is the only solution of (31) orthogonal to $y_1, \ldots, y_r$.

Analogously, the equation

$$(31')\qquad Ax = x_0$$

has a solution x if and only if x_0 is orthogonal to $\zeta_1, \ldots, \zeta_r$. Then the general form of the solution x is

$$x = c_1 z_1 + \ldots + c_{r+d} z_{r+d} + Bx_0$$

and Bx_0 is the only solution of (31′) orthogonal to $\eta_1, \ldots, \eta_{r+d}$.

It follows from (d_3) that such elements $\zeta_1, \ldots, \zeta_r$ exist. Furthermore, it follows from (28) and (d_2) that $\zeta_i y_j = \delta_{ij}$ $(i, j = 1, \ldots, r)$. Thus $\zeta_1, \ldots, \zeta_r$ and $y_1, \ldots, y_r$ are linearly independent. We can establish in the same way that $z_1, \ldots, z_{r+d}$ and $\eta_1, \ldots, \eta_{r+d}$ are also linearly independent. The elements $\zeta_1, \ldots, \zeta_r$ and $z_1, \ldots, z_{r+d}$ are solutions of the equations $\xi A = 0$ and $Ax = 0$, respectively, because it follows from (28), (d_2) and (D'_{r-1}) (or from (28′), (d_2) and (D_{r-1})) that $\zeta_i Ax = 0$ for each $x \in X$ and $\xi A z_i = 0$ for each $\xi \in \Xi$.

Replacing ξ by ξA in (30) and then using (D_n) $(n = r)$ we obtain by (d_2) that $AB = I - \sum_{i=1}^{r} y_i \cdot \zeta_i$. Analogously, replacing x by Ax in (30) and then using (D'_m) $(n = r)$ we obtain $BA = I - \sum_{i=1}^{r+d} z_i \cdot \eta_i$. It follows from this that B is a quasi-inverse of A. The rest of theorem (x) follows from theorem (iii).

(xi) *If $A \in \mathfrak{A}$ is a generalized Fredholm operator, then A has a determinant system $\{D_n\}$ and $r\{D_n\} = r(A)$, $d\{D_n\} = d(A)$. This system is determined by A uniquely up to a scalar factor $\neq 0$.*

Suppose that $r(A) = r > 0$, $d(A) = d$, and B is a quasi-inverse of A. Let $z_1, \ldots, z_{r+d}$, $\zeta_1, \ldots, \zeta_r$ and $y_1, \ldots, y_r$, $\eta_1, \ldots, \eta_{r+d}$ have the same meaning as in section 2.

Let us set

$$\mathscr{D}_n = 0 \quad \text{for} \quad n = 0, \ldots, r-1, \tag{32}$$

$$\mathscr{D}_r\begin{pmatrix}\xi_1, \ldots, \xi_{r+d}\\ x_1, \ldots, x_r\end{pmatrix} = \begin{vmatrix}\xi_1 z_1 & \ldots & \xi_1 z_{r+d}\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\\ \xi_{r+d} z_1 & \ldots & \xi_{r+d} z_{r+d}\end{vmatrix} \cdot \begin{vmatrix}\zeta_1 x_1 & \ldots & \zeta_1 x_r\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\\ \zeta_r x_1 & \ldots & \zeta_r x_r\end{vmatrix}, \tag{33}$$

and for $r = 1, 2, \ldots$

$$\mathscr{D}_{r+k}\begin{pmatrix}\xi_1, \ldots, \xi_{r+d+k}\\ x_1, \ldots, x_{r+k}\end{pmatrix} \tag{34}$$

$$= \sum_{\mathfrak{p},\mathfrak{q}} \operatorname{sgn}\mathfrak{p}\operatorname{sgn}\mathfrak{q} \begin{vmatrix}\xi_{p_1} Bx_{q_1} & \ldots & \xi_{p_1} Bx_{q_k}\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\\ \xi_{p_k} Bx_{q_1} & \ldots & \xi_{p_k} Bx_{q_k}\end{vmatrix} \cdot \mathscr{D}_r\begin{pmatrix}\xi_{p_{k+1}}, \ldots, \xi_{p_{k+r+d}}\\ x_{q_{k+1}}, \ldots, x_{q_{k+r}}\end{pmatrix},$$

where $\sum\limits_{\mathfrak{p},\mathfrak{q}}$ is extended over all permutations $\mathfrak{p} = (p_1, \ldots, p_{r+d+k})$ and $\mathfrak{q} = (q_1, \ldots, q_{r+k})$ of the integers $1, \ldots, r+d+k$, and $1, \ldots, r+k$, respectively, such that

$$p_1 < p_2 < \ldots < p_k, \qquad p_{k+1} < p_{k+2} < \ldots < p_{k+r+d},$$
$$q_1 < q_2 < \ldots < q_k, \qquad q_k < q_{k+2} < \ldots < q_{k+r}.$$

The sequence $\mathscr{D}_0, \mathscr{D}_1, \ldots,$ just defined, is a determinant system for A [9]. To prove this we have to verify that all conditions (d_1)-(d_5) in the definition of the determinant system are satisfied. Obviously $\mathscr{D}_0, \mathscr{D}_1, \ldots$ satisfies conditions (d_1), (d_2), (d_3). Since

$$\mathscr{D}_r\begin{pmatrix}\eta_1, \ldots, \eta_{r+d}\\ y_1, \ldots, y_r\end{pmatrix} = 1,$$

(d_4) is also satisfied.

[9] It is easy to prove that $(2n+d)$-linear functionals $\mathscr{D}_n$ do not depend on the choice of B.

It follows immediately from (32) that (D_n) and (D'_n) hold for $n = 0, \ldots, n-2$. It is easy to see that they also hold for $n = r-1$, because

$$D_r\begin{pmatrix}\xi_1, \ldots, \xi_{r+d}\\ x_1, \ldots, x_r\end{pmatrix} = 0 \tag{35}$$

if one of the points $\xi_1, \ldots, \xi_{r+d}$ belongs to $\mathscr{Y}(A)$ or one of the points $x_1, \ldots, x_r$ belongs to $Y(A)$.

The proof of condition (d_5) for $n > r$ is based on the formula

$$\sum_{\mathfrak{p}} \operatorname{sgn}\mathfrak{p} \begin{vmatrix}\xi_{p_1} x_1 & \ldots & \xi_{p_1} x_n\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\\ \xi_{p_n} x_1 & \ldots & \xi_{p_n} x_n\end{vmatrix} \cdot \begin{vmatrix}\xi_{p_{n+1}} x_{n+1} & \ldots & \xi_{p_{n+1}} x_{n+d}\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\\ \xi_{p_{n+d}} x_{n+1} & \ldots & \xi_{p_{n+d}} x_{n+d}\end{vmatrix} \tag{36}$$

$$= \sum_{\mathfrak{q}} \operatorname{sgn}\mathfrak{q} \begin{vmatrix}\xi_1 x_{q_1} & \ldots & \xi_1 x_{q_n}\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\\ \xi_n x_{q_1} & \ldots & \xi_n x_{q_n}\end{vmatrix} \cdot \begin{vmatrix}\xi_{n+1} x_{q_{n+1}} & \ldots & \xi_{n+1} x_{q_{n+d}}\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\\ \xi_{n+d} x_{q_{n+1}} & \ldots & \xi_{n+d} x_{q_{n+d}}\end{vmatrix}$$

$$= \begin{vmatrix}\xi_1 x_1 & \ldots & \xi_1 x_{n+d}\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\\ \xi_{n+d} x_1 & \ldots & \xi_{n+d} x_{n+d}\end{vmatrix},$$

where $\sum\limits_{\mathfrak{p}}$ and $\sum\limits_{\mathfrak{q}}$ are extended over all permutations $\mathfrak{p} = (p_1, \ldots, p_{n+d})$ and $\mathfrak{q} = (q_1, \ldots, q_{n+d})$ of the integers $1, \ldots, n+d$, such that

$$p_1 < p_2 < \ldots < p_n, \qquad p_{n+1} < \ldots < p_{n+d},$$
$$q_1 < q_2 < \ldots < q_n, \qquad q_{n+1} < \ldots < q_{n+d}.$$

We obtain (see [10])

$$\mathscr{D}_{r+k+1}\begin{pmatrix}\xi_0, \xi_1, \ldots, \xi_{r+d+k}\\ x_0, x_1, \ldots, x_{r+k}\end{pmatrix}$$

$$= \sum_{\mathfrak{p}',\mathfrak{q}} \operatorname{sgn}\mathfrak{p}'\operatorname{sgn}\mathfrak{q} \begin{vmatrix}\xi_0\ Bx_{q_0} & \ldots & \xi_0\ Bx_{q_k}\\ \xi_{p_1}\ Bx_{q_0} & \ldots & \xi_{p_1}\ Bx_{q_k}\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\\ \xi_{p_k}\ Bx_{q_0} & \ldots & \xi_{p_k}\ Bx_{q_k}\end{vmatrix} \cdot \mathscr{D}_r\begin{pmatrix}\xi_{p_{k+1}}, \ldots, \xi_{p_{k+r+d}}\\ x_{q_{k+1}}, \ldots, x_{q_{k+r}}\end{pmatrix} +$$

$$+\sum_{\mathfrak{p}'',\mathfrak{q}}\operatorname{sgn}\mathfrak{p}''\operatorname{sgn}\mathfrak{q}\begin{vmatrix}\xi_{p_0}Bx_{q_0} & \dots & \xi_{p_0}Bx_{q_k}\\ \xi_{p_1}Bx_{q_0} & \dots & \xi_{p_1}Bx_{q_k}\\ \dots & \dots & \dots\\ \dots & \dots & \dots\\ \xi_{p_k}Bx_{q_0} & \dots & \xi_{p_k}Bx_{q_k}\end{vmatrix}\cdot\mathscr{D}_r\begin{pmatrix}\xi_0, & \xi_{p_{k+2}},\dots,\xi_{p_{k+r+d}}\\ x_{q_{k+1}}, & x_{q_{k+2}},\dots,x_{k+r}\end{pmatrix},$$

where $\mathfrak{p}'$, $\mathfrak{p}''$, $\mathfrak{q}$ denote arbitrary permutations (of the integers $0,1,\dots,r+k$) of the form:

$$\mathfrak{p}' = (0,p_1,\dots,p_k,p_{k+1},\dots,p_{k+r+d}),$$
$$p_1<p_2<\dots<p_k,\quad p_{k+1}<\dots<p_{k+r+d};$$
$$\mathfrak{p}'' = (p_0,p_1,\dots,p_k,0,p_{k+2},\dots,p_{k+r+d}),$$
$$p_0<p_1<\dots<p_k,\quad p_{k+2}<\dots<p_{k+r+d};$$
$$\mathfrak{q} = (q_0,q_1,\dots,q_k,q_{k+1},\dots,q_{k+r}),$$
$$q_0<q_1<\dots<q_k,\quad q_{k+1}<\dots<q_{k+r}.$$

Hence, by (35), (36) and by the formula $AB = I-\sum_{i=1}^{r}y_i\cdot\zeta_i$ (see (5)), we obtain

$$\mathscr{D}_{r+k+1}\begin{pmatrix}\xi_0A, & \xi_1,\dots,\xi_{r+d+k}\\ x_0, & x_1,\dots,x_{r+k}\end{pmatrix}$$

$$=\sum_{\mathfrak{p}',\mathfrak{q}}\operatorname{sgn}\mathfrak{p}'\operatorname{sgn}\mathfrak{q}\begin{vmatrix}\xi_0ABx_{q_0} & \dots & \xi_0ABx_{q_k}\\ \xi_{p_1}Bx_{q_0} & \dots & \xi_{p_1}Bx_{q_k}\\ \dots & \dots & \dots\\ \dots & \dots & \dots\\ \xi_{p_k}Bx_{q_0} & \dots & \xi_{p_k}Bx_{q_k}\end{vmatrix}\cdot\mathscr{D}_r\begin{pmatrix}\xi_{p_{k+1}},\dots,\xi_{p_{k+r+d}}\\ x_{q_{k+1}},\dots,q_{k+r}\end{pmatrix}$$

$$=\sum_{\mathfrak{p}'}\operatorname{sgn}\mathfrak{p}'\begin{vmatrix}\xi_0x_0-\sum_{i=1}^{r}\xi_0y_i\cdot\zeta_ix_0 & \dots & \xi_0x_{k+r}-\sum_{i=1}^{r}\xi_0y_i\cdot\zeta_ix_{k+r}\\ \xi_{p_1}Bx_0 & \dots & \xi_{p_1}Bx_{k+r}\\ \dots & \dots & \dots\\ \xi_{p_k}Bx_0 & \dots & \xi_{p_k}Bx_{k+r}\\ \zeta_1x_0 & \dots & \zeta_1x_{k+r}\\ \dots & \dots & \dots\\ \zeta_rx_0 & \dots & \zeta_rx_{k+r}\end{vmatrix}\times$$

$$\times\begin{vmatrix}\xi_{p_{k+1}}z_1 & \dots & \xi_{p_{k+1}}z_{r+d}\\ \dots & \dots & \dots\\ \xi_{p_{k+r+d}}z_1 & \dots & \xi_{p_{k+r+d}}z_{r+d}\end{vmatrix}$$

$$=\sum_{\mathfrak{p}'}\operatorname{sgn}\mathfrak{p}'\sum_{i=0}^{k+r}(-1)^i\xi_0x_i\begin{vmatrix}\xi_{p_1}Bx_0 & \dots & \xi_{p_1}Bx_{i-1} & \xi_{p_1}Bx_{i+1} & \dots & \xi_{p_1}Bx_{k+r}\\ \dots & \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots\\ \xi_{p_k}Bx_0 & \dots & \xi_{p_k}Bx_{i-1} & \xi_{p_k}Bx_{i+1} & \dots & \xi_{p_k}Bx_{k+r}\\ \zeta_1x_0 & \dots & \zeta_1x_{i-1} & \zeta_1x_{i+1} & \dots & \zeta_1x_{k+r}\\ \dots & \dots & \dots & \dots & \dots & \dots\\ \zeta_rx_0 & \dots & \zeta_rx_{i-1} & \zeta_rx_{i+1} & \dots & \zeta_rx_{k+r}\end{vmatrix}\times$$

$$\times\begin{vmatrix}\xi_{p_{k+1}}z_1 & \dots & \xi_{p_{k+1}}z_{r+d}\\ \dots & \dots & \dots\\ \xi_{p_{k+r+d}}z_1 & \dots & \xi_{p_{k+r+d}}z_{r+d}\end{vmatrix}$$

$$=\sum_{i=0}^{r+k}(-1)^i\xi_0x_i\sum_{\mathfrak{p},\mathfrak{q}_i}\operatorname{sgn}\mathfrak{p}\operatorname{sgn}\mathfrak{q}_i\begin{vmatrix}\xi_{p_1}Bx_{q_{i,1}} & \dots & \xi_{p_1}Bx_{q_{i,k+r}}\\ \dots & \dots & \dots\\ \xi_{p_k}Bx_{q_{i,1}} & \dots & \xi_{p_k}Bx_{q_{i,k+r}}\end{vmatrix}\times$$

$$\times\mathscr{D}_r\begin{pmatrix}\xi_{p_{k+1}}, & \dots, & \xi_{p_{k+r+d}}\\ x_{q_{i,k+1}}, & \dots, & x_{q_{i,k+r}}\end{pmatrix}=\sum_{i=0}^{r+k}(-1)^i\xi_0x_i\cdot\mathscr{D}_{r+k}\begin{pmatrix}\xi_1,\ \dots\dots\dots\dots, & \xi_{k+r+d}\\ x_0,\dots,x_{i-1},x_{i+1},\dots, & x_{k+r}\end{pmatrix},$$

where $\sum_{\mathfrak{p},\mathfrak{q}_i}$ is extended over all permutations $\mathfrak{p}=(p_1,\dots,p_{r+d+k})$ of the integers $(1,\dots,r+d+k)$ such that $p_1<\dots<p_k$, $p_{k+1}<\dots<p_{k+r+d}$ and all permutations $\mathfrak{q}_i=(q_{i,1},\dots,q_{i,k+r})$ of the integers $0,\dots,i-1,i+1,\dots,k+r$ such that $q_{i,1}<\dots<q_{i,k}$, $q_{i,k+1}<\dots<q_{i,k+r}$. This proves (D_n). The proof of (D_n') is similar.

Now, we shall prove the last part of theorem (xi).

It has been proved by Sikorski [10] in the case $d(A)=0$ that the determinant system for A is determined by A uniquely up to a scalar factor $\neq 0$. The proof in the case $d(A)>0$ is analogous to that in above-mentioned paper [10]. Let $D_0, D_1, \dots$ be any determinant system for A. We have to prove that there exists a scalar $c\neq 0$ such that

(37) $$D_n = c\mathscr{D}_n\quad\text{for}\quad n=0,1,2,\dots,$$

where $\mathscr{D}_n$ is defined by (32), (33), and (34).

By theorem (x), we obtain $r\{D_n\}=r\{\mathscr{D}_n\}$, $d\{D_n\}=d\{\mathscr{D}_n\}$, i. e.

(38) $$D_n = 0\quad\text{for}\quad n=0,\dots,r-1,$$

and $D_r \neq 0$. If one of the points $x_1, \ldots, x_r$ belongs to $Y(A)$ or one of the points $\xi_1, \ldots, \xi_{r+d}$ belongs to $\mathscr{Y}(A)$, then it follows from (d_2), (D_{r-1}), (D'_{r-1}) and from (38) that

$$D_r\begin{pmatrix}\xi_1, \ldots, \xi_{r+d}\\ x_1, \ldots, x_r\end{pmatrix} = 0. \tag{39}$$

We know that $(r+d)$-dimensional subspace $\mathscr{Z}(B)$ is spanned by elements $\eta_1, \ldots, \eta_{r+d}$. Similarly, r-dimensional subspace $Z(B)$ is spanned by elements $y_1, \ldots, y_r$. Each point $\xi_i \in \Xi$ can be uniquely represented in the form $\xi_i = \xi_i' + \xi_i''$, where $\xi' \in \mathscr{Y}(A)$ and $\xi'' \in \mathscr{Z}(B)$. Analogously, each point $x_i \in X$ can be uniquely represented in the form $x_i = x_i' + x_i''$, where $x' \in Y(A)$ and $x_i'' \in Z(B)$. By (39), we obtain

$$\mathscr{D}_r\begin{pmatrix}\xi_1, \ldots, \xi_{r+d}\\ x_1, \ldots, x_r\end{pmatrix} = \mathscr{D}_r\begin{pmatrix}\xi_1'', \ldots, \xi_{r+d}''\\ x_1'', \ldots, x_r''\end{pmatrix} \tag{40}$$

for arbitrary $\xi_1, \ldots, \xi_{r+d} \in \Xi$ and $x_1, \ldots, x_r \in X$. The same is true for D_r, i. e.

$$D_r\begin{pmatrix}\xi_1, \ldots, \xi_{r+d}\\ x_1, \ldots, x_r\end{pmatrix} = D_r\begin{pmatrix}\xi_1'', \ldots, \xi_{r+d}''\\ x_1'', \ldots, x_r''\end{pmatrix}. \tag{41}$$

Since two arbitrary $(2r+d)$-linear functionals defined on $Z(B)^{r+d} \times \times Z(B)^r$ and skew symmetric in $\xi_1'', \ldots, \xi_{r+d}'' \in \mathscr{Z}(B)$ and $x_1'', \ldots, x_r'' \in Z(B)$ differ only by a scalar factor, there exists a scalar $c \neq 0$ (since $D_r \neq 0 \neq \mathscr{D}_r$) such that

$$D_r\begin{pmatrix}\xi_1'', \ldots, \xi_{r+d}''\\ x_1'', \ldots, x_r''\end{pmatrix} = c\mathscr{D}_r\begin{pmatrix}\xi_1'', \ldots, \xi_{r+d}''\\ x_1'', \ldots, x_r''\end{pmatrix}$$

for $\xi_1'', \ldots, \xi_{r+d}'' \in \mathscr{Z}(B)$ and $x_1'', \ldots, x_r'' \in Z(B)$. It follows from this, (40), and (41), that

$$D_r\begin{pmatrix}\xi_1, \ldots, \xi_{r+d}\\ x_1, \ldots, x_r\end{pmatrix} = c\mathscr{D}_r\begin{pmatrix}\xi_1, \ldots, \xi_{r+d}\\ x_1, \ldots, x_r\end{pmatrix}$$

for arbitrary $\xi_1, \ldots, \xi_{r+d} \in \Xi$ and $x_1, \ldots, x_r \in X$. This and (38) proves that (37) is true for $n = 0, \ldots, r-1$.

For $n > r$ formula (37) can be proved by induction. Suppose (37) to be valid for $n \geqslant r$. We have to prove that it holds for $n+1$, i. e.

$$D_{n+1}\begin{pmatrix}\xi_1, \ldots, \xi_{n+d+1}\\ x_1, \ldots, x_{n+1}\end{pmatrix} = c\mathscr{D}_{n+1}\begin{pmatrix}\xi_1, \ldots, \xi_{n+d+1}\\ x_1, \ldots, x_{n+1}\end{pmatrix}. \tag{42}$$

Since D_{n+1} and $\mathscr{D}_{n+1}$ are linear in each variable, it is sufficient to prove (42) only in the case when each of the points $\xi_1, \ldots, \xi_{n+d+1}$, either belongs to $\mathscr{Y}(A)$ or is equal to one of the points $\eta_1, \ldots, \eta_{r+d}$.

If the sequence $\xi_1, \ldots, \xi_{n+d+1}$ contains one point belonging to $\mathscr{Y}(A)$, then it follows from the induction hypothesis and from the identity (D_n) for D_{n+1} and $\mathscr{D}_{n+1}$ that (42) holds. If the sequence $\xi_1, \ldots, \xi_{n+d+1}$ contains only points $\eta_1, \ldots, \eta_{r+d}$, one of them appears at least twice. Hence, by the skew symmetry, both sides of (42) are equal to zero.

Theorem (xi) implies the following:

Corollary. *Let $A \in \mathfrak{A}$ be a generalized Fredholm operator such that $r(A) = 0$ and $d(A) = d > 0$. If $\{D_n\}$ is a determinant system for A, then*

$$D_n\begin{pmatrix}\xi_1, \ldots, \xi_{n+d}\\ x_1, \ldots, x_n\end{pmatrix} = \sum_{\mathfrak{p}} \operatorname{sgn}\mathfrak{p}\, D_0(\xi_{p_{n+1}}, \ldots, \xi_{p_{n+d}})\cdot \begin{vmatrix} \xi_{p_1}Bx_1 \ldots \xi_{p_1}Bx_n \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ \xi_{p_n}Bx_1 \ldots \xi_{p_n}Bx_n \end{vmatrix} \quad \text{for } n = 0, 1, \ldots, \tag{43}$$

where $B \in \mathfrak{A}$ is a quasi-inverse of A and $\sum_{\mathfrak{p}}$ is extended over all permutations $\mathfrak{p} = (p_1, \ldots, p_{n+d})$ of the integers $1, \ldots, n+d$ such that $p_1 < \ldots < p_n$, $p_{n+1} < \ldots < p_{n+d}$.

(xii) *Let $A = S+T$ be such that $r(A) = r$, $d(S) = d > 0$ and let $s_1, \ldots, s_d$, be all linearly independent solutions of $Sx = 0$. Let U be a quasi-inverse of S and let $\{\bar{D}_n\}$ be a determinant system for the Fredholm operator $I+UT$.*

The sequence $D_0, D_1, \ldots$, defined by the formula

$$D_n\begin{pmatrix}\xi_1, \ldots, \xi_{n+d}\\ x_1, \ldots, x_n\end{pmatrix} = \bar{D}_{n+d}\begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots, \xi_{n+d}\\ Ux_1, \ldots, Ux_n, s_1, \ldots, s_d\end{pmatrix} \quad (n = 0, 1, \ldots), \tag{44}$$

is a determinant system for $S+T$ which does not depend on the choice of U.

By (18), $r(S+T) = r(I+UT)$. Let $\bar{\zeta}_1, \ldots, \bar{\zeta}_r$ and $\bar{z}_1, \ldots, \bar{z}_r$ be linearly independent solutions of the equations $\xi(I+UT) = 0$ and $(I+UT)x = 0$, respectively, and let $\bar{B}$ be a quasi-inverse of $I+UT$.

It suffices to prove (xii) in the case where $\{\bar{D}_n\}$ is defined by the formulae:

$$\bar{D}_n = 0 \quad \text{for} \quad n = 0, 1, \ldots, r-1, \tag{45}$$

$$\bar{D}_r\begin{pmatrix}\xi_1, \ldots, \xi_r\\ x_1, \ldots, x_r\end{pmatrix} = \begin{vmatrix} \xi_1\bar{z}_1 \ldots \xi_r\bar{z}_r \\ \ldots\ldots \\ \ldots\ldots \\ \xi_r\bar{z}_1 \ldots \xi_r\bar{z}_r \end{vmatrix} \cdot \begin{vmatrix} \bar{\zeta}_1x_1 \ldots \bar{\zeta}_rx_r \\ \ldots\ldots \\ \ldots\ldots \\ \bar{\zeta}_rx_1 \ldots \bar{\zeta}_rx_r \end{vmatrix}, \tag{46}$$

and for $k = 1, 2, \dots$

$$(47)\qquad \bar{D}_{r+k}\begin{pmatrix}\xi_1, \dots, \xi_{r+k}\\ x_1, \dots, x_{r+k}\end{pmatrix} = \sum_{\mathfrak{p},\mathfrak{q}} \operatorname{sgn}\mathfrak{p}\,\operatorname{sgn}\mathfrak{q}\begin{vmatrix}\xi_{p_1}\bar{B}x_{q_1} & \dots & \xi_{p_1}\bar{B}x_{q_k}\\ \dots & \dots & \dots\\ \dots & \dots & \dots\\ \xi_{p_k}\bar{B}x_{q_1} & \dots & \xi_{p_k}\bar{B}x_{q_k}\end{vmatrix}\cdot\bar{D}_r\begin{pmatrix}\xi_{p_{k+1}}, \dots, \xi_{p_{k+r}}\\ x_{q_{k+1}}, \dots, x_{p_{k+r}}\end{pmatrix},$$

where $\sum\limits_{\mathfrak{p},\mathfrak{q}}$ is extended over all permutations $\mathfrak{p} = (p_1, \dots, p_{k+r})$, and $\mathfrak{q} = (q_1, \dots, q_{k+r})$ of the integers $1, \dots, r+k$ such that

$$(47')\qquad \begin{matrix} p_1 < p_2 < \dots < p_k, & p_{k+1} < p_{k+2} < \dots < p_{k+r},\\ q_1 < q_2 < \dots < q_r, & q_{r+1} < q_{r+2} < \dots < q_{k+r}.\end{matrix}$$

It follows immediately from (44) that $\{D_n\}$ satisfies conditions (d_1), (d_2), (d_3). If $n+d = r$, then it follows from (46) that

$$D_n\begin{pmatrix}\xi_1, \dots, \xi_{n+d}\\ x_1, \dots, x_n\end{pmatrix} = \begin{vmatrix}\xi_1\bar{z}_1 & \dots & \xi_1\bar{z}_r\\ \dots & \dots & \dots\\ \dots & \dots & \dots\\ \xi_r\bar{z}_1 & \dots & \xi_r\bar{z}_r\end{vmatrix}\cdot\begin{vmatrix}\bar{\zeta}_1Ux_1 & \dots & \bar{\zeta}_1Ux_n & \bar{\zeta}_1s_1 & \dots & \bar{\zeta}_1s_d\\ \dots & \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots\\ \bar{\zeta}_rUx_1 & \dots & \bar{\zeta}_rUx_n & \bar{\zeta}_rs_1 & \dots & \bar{\zeta}_rs_d\end{vmatrix} = 0,$$

because all the solutions $\bar{\zeta}_i$ of the equation $\xi(I+UT) = 0$ are orthogonal to all the solutions s_i of the equation $Sx = 0$. If $n < r$ but $n+d > r$, then $D_n\begin{pmatrix}\xi_1, \dots, \xi_{n+d}\\ x_1, \dots, x_n\end{pmatrix} = 0$, since each term [10] of the sum (47) is equal to zero.

If $n = r$, then by (46) and (47) we obtain

$$(48)\qquad D_r\begin{pmatrix}\xi_1, \dots, \xi_{r+d}\\ x_1, \dots, x_r\end{pmatrix} = \sum_{\mathfrak{p}}\operatorname{sgn}\mathfrak{p}\begin{vmatrix}\xi_{p_1}\bar{B}s_1 & \dots & \xi_{p_1}\bar{B}s_d\\ \dots & \dots & \dots\\ \dots & \dots & \dots\\ \xi_{p_d}\bar{B}s_1 & \dots & \xi_{p_d}\bar{B}s_d\end{vmatrix}\cdot\bar{D}_r\begin{pmatrix}\xi_{p_{d+1}}, & \dots, & \xi_{p_{d+r}}\\ Ux_1, & \dots, & Ux_n\end{pmatrix},$$

where $\sum\limits_{\mathfrak{p}}$ is extended over all permutations $\mathfrak{p}$ satisfying (47′). Applying

[10] Each term of that sum contains as a factor a determinant with at least one zero column $\bar{\zeta}_i s_j$, where $i = 1, \dots, r$, and j is fixed.

formula (36) and elementary properties of determinants we obtain by theorem (viii)

$$D_r\begin{pmatrix}\xi_1, \dots, \xi_{r+d}\\ x_1, \dots, x_r\end{pmatrix} = \begin{vmatrix}\xi_1z_1 & \dots & \xi_1z_{r+d}\\ \dots & \dots & \dots\\ \dots & \dots & \dots\\ \xi_{r+d}z_1 & \dots & \xi_{r+d}z_{r+d}\end{vmatrix}\cdot\begin{vmatrix}\zeta_1x_1 & \dots & \zeta_1x_r\\ \dots & \dots & \dots\\ \dots & \dots & \dots\\ \zeta_rx_1 & \dots & \zeta_rx_r\end{vmatrix},$$

where $z_1, \dots, z_{r+d}$ and $\zeta_1, \dots, \zeta_r$ are solutions of the equations $(S+T)x = 0$ and $\xi(S+T) = 0$, respectively. Hence the condition (d_4) is also satisfied (see (34)).

The proof of (d_5) is based on the identities

$$SU = I, \qquad US = I - \sum_{i=1}^{d} s_i\cdot\omega_i$$

(see (6), (6′)). We have

$$D_{n+1}\begin{pmatrix}\xi_0(S+T), \xi_1, \dots, \xi_{n+d}\\ x_0, \quad x_1, \dots, x_n\end{pmatrix}$$

$$= \bar{D}_{n+d+1}\begin{pmatrix}\xi_0S(I+UT), \xi_1, \dots\dots, \xi_{n+d}\\ Ux_0, \quad Ux_1, \dots, Ux_n, s_1, \dots, s_d\end{pmatrix}$$

$$= \sum_{i=0}^{n}(-1)^i\xi_0SUx_i\cdot\bar{D}_{n+d}\begin{pmatrix}\xi_1, \dots\dots, \xi_{n+d}\\ Ux_0, \dots, Ux_{i-1}, Ux_{i+1}, \dots, Ux_n, s_1, \dots, s_d\end{pmatrix} +$$

$$+ \sum_{j=1}^{d}(-1)^{n+j}\xi_0Ss_j\cdot\bar{D}_{n+d}\begin{pmatrix}\xi_1, \dots\dots, \xi_{n+d}\\ Ux_0, \dots, Ux_n, s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_d\end{pmatrix}$$

$$= \sum_{i=0}^{n}(-1)^i\xi_0x_i\cdot D_n\begin{pmatrix}\xi_1, \dots\dots, \xi_{n+d}\\ x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n\end{pmatrix};$$

$$D_{n+1}\begin{pmatrix}\xi_0, \xi_1, \dots, \xi_{n+d}\\ (S+T)x_0, x_1, \dots, x_n\end{pmatrix}$$

$$= \bar{D}_{n+d+1}\begin{pmatrix}\xi_0, \quad \xi_1, \dots\dots, \xi_{n+d}\\ \left(I+UT-\sum_{j=1}^{d}s_j\cdot\omega_j\right)x_0, Ux_1, \dots, Ux_n, s_1, \dots, s_d\end{pmatrix}$$

$$= \bar{D}_{n+d+1}\begin{pmatrix}\xi_0, \quad \xi_1, \dots\dots, \xi_{n+d}\\ (I+UT)x_0, Ux_1, \dots, Ux_n, s_1, \dots, s_d\end{pmatrix} -$$

$$- \sum_{j=1}^{d}\omega_jx_0\cdot\bar{D}_{n+d+1}\begin{pmatrix}\xi_0, \quad \xi_1, \dots\dots, \xi_{n+d}\\ s_j, Ux_1, \dots, Ux_n, s_1, \dots, s_d\end{pmatrix}$$

$$= \sum_{i=0}^{n+d} (-1)^i \xi_i x_0 \cdot \bar{D}_{n+d} \begin{pmatrix} \xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, & & \xi_{n+d} \\ Ux_1, \ldots\ldots\ldots\ldots, & Ux_n, s_1, \ldots, & s_d \end{pmatrix}$$

$$= \sum_{i=0}^{n+d} (-1)^i \xi_i x_0 \cdot D_n \begin{pmatrix} \xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \xi_{n+d} \\ x_1, \ldots\ldots\ldots\ldots, x_n \end{pmatrix}.$$

Thus $\{D_n\}$ defined by formula (44) is a determinant system for $S+T$. It remains to establish that $(2n+d)$-linear functionals D_n defined by (44) for fixed $s_1, \ldots, s_d$ do not depend on U.

Suppose that $V \in \mathfrak{A}$ is another quasi-inverse of S. We have $SV = I$, and $SU = I$. Hence, there exists a finitely dimensional operator $\sum_{i=1}^{d} s_i \cdot \sigma_i$ such that $V = U + \sum_{i=1}^{d} s_i \cdot \sigma_i$. Replacing U by V in (44) we verify by the skew symmetry and linearity of D_n that the $(2n+d)$-linear functionals D_n obtained by means of V are the same as those obtained by means of U. This completes the proof.

(xiii) *Let $\{\bar{D}_n\}$ be a determinant system for $I+UT$ and let $\eta_1, \ldots, \eta_r \in \Xi$, $y_1, \ldots, y_r \in X$ be such that*

$$\bar{D}_r \begin{pmatrix} \eta_1, \ldots, \eta_r \\ y_1, \ldots, y_r \end{pmatrix} \neq 0.$$

Then elements $\zeta_1, \ldots, \zeta_r \in \Xi$ and $z_1, \ldots, z_{r+d} \in X$ such that

$$\zeta_i x = \frac{\bar{D}_r \begin{pmatrix} \eta_1, \ldots\ldots\ldots\ldots, \eta_r \\ y_1, \ldots, y_{i-1}, Ux, y_{i+1}, \ldots, y_r \end{pmatrix}}{\bar{D}_r \begin{pmatrix} \eta_1, \ldots, \eta_r \\ y_1, \ldots, y_r \end{pmatrix}} \quad \text{for every} \quad x \in X,$$

$$\xi z_i = \frac{\bar{D}_r \begin{pmatrix} \eta_1, \ldots, \eta_{i-1}, \xi, \eta_{i+1}, \ldots, \eta_r \\ y_1, \ldots\ldots\ldots\ldots, y_r \end{pmatrix}}{\bar{D}_r \begin{pmatrix} \eta_1, \ldots, \eta_r \\ y_1, \ldots, y_r \end{pmatrix}} \quad \text{for every} \quad \xi \in \Xi,$$

$$\xi z_{r+j} = \frac{\bar{D}_{r+1} \begin{pmatrix} \xi, \eta_1, \ldots, \eta_r \\ s_j, y_1, \ldots, y_r \end{pmatrix}}{\bar{D}_r \begin{pmatrix} \eta_1, \ldots, \eta_r \\ y_1, \ldots, y_r \end{pmatrix}} \quad \text{for every} \quad \xi \in \Xi$$

(where $i = 1, \ldots, r, j = 1, \ldots, d$), are linearly independent solutions of the equations $\xi A = 0$ and $Ax = 0$, respectively.

If ξ_0 is orthogonal to $z_1, \ldots, z_{r+d}$, then the element ξ such that

$$\xi x = \frac{\bar{D}_{r+1} \begin{pmatrix} \xi_0, \eta_1, \ldots, \eta_r \\ Ux, y_1, \ldots, y_r \end{pmatrix}}{\bar{D}_r \begin{pmatrix} \eta_1, \ldots, \eta_r \\ y_1, \ldots, y_r \end{pmatrix}} \quad \text{for every} \quad x \in X$$

is the only solution of the equation $\xi A = \xi_0$ orthogonal to $y_1, \ldots, y_r$.

Analogously, if x_0 is orthogonal to $\zeta_1, \ldots, \zeta_r$, then the element x such that

$$\xi x = \frac{\bar{D}_{r+1} \begin{pmatrix} \xi, \eta_1, \ldots, \eta_r \\ Ux_0, y_1, \ldots, y_r \end{pmatrix}}{\bar{D}_r \begin{pmatrix} \eta_1, \ldots, \eta_r \\ y_1, \ldots, y_r \end{pmatrix}} \quad \text{for every} \quad \xi \in \Xi$$

is the only solution of the equation $Ax = x_0$ orthogonal to $\eta_1, \ldots, \eta_r, \omega_1, \ldots \ldots, \omega_d$, where $\omega_1, \ldots, \omega_d$ are linearly independent solutions of the equation $\xi U = 0$.

Theorem (xiii) follows immediately from (x) and (xiii).

7. Analytic formulae for determinant systems. So far we have dealt with algebraic properties of determinant systems. If we know a determinant system for a generalized Fredholm operator A, then we can solve the equations (see (x))

$$\xi A = \xi_0, \qquad Ax = x_0.$$

Formulae (32), (33) and (34) defining a determinant system have no practical value from the point of view of solving equations because they are obtained by means of quasi-inverse B of A and of the solutions of the equations $\xi A = 0$, $Ax = 0$. However, for a large class of generalized Fredholm operators, it is possible to give an analytic formula for determinant systems in Banach spaces. This will be done in this and the next sections.

From now on, let Ξ and X be conjugate (in the sense explained above) Banach spaces and let ξx be a bilinear functional such that

$$\|x\| = \sup_{\|\xi\| \leq 1} |\xi x|, \qquad \|\xi\| = \sup_{\|x\| \leq 1} |\xi x| \qquad \text{for every} \quad \xi \in \Xi,\ x \in X.$$

Suppose that $A \in \mathfrak{A}$ is bounded. It is easy to verify the equations of the norms:

$$\sup_{\|x\| \leq 1} \|Ax\| = \sup_{\|\xi\| \leq 1} \|\xi A\| = \sup_{\substack{\|\xi\| \leq 1 \\ \|x\| \leq 1}} |\xi A x| = \|A\|.$$

Let us denote by $\mathfrak{M}$ the class of all linear bounded functionals $\mathscr{F}$ on $\mathfrak{A}$ such that operators (bilinear functionals) $T_{\mathscr{F}}$ defined by the formula [11]

$$\xi T_{\mathscr{F}} x = \mathscr{F}(x \cdot \xi) \tag{49}$$

belong to A.

Obviously $T_{\mathscr{F}}$ is also bounded, since $\|T_{\mathscr{F}}\| \leqslant \|\mathscr{F}\|$.

Following Sikorski [12], elements $\mathscr{F} \in \mathfrak{M}$ will be called *quasi-nuclei* [12]. If for an operator T there exists a quasi-nucleus $\mathscr{F}$ such that $T = T_{\mathscr{F}}$, then T is said to be *quasi-nuclear* and $\mathscr{F}$ is said to be a *quasi-nucleus* of T.

If $\xi_1, \ldots, \xi_n$ and $x_1, \ldots, x_n$ are fixed, then the quasi-nucleus $\mathscr{F}$ defined by the formula

$$\mathscr{F}(A) = \sum_{i=1}^{m} \xi_i A x_i \quad \text{for every} \quad A \in \mathfrak{A} \tag{50}$$

will be called *finitely dimensional*, and it will be denoted by

$$\mathscr{F} = \sum_{i=1}^{m} \xi_i \otimes x_i. \tag{50'}$$

Evidently the finitely dimensional operator $\sum_{i=1}^{m} x_i \cdot \xi_i$ is determined by the quasi-nucleus (50′).

In the sequel, if a quasi-nucleus $\mathscr{F}$ is fixed, then, for brevity, instead of $T_{\mathscr{F}}$ we shall write T.

If $\mathscr{F}$ is a quasi-nucleus and $A \in \mathfrak{A}$, then, following Leżański [3], we shall also write $\mathscr{F}_{\eta y}(\eta A y)$ instead of $\mathscr{F}(A)$.

According to this notation, formula (49) can be written in the form

$$\xi T x = \mathscr{F}_{\eta y}(\eta x \cdot \xi y) \qquad (T = T_{\mathscr{F}}), \tag{51}$$

or more generally

$$\xi A_2 T A_1 x = \mathscr{F}_{\eta y}(\eta A_1 x \cdot \xi A_2 y) \quad \text{where} \quad A_1, A_2 \in \mathfrak{A}. \tag{51'}$$

Notice also that

$$|\mathscr{F}_{\eta y}(\eta A y)| \leqslant \|\mathscr{F}\| \|A\| \quad \text{for every} \quad A \in \mathfrak{A}. \tag{52}$$

Let us consider the following expression for $m \leqslant n$:

$$\mathscr{G}\begin{pmatrix} \xi_{m+1}, \ldots, \xi_{n+d} \\ x_{m+1}, \ldots, x_n \end{pmatrix} = \mathscr{F}_{\xi_1 x_1} \ldots \mathscr{F}_{\xi_m x_m}(\xi_{p_1} U x_1 \cdot \ldots \cdot \xi_{p_n} U x_n \cdot \xi_{p_{n+1}} s_1 \cdot \ldots \cdot \xi_{p_{n+d}} s_d), \tag{53}$$

[11] Operators $T_{\mathscr{F}}$ have been introduced by Leżański [3].
[12] For the properties of a quasi-nucleus see Sikorski [12], [14].

where $\mathscr{F} \in \mathfrak{M}$, $U \in \mathfrak{A}$, $s_1, \ldots, s_d$ are fixed and $p_1, \ldots, p_{n+d}$ is an arbitrary permutation of the integers $1, \ldots, n+d$.

Similarly as in paper [3], it can be proved that expression (53) is a $(2(n-m)+d)$-linear functional of the variables $\xi_{m+1}, \ldots, \xi_{n+d}, x_{m+1}, \ldots, x_n$, whose value does not depend on the ordering of $\mathscr{F}_{x_i \xi_i}$. Clearly it satisfies condition (d_3) in the definition of the determinant system.

(xiv) *Let $S \in \mathfrak{A}$ be a generalized Fredholm operator of order zero, let $U \in \mathfrak{A}$ be a quasi-inverse of S, let $s_1, \ldots, s_d$ be linearly independent solutions of $Sx = 0$, and let $\theta_0, \theta_1, \ldots$, be the determinant system for S defined by the formula* (26). *For any $\mathscr{F} \in \mathfrak{M}$, let*

$$D_n(\mathscr{F}) = \sum_{m=0}^{\infty} D_{n,m}(\mathscr{F}), \tag{54}$$

where

$$D_{n,m}(\mathscr{F})\begin{pmatrix} \xi_1, \ldots, \xi_{n+d} \\ x_1, \ldots, x_n \end{pmatrix} = \frac{1}{m!} \mathscr{F}_{\eta_1 y_1} \ldots \mathscr{F}_{\eta_m y_m} \theta_{n+m}\begin{pmatrix} \eta_1, \ldots, \eta_m, \xi_1, \ldots, \xi_{n+d} \\ y_1, \ldots, y_m, x_1, \ldots, x_n \end{pmatrix} \tag{55}$$

for $n = 0, 1, \ldots$

Then the sequence $D_0(\mathscr{F}), D_1(\mathscr{F}), \ldots$ is a determinant system for $S + T_{\mathscr{F}}$ and, by fixed $s_1, \ldots, s_d$, it does not depend on U.

The proof is analogous to that of Leżański [3].

To show that series (54) for $n = 0, 1, \ldots$ are well defined we have first to prove that the series of norms of $(2n+d)$-linear functionals $D_{n,m}(\mathscr{F})$ are convergent.

By Hadamard's inequality and by (52), we obtain

$$\|D_{n,m}(\mathscr{F})\| = \sup_{\substack{\|\xi_i\| \leqslant 1 \\ \|x_j\| \leqslant 1 \\ i=1,\ldots,n+d \\ j=1,\ldots,n}} \left| D_{n,m}(\mathscr{F})\begin{pmatrix} \xi_1, \ldots, \xi_{n+d} \\ x_1, \ldots, x_n \end{pmatrix} \right|$$

$$= \frac{1}{m!} \sup_{\substack{\|\xi_i\| \leqslant 1 \\ \|x_j\| \leqslant 1 \\ i=1,\ldots,n+d \\ j=1,\ldots,n}} \left| \mathscr{F}_{\eta_1 y_1} \ldots \mathscr{F}_{\eta_m y_m} \theta_{n+m}\begin{pmatrix} \eta_1, \ldots, \eta_m, \xi_1, \ldots, \xi_{n+d} \\ y_1, \ldots, y_m, x_1, \ldots, x_n \end{pmatrix} \right|$$

$$\leqslant \frac{\|\mathscr{F}\|^m}{m!} \sup_{\substack{\|\xi_i\| \leqslant 1 \\ \|x_j\| \leqslant 1 \\ i=1,\ldots,n+d \\ j=1,\ldots,n}} \sup_{\substack{\|\eta_p\| \leqslant 1 \\ \|y_q\| \leqslant 1 \\ p,q=1,\ldots,m}} \left| \theta_{n+m}\begin{pmatrix} \eta_1, \ldots, \eta_m, \xi_1, \ldots, \xi_{n+d} \\ y_1, \ldots, y_m, x_1, \ldots, x_n \end{pmatrix} \right|$$

$$\leqslant \frac{\|\mathscr{F}\|^m}{m!} (n+d+m)^{(n+d+m)/2} \|U\|^{n+m} \|s_1\| \ldots \|s_d\|.$$

Thus we have proved

$$(56)\qquad \|D_{n,m}\| \leqslant \frac{\|\mathscr{F}\|^m}{m!}(n+d+m)^{(n+d+m)/2}\|U\|^{n+m}\|s_1\|\dots\|s_d\|.$$

It follows from this immediately that

$$\sum_{m=0}^{\infty}\|D_{n,m}\| < \infty \quad \text{for} \quad n = 0, 1, \dots$$

Clearly, multilinear functionals $D_n(\mathscr{F})$ $(n = 0, 1, \dots)$ satisfy conditions (d_1), (d_2), (d_3).

To prove condition (d_4) let us consider the series

$$(57)\qquad D_0(\lambda\mathscr{F})(\omega_1, \dots, \omega_d) = \sum_{m=0}^{\infty}\lambda^m D_{0\,m}(F)(\omega_1, \dots, \omega_d),$$

where $\omega_1, \dots, \omega_d$ are solutions of $\xi U = 0$ such that $\omega_i s_j = \delta_{i,j}$ for $i, j = 1, \dots, d$. Series (57) is convergent for every complex λ and it is a holomorphic function of the variable λ. It can also be verified that the following identity holds:

$$(58)\qquad \frac{d^n}{d\lambda^n}D_0(\lambda\mathscr{F})(\omega_1, \dots, \omega_d)$$
$$= \mathscr{F}_{\xi_1 x_1}, \dots, \mathscr{F}_{\xi_n x_m} D_n(\lambda\mathscr{F})\begin{pmatrix}\xi_1, \dots, \xi_n, \omega_1, \dots, \omega_d\\ x_1, \dots\dots\dots\dots, x_n\end{pmatrix}.$$

Since $D_0(0)(\omega_1, \dots, \omega_d) = 1$, the holomorphic function (57) is not identically equal to zero. Thus by the well-known property of holomorphic functions, there exists an integer $r \geqslant 0$ such that

$$\left[\frac{d^r}{d\lambda^r}D_0(\lambda\mathscr{F})(\omega_1, \dots, \omega_d)\right]_{\lambda=1} \neq 0.$$

Hence it follows from (58) that $D_r(\mathscr{F}) \neq 0$.

Expanding the determinant $\theta_{n+m+1}\begin{pmatrix}\eta_1, \dots, \eta_m, \xi_0 S, \xi_1, \dots, \xi_{n+d}\\ y_1, \dots, y_m, x_0, x_1, \dots, x_n\end{pmatrix}$ in terms of its $(m+1)$-st row, remembering that $s_1, \dots, s_d$ are solutions of $Sx = 0$ and applying the formula $SU = I$, we obtain

$$(59)\qquad \theta_{n+m+1}\begin{pmatrix}\eta_1, \dots, \eta_m, \xi_0 S, \xi_1, \dots, \xi_{n+d}\\ y_1, \dots, y_m, x_0, \quad x_1, \dots, x_n\end{pmatrix}$$
$$= \sum_{j=0}^{m}(-1)^{m+j}\xi_0 y_j\,\theta_{n+m}\begin{pmatrix}\eta_1, \dots\dots\dots\dots, \eta_m, \xi_1, \dots\dots, \xi_{n+d}\\ y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_m, x_0, x_1, \dots, x_n\end{pmatrix} +$$
$$+ \sum_{i=0}^{n}(-1)^i\xi_0 x_i\,\theta_{n+m}\begin{pmatrix}\eta_1, \dots, \eta_m, \xi_1, \dots\dots\dots\dots\dots, \xi_{n+d}\\ y_1, \dots, y_m, x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n\end{pmatrix}.$$

Applying the operator $\frac{1}{m!}\mathscr{F}_{\eta_1 y_1}\dots\mathscr{F}_{\eta_m y_m}$ to both sides of (59) and performing the calculation, we obtain according to (55)

$$(60)\qquad D_{n+1,m}(\mathscr{F})\begin{pmatrix}\xi_0 S, \xi_1, \dots, \xi_{n+d}\\ x_0, \quad x_1, \dots, x_n\end{pmatrix}$$
$$= -D_{n+1,m-1}(\mathscr{F})\begin{pmatrix}\xi_0 T, \xi_1, \dots, \xi_{n+d}\\ x_0, \quad x_1, \dots, x_n\end{pmatrix} +$$
$$+ \sum_{i=0}^{n}(-1)^i\xi_0 x_i\cdot D_{n,m}(\mathscr{F})\begin{pmatrix}\xi_1, \dots\dots\dots\dots\dots, \xi_{n+d}\\ x_0, x_{i-1}, x_{i+1}, \dots, x_n\end{pmatrix},$$

where $D_{n+1,m-1}(\mathscr{F}) = 0$ for $m = 0$.

Adding the terms of both sides of (60) (from $m = 0$ up to ∞), we obtain

$$D_{n+1}(\mathscr{F})\begin{pmatrix}\xi_0 S, \xi_1, \dots, \xi_{n+d}\\ x_0, \quad x_1, \dots, x_n\end{pmatrix} = -D_{n+1}(\mathscr{F})\begin{pmatrix}\xi_0 T, \xi_1, \dots, \xi_{n+d}\\ x_0, \quad x_1, \dots, x_n\end{pmatrix} +$$
$$+ \sum_{i=0}^{n}(-1)^i\xi_0 x_i D_n(\mathscr{F})\begin{pmatrix}\xi_1, \dots\dots\dots\dots\dots, \xi_{n+d}\\ x_0, \dots, x_{i-1}, x_{i+1}, x_n\end{pmatrix}.$$

Hence the condition (D_n) is satisfied.

The proof of condition (D_n') is analogous.

Expanding the determinant $\theta_{n+m+1}\begin{pmatrix}\eta_1, \dots, \eta_m, \quad \xi_0, \xi_1, \dots, \xi_{n+d}\\ y_1, \dots, y_m, Sx_0, x_1, \dots, x_n\end{pmatrix}$ in terms of its $m+1$-st column, applying the formula $US = I - \sum_{i=1}^{d}s_i\cdot\omega_i$ and basic properties of determinants, we obtain

$$(61)\qquad \theta_{n+m+1}\begin{pmatrix}\eta_1, \dots, \eta_m, \quad \xi_0, \xi_1, \dots, \xi_{n+d}\\ y_1, \dots, y_m, Sx_0, x_1, \dots, x_n\end{pmatrix}$$
$$= \sum_{j=0}^{m}(-1)^{m+j}\eta_j x_0\cdot\theta_{n+m}\begin{pmatrix}\eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_m, \xi_0, \dots, \xi_{n+d}\\ y_1, \dots, y_m, \quad x_1, \dots\dots\dots\dots\dots, x_n\end{pmatrix} +$$
$$+ \sum_{i=0}^{n+d}(-1)^i\xi_i x_0\cdot\theta_{n+m}\begin{pmatrix}\eta_1, \dots, \eta_m, \xi_0, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_{n+d}\\ y_1, \dots, y_m, x_1, \dots\dots\dots\dots\dots, x_n\end{pmatrix}.$$

Applying the operator $\frac{1}{m!}\mathscr{F}_{\eta_1 y_1}\dots\mathscr{F}_{\eta_m y_m}$ to both sides of (61) and calculating, we obtain

$$(62)\quad D_{n+1,m}(\mathscr{F})\begin{pmatrix}\xi_0, \xi_1, \dots, \xi_{n+d}\\ Sx_0, x_1, \dots, x_n\end{pmatrix} = -D_{n+1,m-1}(\mathscr{F})\begin{pmatrix}\xi_0, \xi_1, \dots, \xi_{n+d}\\ Sx_0, x_1, \dots, x_n\end{pmatrix} + \sum_{i=0}^{n+d}(-1)^i \xi_i x_0 \cdot D_{n,m}(\mathscr{F})\begin{pmatrix}\xi_0, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_{n+d}\\ x_1, \dots\dots\dots, x_n\end{pmatrix}.$$

Adding the terms of both sides of (62) (from $m = 0$ up to ∞), we obtain

$$D_{n+1}(\mathscr{F})\begin{pmatrix}\xi_0, \xi_1, \dots, \xi_{n+d}\\ Sx_0, x_1, \dots, x_n\end{pmatrix} = -D_{n+1}(\mathscr{F})\begin{pmatrix}\xi_0, \xi_1, \dots, \xi_{n+d}\\ Tx_0, x_1, \dots, x_n\end{pmatrix} + \sum_{i=0}^{n+d}(-1)^i \xi_i x_0 \cdot D_n(\mathscr{F})\begin{pmatrix}\xi_0, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_{n+d}\\ x_1, \dots\dots\dots, x_n\end{pmatrix}.$$

Thus the condition (D'_n) is also satisfied.

Since the determinant system $\{\theta_n\}$ for S does not depend on U by fixed $s_1, \dots, s_d$, the determinant system (54) for $S+T$ does not depend on U either. This completes the proof.

Theorem (xiv) is very useful. If a quasi-inverse U of S and solutions $s_1, \dots, s_d$ of $Sx = 0$ are known, then for arbitrary quasi-nucleus $\mathscr{F} \in \mathfrak{M}$ we can obtain the determinant system for $S+T_{\mathscr{F}}$ and, applying theorem (x), we can solve the equations:

$$\xi(S+T_{\mathscr{F}}) = \xi_0, \qquad (S+T_{\mathscr{F}})x = x_0.$$

8. Another form of formulae for the determinant system. If $\bar{D}_0(\mathscr{F})$, $\bar{D}_1(\mathscr{F}), \dots$ is a determinant system for Fredholm operator $I+T$, where T is a quasi-nuclear operator determined by quasi-nucleus $\mathscr{F}$, then the following formula holds (see Sikorski [14]):

$$(63)\quad \bar{D}_n(\mathscr{F}) = \sum_{m=0}^{\infty}\frac{1}{m!}\begin{vmatrix} T_n^0 & n & 0 & 0 & 0 \dots & 0\\ T_n^1 & \sigma_1 & m-1 & 0 & 0 \dots & 0\\ T_n^2 & \sigma_2 & \sigma_1 & m-2 & 0 \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ T_n^{m-1} & \sigma_{m-1} & \sigma_{m-2} & \sigma_{m-3} & \dots \sigma_1 & 1\\ T_n^m & \sigma_m & \sigma_{m-1} & \sigma_{m-2} & \dots \sigma_2 & \sigma_1\end{vmatrix}\quad (n = 0, 1, \dots),$$

where $\sigma_m = \mathscr{F}(T^{m-1})$ for $m = 1, 2, \dots$, T_m^n is a $2n$-linear functional

$$(64)\quad T_n^m\begin{pmatrix}\xi_1, \dots, \xi_n\\ x_1, \dots, x_n\end{pmatrix} = \sum\begin{vmatrix}\xi_1 T^{i_1}x_1 \dots \xi_1 T^{i_1}x_n\\ \dots\dots\dots\\ \xi_n T^{i_n}x_1 \dots \xi_n T^{i_n}x_n\end{vmatrix} = \sum\begin{vmatrix}\xi_1 T^{i_1}x_1 \dots \xi_1 T^{i_n}x_n\\ \dots\dots\dots\\ \xi_n T^{i_1}x_1 \dots \xi_n T^{i_n}x_n\end{vmatrix},$$

and $\sum$ is extended over all sequences of non-negative integers $i_1, \dots, i_n$, such that $i_1+i_2+\dots+i_n = m$.

There is a formula for the determinant system (54) for $S+T$, analogous to (63). It can be obtained from the determinant system for $I+UT$ by means of theorem (xii).

For this purpose we introduce the following notation.

If $\mathscr{F} \in \mathfrak{M}$ and $C \in \mathfrak{A}$, then $C\mathscr{F}$ and $\mathscr{F}C$ denote the functionals defined by the formulae [13]

$$C\mathscr{F}(A) = \mathscr{F}(CA), \qquad \mathscr{F}C(A) = \mathscr{F}(AC) \quad \text{for all} \quad A \in \mathfrak{A}.$$

It is obvious that $C\mathscr{F}$ and $\mathscr{F}C$ are quasi-nuclei.

Observe that $C\mathscr{F}$ and $\mathscr{F}C$ determine the quasi-nuclear operators $CT_{\mathscr{F}}$ and $T_{\mathscr{F}}C$, respectively.

Let U be a quasi-inverse of S. If a quasi-nucleus $\mathscr{F}$ determines the quasi-nuclear operator T, then $U\mathscr{F}$ determines the operator $UT_{\mathscr{F}}$ and the determinant system for $I+UT$

$$(65)\quad \bar{D}_n(U\mathscr{F}) = \sum_{m=0}^{\infty}\bar{D}_{n,m}(U\mathscr{F}) \quad (n = 0, 1, \dots),$$

where

$$(65')\quad \bar{D}_{n,m}(U\mathscr{F})\begin{pmatrix}\xi_1, \dots, \xi_n\\ x_1, \dots, x_n\end{pmatrix} = \frac{1}{m!}U\mathscr{F}_{\eta_1 y_1}\dots U\mathscr{F}_{\eta_m y_m}\begin{vmatrix}\eta_1 y_1 \dots \eta_1 y_m & \eta_1 x_1 \dots \eta_1 x_n\\ \dots\dots & \dots\dots\\ \eta_m y_1 \dots \eta_m y_m & \eta_m x_1 \dots \eta_m x_n\\ \xi_1 y_1 \dots \xi_1 y_m & \xi_1 x_1 \dots \xi_1 x_n\\ \dots\dots & \dots\dots\\ \xi_n y_1 \dots \xi_n y_m & \xi_n x_1 \dots \xi_n x_n\end{vmatrix}$$

$$= \frac{1}{m!}\mathscr{F}_{\eta_1 y_1}\dots\mathscr{F}_{\eta_1 y_1}\begin{vmatrix}\eta_1 Uy_1 \dots \eta_1 Uy_m & \eta_1 x_1 \dots \eta_1 x_n\\ \dots\dots & \dots\dots\\ \eta_m Uy_1 \dots \eta_m Uy_m & \eta_m x_1 \dots \eta_m x_n\\ \xi_1 Uy_1 \dots \xi_1 Uy_m & \xi_1 x_1 \dots \xi_1 x_n\\ \dots\dots & \dots\dots\\ \xi_n Uy_1 \dots \xi_n Uy_m & \xi_n x_1 \dots \xi_n x_n\end{vmatrix}.$$

Similarly, $\mathscr{F}U$ determines the determinant system for $I+TU$

$$(66)\quad \bar{D}_n(\mathscr{F}U) = \sum_{m=0}^{\infty}\bar{D}_{n,m}(\mathscr{F}U) \quad (n = 0, 1, \dots),$$

[13] See Sikorski [14].

where

$$(66')\qquad D_{n,m}(\mathscr{F}U)\begin{pmatrix}\xi_1 \ldots \xi_n\\ x_1 \ldots x_n\end{pmatrix}$$

$$=\frac{1}{m!}\mathscr{F}U_{\eta_1 y_1}\ldots\mathscr{F}U_{\eta_m y_m}\begin{vmatrix}\eta_1 y_1\ldots\eta_1 y_m & \eta_1 x_1\ldots\eta_1 x_n\\ \ldots\ldots & \ldots\ldots\\ \eta_m y_1\ldots\eta_m y_m & \eta_m x_1\ldots\eta_m x_n\\ \xi_1 y_1\ldots\xi_1 y_m & \xi_1 x_1\ldots\xi_1 x_n\\ \ldots\ldots & \ldots\ldots\\ \xi_n y_1\ldots\xi_n y_m & \xi_n x_1\ldots\xi_n x_n\end{vmatrix}$$

$$=\frac{1}{m!}\mathscr{F}_{\eta_1 y_1}\ldots\mathscr{F}_{\eta_m y_m}\begin{vmatrix}\eta_1 Uy_1\ldots\eta_1 Uy_m & \eta_1 Ux_1\ldots\eta_1 Ux_n\\ \ldots\ldots & \ldots\ldots\\ \eta_m Uy_1\ldots\eta_m Uy_m & \eta_m Ux_1\ldots\eta_m Ux_n\\ \xi_1 y_1\ldots\xi_1 y_m & \xi_1 x_1\ldots\xi_1 x_n\\ \ldots\ldots & \ldots\ldots\\ \xi_n y_1\ldots\xi_n y_m & \xi_n x_1\ldots\xi_n x_n\end{vmatrix}.$$

Evidently, formula (63) for determinant systems (65) is of the same form. The operator T is replaced by UT and the numbers σ_m are equal

$$(67)\qquad \sigma_m = U\mathscr{F}\big((UT)^{m-1}\big) = \mathscr{F}\big((UT)^{m-1}U\big)\qquad (m = 1, 2, \ldots).$$

If we replace x_i by Ux_i for $i = 1, \ldots, n$ and x_{n+j} by s_j for $j = 1, \ldots, d$ in (65′), then we obtain the determinant system (54) for $S+T$ (see (xii)).

Hence by (63), (64), we obtain the following theorem:

(xv) *The determinant system* (54) *for* $S+T$ *satisfies the identities*

$$D_n(\mathscr{F}) = \sum_{m=0}^{\infty}\frac{1}{m!}\begin{vmatrix}T_n^0 & n & 0 & 0 & 0\ldots 0\\ T_n^1 & \sigma_1 & m-1 & 0 & 0\ldots 0\\ T_n^2 & \sigma_2 & \sigma_1 & m-2 & 0\ldots 0\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ T_n^{m-1} & \sigma_{m-1} & \sigma_{m-2} & \sigma_{m-3} & \ldots\ \sigma_1\, 1\\ T_n^m & \sigma_m & \sigma_{m-1} & \sigma_{m-2} & \ldots\ \sigma_2\,\sigma_1\end{vmatrix}\qquad (n = 0, 1, \ldots),$$

where σ_m $(m = 1, 2, \ldots)$ *are defined by* (67) *and* T_n^m *is the* $(2n+d)$*-linear functional*

$$T_n^m\begin{pmatrix}\xi_1, \ldots, \xi_{n+d}\\ x_1, \ldots, x_n\end{pmatrix} = \sum\det(a_{k,l}) = \sum\det(b_{k,l}),$$

where for $k, l = 1, \ldots, n+d$

$$a_{k,l} = \begin{cases}\xi_k(UT_{\mathscr{F}})^{i_k}Ux_l & \text{for}\quad l \leqslant n,\\ \xi_k(UT_{\mathscr{F}})^{i_k}s_{l-n} & \text{for}\quad l > n;\end{cases}$$

$$b_{k,l} = \begin{cases}\xi_k(UT_{\mathscr{F}})^{i_l}Ux_l & \text{for}\quad l \leqslant n,\\ \xi_k(UT_{\mathscr{F}})^{i_l}Us_{l-n} & \text{for}\quad l > n,\end{cases}$$

and $\sum$ *is extended over all finite sequences of non-negative integers* $i_1, \ldots, i_{n+d}$, *such that* $i_1+\ldots+i_{n+d} = m$.

(xvi) *The following connections between the determinant system* $\{D_n(\mathscr{F})\}$ *for* $S+T_{\mathscr{F}}$ (see (54)) *and the determinant systems* $\{\overline{D}_n(U\mathscr{F})\}$, $\{\overline{D}_n(\mathscr{F}U)\}$ *hold* (see (65), (66)):

$$D_n(\mathscr{F})\begin{pmatrix}\xi_1, \ldots, \xi_n, \omega_1, \ldots, \omega_d\\ Sx_1, \ldots\ldots\ldots, Sx_n\end{pmatrix} = \overline{D}_n(U\mathscr{F})\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix}\qquad (n = 0, 1, \ldots),$$

$$D_n(\mathscr{F})\begin{pmatrix}\xi_1 S, \ldots, \xi_n S, \omega_1 \ldots, \omega_d\\ x_1, \ldots\ldots\ldots, x_n\end{pmatrix} = \overline{D}_n(\mathscr{F}U)\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix}\qquad (n = 0, 1, \ldots),$$

where $\omega_1, \ldots, \omega_d$ *are linearly independent solutions of the equation* $\xi U = 0$ *such that* $\omega_i s_j = \delta_{i,j}$ *for* $i, j = 1, \ldots, d$.

These formulae can be proved by applying the identities $SU = I$, $US = I-\sum_{i=1}^{d} s_i\cdot\omega_i$, to their left sides.

9. Determinant systems in spaces of sequences. Now we shall consider two conjugate spaces Ξ and X of infinite sequences

$$\xi = (\varphi_1, \varphi_2, \ldots)\in\Xi,\qquad x = (v_1, v_2, \ldots)\in X.$$

We suppose:

1) bilinear functional ξx is the ordinary scalar product of the sequences ξ, x, i. e.

$$\xi x = \sum_{i=1}^{\infty}\varphi_i v_i,$$

2) the series ξx is absolutely convergent for each $\xi\in\Xi$ and $x\in X$,

3) the sequences

$$e_1 = (1, 0, 0, \ldots),$$
$$e_2 = (0, 1, 0, \ldots),$$
$$\ldots\ldots\ldots$$

form a basis in Ξ and in X.

Each operator $A \in \mathfrak{A}$ is uniquely represented by an infinite square matrix $a = (a_{i,j})$. Thus we can identify operators A with the corresponding matrices a, writing $A = a = (a_{i,j})$.

In particular, the unit operator is represented by the matrix $\delta = (\delta_{i,j})$.

By a *matrix quasi-nucleus* we understand any quasi-nucleus $\mathscr{T}$ of the form

$$(68) \qquad \mathscr{T}(A) = \sum_{i,j=1}^{\infty} \tau_{j,i} a_{i,j} \quad \text{for} \quad (a_{i,j}) \in \mathfrak{A},$$

where $\tau = (\tau_{i,j})$ is an infinite square matrix. In particular,

$$\mathscr{T}(x \cdot \xi) = \sum_{i,j=1}^{\infty} \varphi_j \tau_{j,i} v_i \quad \text{for} \quad \xi = (\varphi_1, \varphi_2, \ldots), \quad x = (v_1, v_2, \ldots).$$

The class of all matrix quasi-nuclei will be denoted by $\mathfrak{M}_0$. If $\mathscr{T} \in \mathfrak{M}_0$, then the corresponding quasi-nuclear operator will be denoted by T.

It is easily seen that both correspondences

$$T \leftrightarrow \tau, \quad \mathscr{T} \leftrightarrow \tau$$

are one-to-one correspondences. Consequently we can identify T with $\mathscr{T}$:

$$T = \tau = \mathscr{T}.$$

However, this identification is not extended over the norms. In general, the norm of a matrix quasi-nucleus $\mathscr{T} = \tau$ is not equal to the norm of a quasi-nuclear operator $T = \tau$.

Let us consider the operator S defined by the formula

$$Sx = (v_{d+1}, v_{d+2}, \ldots), \quad x = (v_1, v_2, \ldots) \in X.$$

Clearly, S is a generalized Fredholm operator such that $r(S) = 0$, $d(S) = d > 0$.

The operator U defined by the formula

$$Ux = (0, \ldots, 0, v_1, v_2, \ldots), \quad x = (v_1, v_2, \ldots) \in X$$

is a quasi-inverse of S. The elements $e_1, \ldots, e_d$ form bases of solutions of the equations $Sx = 0$ and $\xi U = 0$. It is easy to see that

$$(69) \qquad S = (\delta_{i+d,j}) = s, \quad U = (\delta_{i,j+d}) = u.$$

Let S be fixed and let $\mathfrak{A}_S$ denote the class of matrices (operators) $a \in \mathfrak{A}$ such that

$$(70) \qquad a = S + T = s + \tau = (\delta_{i+d,j}) + (\tau_{i,j}),$$

where $\tau \in \mathfrak{M}_0$.

Every matrix $a \in \mathfrak{A}_S$ has exactly one determinant system (54), provided the solutions $e_1, \ldots, e_d$ of the equation $Sx = 0$ are fixed. This follows from the identification of the matrix τ as a quasi-nuclear operator T with the quasi-nucleus $\mathscr{T} \in \mathfrak{M}_0$ and from the fact that the determinant system does not depend on U. Consequently we can denote the determinant system by $D_0(a), D_1(a), \ldots$ for $a \in \mathfrak{A}_S$.

It follows from theorem (xii) that for every $a \in \mathfrak{A}_S$ we can obtain the determinant system $\{D_n(a)\}$ from $\{D_n(\delta + u\tau)\}$.

For this purpose let us denote by $\mathfrak{A}_0$ the class of matrices (operators) β such that

$$\beta = \delta + \tau = (\delta_{i,j} + \tau_{i,j}), \quad \text{where} \quad \tau = (\tau_{i,j}) \in M_0.$$

For every matrix $\beta \in \mathfrak{A}_0$, the determinant $D_0(\beta)$ will be denoted by $D(\beta)$ or by

$$\begin{vmatrix} \beta_{1,1} & \beta_{1,2} & \cdots \\ \beta_{2,1} & \beta_{2,2} & \cdots \\ \cdot & \cdot & \cdots \end{vmatrix}.$$

If we replace j-th column in the matrix β by terms of a sequence $x \in X$, then the matrix [14] β_x obtained in such a way belongs also to $\mathfrak{A}_0$. Analogously, any matrix ${}_\xi\beta$ obtained from β by replacing the i-th row by terms of a sequence $\xi \in \Xi$ belongs also to $\mathfrak{A}_0$.

Moreover, the determinants $D(\beta_x)$, $D({}_\xi\beta)$ are linear functionals of the variables x and ξ, respectively.

Observe also that every column in β is an element of X and every row in β is an element of Ξ.

Let $\beta \in \mathfrak{A}_0$ and let $i_1, \ldots, i_n$ and $j_1, \ldots, j_n$ be two finite sequences of positive integers. To quote several properties of determinant systems for $a \in \mathfrak{A}_S$, we define, following Sikorski [14], for every $\beta \in \mathfrak{A}_0$, the numbers $\beta\begin{pmatrix} i_1, \ldots, i_n \\ j_1, \ldots, j_n \end{pmatrix}$ as follows: if either in the sequence $i_1, \ldots, i_n$, or in $j_1, \ldots, j_n$, two of the integers are equal, then $\beta\begin{pmatrix} i_1, \ldots, i_n \\ j_1, \ldots, j_n \end{pmatrix} = 0$; otherwise $\beta\begin{pmatrix} i_1, \ldots, i_n \\ j_1, \ldots, j_n \end{pmatrix}$ is the determinant of the matrix belonging to $\mathfrak{A}_0$ and obtained from the matrix β by replacing the i_1-st$, \ldots, i_n$-th column in β by the columns $(\underset{1}{0}, \ldots, \underset{j_1}{1}, 0, \ldots), \ldots, (\underset{1}{0}, \ldots, 0, \underset{j_n}{1}, \ldots)$, respectively, and the j_1-st$, \ldots, j_n$-th row by the rows $(\underset{1}{0}, \ldots, 1, \underset{i_1}{0}, \ldots), \ldots, (\underset{1}{0}, \ldots, 1, \underset{i_n}{0}, \ldots)$.

The following theorem of Sikorski [14] holds:

[14] See Sikorski [14].

For any $\beta \in \mathfrak{A}_0$ the following formulae hold:

$$\beta\binom{i_1, \ldots, i_n}{j_1, \ldots, j_n} = D_n(\beta)\binom{e_{i_1}, \ldots, e_{i_n}}{e_{j_1}, \ldots, e_{j_n}}, \tag{71}$$

$$D_n(\beta)\binom{\xi_1, \ldots, \xi_n}{x_1, \ldots, x_n} \tag{72}$$

$$= \sum_{i_1, \ldots, i_n = 1}^{\infty} \sum_{j_1, \ldots, j_n = 1}^{\infty} \varphi_{1,i_1} \ldots \varphi_{n,i_n} \beta\binom{i_1, \ldots, i_n}{j_1, \ldots, j_n} v_{1,j_1} \ldots v_{n,j_n},$$

where $x_k = (v_{k,1}, v_{k,2}, \ldots)$, $\xi_k = (\varphi_{k,1}, \varphi_{k,2}, \ldots)$ for $k = 1, \ldots, n$;

$$\beta\binom{i_1, \ldots, i_n}{j_1, \ldots, j_n} = \sum_{k=0}^{\infty} \binom{i_1, \ldots, i_n}{j_1, \ldots, j_n}, \tag{73}$$

where

$$\beta_{n,k}\binom{i_1, \ldots, i_n}{j_1, \ldots, j_n} = \sum_{r_1 < r_2 < \ldots < r_k} \begin{vmatrix} \delta_{i_1,j_1} & \ldots & \delta_{i_1,j_n} & \delta_{i_r,r_1} & \ldots & \delta_{i_1,r} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \delta_{i_m,j_1} & \ldots & \delta_{i_n,j_n} & \delta_{i_n,r_1} & \ldots & \delta_{i_1,r_k} \\ \tau_{r_1,j_1} & \ldots & \tau_{r_1,j_n} & \tau_{r_1,r_1} & \ldots & \tau_{r_1,r_k} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \tau_{r_k,j_1} & \ldots & \tau_{r_k,j_n} & \tau_{r_k,r_1} & \ldots & \tau_{r_k,r_k} \end{vmatrix}. \tag{74}$$

Now, let us consider the matrix $\alpha = s + \tau \in \mathfrak{A}_S$ (see (70)) and the Fredholm matrix $\beta = \delta + u\tau \in \mathfrak{A}_0$ where u is a quasi-inverse of s.

It is easy to verify that

$$\beta = \delta + u\tau = \begin{vmatrix} 1 & 0 & 0 & \ldots & & \\ 0 & 1 & 0 & \ldots & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 1 & 0 & \ldots \\ \alpha_{1,1} & \alpha_{1,2} & \ldots & \alpha_{1,d} & \alpha_{1,d+1} & \ldots \\ \alpha_{2,1} & \alpha_{2,2} & \ldots & \alpha_{2,d} & \alpha_{2,d+1} & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}. \tag{75}$$

We introduce the following notation for every $\alpha \in \mathfrak{A}_S$:

$$\alpha\binom{i_1, \ldots, i_{n+d}}{j_1, \ldots, j_n} = \beta\binom{i_1, \ldots \ldots \ldots \ldots, i_{n+d}}{j_1+d, \ldots, j_n+d, 1, \ldots, d} \tag{76}$$

$$(\alpha = s + \tau,\ \beta = \delta + u\tau).$$

Applying theorem (xii) to the theorem of Sikorski, in view of $Ue_i = e_{i+d}$, we obtain the following formulae for every $\alpha \in \mathfrak{A}_S$:

$$\alpha\binom{i_1, \ldots, i_{n+d}}{j_1, \ldots, j_n} = D_n(\alpha)\binom{e_{i_1}, \ldots, e_{i_{n+d}}}{e_{j_1}, \ldots, e_{j_n}} \quad (n = 0, 1, \ldots). \tag{77}$$

Hence, by (72),

$$D_n(\alpha)\binom{\xi_1, \ldots, \xi_{n+d}}{x_1, \ldots, x_n} \tag{78}$$

$$= \sum_{i_1, \ldots, i_{n+d} = 1}^{\infty} \sum_{j_1, \ldots, j_n = 1}^{\infty} \varphi_{1,i_1} \ldots \varphi_{n+d,i_{n+d}} \alpha\binom{i_1, \ldots, i_{n+d}}{j_1, \ldots, j_n} v_{1,j_1} \ldots v_{n,j_n},$$

where $x_k = (v_{k,1}, v_{k,2}, \ldots)$, $\xi_l = (\varphi_{l,n}, \varphi_{l,2} \ldots)$ for $k = 1, \ldots, n$ and $l = 1, \ldots, n+d$. The numbers (76) will be said to be the *coordinates* [15] of the n-subdeterminant $D_n(\alpha)$.

There are formulae for the coordinates of $D_n(\alpha)$ which follow immediately from (73), (74), and (76):

$$\alpha\binom{i_1, \ldots, i_{n+d}}{j_1, \ldots, j_n} = \sum_{m=0}^{\infty} \alpha_{n,m}\binom{i_1, \ldots, i_{n+d}}{j_1, \ldots, j_n}, \tag{79}$$

where

$$\alpha_{n,m}\binom{i_1, \ldots, i_{n+d}}{j_1, \ldots, j_n} \tag{80}$$

$$= \sum_{r_1 < r_2 < \ldots < r_m} \begin{vmatrix} \delta_{i_1,j_1+d} & \ldots \delta_{i_1,j_n+d} & \delta_{i_1,1} & \ldots \delta_{i_1,d} & \delta_{i_1,r_1} & \ldots \delta_{i_1,r_m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \delta_{i_{n+d},j_1+d} & \ldots \delta_{i_{n+d},j_n+d} & \delta_{i_{n+d},1} & \ldots \delta_{i_{n+d},d} & \delta_{i_{n+d},r_1} & \ldots \delta_{i_{n+d},r_m} \\ \tau_{r_1,j_1+d} & \ldots \tau_{r_1,j_n+d} & \tau_{r_1,1} & \ldots \tau_{r_1,d} & \tau_{r_1,r_1} & \ldots \tau_{r_1,r_m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \tau_{r_m,j_1+d} & \ldots \tau_{r_m,j_n+d} & \tau_{r_m,1} & \ldots \tau_{r_m,d} & \tau_{r_m,r_1} & \ldots \tau_{r_m,r_m} \end{vmatrix}.$$

If we take $n = 0$, then we obtain the formulae for $D_0(\alpha)$. Applying the conditions (D_n) and (D'_n) to the expressions $D_1(\alpha)\binom{e_{j_0}A, e_{i_1}, \ldots, e_{i_d}}{e_{j_0}}$,

[15] See Sikorski [14].

$D_1(a)\begin{pmatrix} e_{i_0}, e_{i_1}, \ldots, e_{i_d} \\ Ae_{i_0} \end{pmatrix}$, where $A = a \epsilon \mathfrak{A}_S$, we obtain the following formula:

$$(81)\qquad D_0(a) = \sum_{i,i_1,\ldots,i_d=1}^{\infty} a_{i,j_0}\, a\begin{pmatrix} i, i_1 \ldots, i_d \\ j_0 \end{pmatrix} = \sum_{i_1,\ldots,i_d,j=1}^{\infty} a_{i_0,j}\, a\begin{pmatrix} i_0, i_1, \ldots, i_d \\ j \end{pmatrix}.$$

It is easy to establish by (81) that the value of $D(a)$ at the point $(\xi_1, \ldots, \xi_d)$ is the determinant of the matrix

$$\begin{pmatrix} \varphi_{1,1} & \varphi_{1,2} & \cdots \\ \cdot & \cdot & \cdot \\ \varphi_{d,1} & \varphi_{d,2} & \cdots \\ a_{1,1} & a_{1,2} & \cdots \\ a_{2,1} & a_{2,2} & \cdots \\ \cdot & \cdot & \cdot \end{pmatrix} \qquad (\xi_k = \varphi_{k,1}, \varphi_{k,2}, \ldots \text{ for } k = 1, 2, \ldots, d)$$

belonging to $\mathfrak{A}_0$.

Thus we can write $D(a)(\xi_1, \ldots, \xi_d)$ in the form

$$(82)\qquad D(a)(\xi_1, \ldots, \xi_d) = \begin{vmatrix} \varphi_{1,1} & \varphi_{1,2} & \cdots \\ \cdot & \cdot & \cdot \\ \varphi_{d,1} & \varphi_{d,2} & \cdots \\ a_{1,1} & a_{1,2} & \cdots \\ a_{2,1} & a_{2,2} & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix},$$

or, for brevity,

$$(83)\qquad D(a)(\xi_1, \ldots, \xi_d) = D\Big(ua + \sum_{k=1}^{d} e_k \cdot \xi_k\Big).$$

Using the same method as in proof of (82), we obtain

$$(84)\qquad D_1(a)\begin{pmatrix} \xi, \xi_1, \ldots, \xi_d \\ x \end{pmatrix} = \sum_{i=1}^{\infty} \varphi_i \begin{vmatrix} \varphi_{1,1} & \cdots & \varphi_{1,i-1} & 0 & \varphi_{1,i+1} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi_{d,1} & \cdots & \varphi_{d,i-1} & 0 & \varphi_{d,i+1} & \cdots \\ a_{1,1} & \cdots & a_{1,i-1} & v_1 & a_{1,i+1} & \cdots \\ a_{2,1} & \cdots & a_{2,i-1} & v_2 & a_{2,i+1} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$= \sum_{j=1}^{\infty} \begin{vmatrix} \varphi_{1,1} & \varphi_{1,2} & \cdots \\ \cdot & \cdot & \cdot \\ \varphi_{d,1} & \varphi_{d,2} & \cdots \\ a_{1,1} & a_{1,2} & \cdots \\ \cdot & \cdot & \cdot \\ a_{j-1,1} & a_{j-1,2} & \cdots \\ \varphi_1 & \varphi_2 & \cdots \\ a_{j+1,1} & a_{j+1,2} & \cdots \\ \cdot & \cdot & \cdot \end{vmatrix} v_j,$$

where $x = (v_1, v_2, \ldots)$, $\xi = (\varphi_1, \varphi_2, \ldots)$.

Let $a \epsilon \mathfrak{A}_S$ be of order zero and let $\xi_1, \ldots, \xi_d \epsilon \Xi$ be such that $D(a)(\xi_1, \ldots, \xi_d) \neq 0$. It follows from theorem (x) and from (82), (84), that for every $x_0 = (w_1, w_2, \ldots) \epsilon X$ the system of linear equations

$$(85)\qquad \sum_{j=1}^{\infty} a_{i,j} v_j = w_i \quad (i = 1, 2, \ldots)$$

has a solution of the form

$$(86)\qquad x = \sum_{k=1}^{d} c_k z_k + \overline{x},$$

where c_k are arbitrary constants, $z_k = (z_{k,1}, z_{k,2}, \ldots)$ are linearly independent solutions of a homogeneous system, determined by the formula

$$(87)\qquad z_{k,i} = (-1)^i \begin{vmatrix} \varphi_{1,1} & \cdots & \varphi_{1,i-1} & \varphi_{1,i+1} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi_{k-1,1} & \cdots & \varphi_{k-1,i-1} & \varphi_{k-1,i+1} & \cdots \\ \varphi_{k+1,1} & \cdots & \varphi_{k+1,i-1} & \varphi_{k+1,i+1} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi_{d,1} & \cdots & \varphi_{d,i-1} & \varphi_{d,i+1} & \cdots \\ a_{1,1} & \cdots & a_{1,i-1} & a_{1,i+1} & \cdots \\ a_{2,1} & \cdots & a_{2,i-1} & a_{2,i+1} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \qquad \begin{matrix} (k = 1, \ldots, d, \\ i = 1, 2, \ldots), \end{matrix}$$

and $\overline{x} = (u_1, u_2, \ldots)$ is the only solution of the system (85), determined by the formula

$$u_i = \frac{1}{D_0(\xi_1, \ldots, \xi_d)} \begin{vmatrix} \varphi_{1,1} & \cdots & \varphi_{1,i-1} & 0 & \varphi_{1,i+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \varphi_{d,1} & \cdots & \varphi_{d,i-1} & 0 & \varphi_{d,i+1} & \cdots \\ a_{1,1} & \cdots & a_{1,i-1} & w_1 & a_{1,i+1} & \cdots \\ a_{2,1} & \cdots & a_{2,i-1} & w_2 & a_{2,i+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \quad (i = 1, 2, \ldots)$$

and orthogonal to $\xi_1, \ldots, \xi_d$.

If $\xi_0 = (\psi_1, \psi_2, \ldots) \in \Xi$ is orthogonal to $z_1, \ldots, z$, then there exists the only solution $\xi = (\varphi_1, \varphi_2, \ldots)$ of the adjoint system

$$\sum_{i=1}^{\infty} \varphi_i a_{i,j} = \psi_j \quad (j = 1, 2, \ldots)$$

given by the formula

$$\varphi_j = \frac{1}{D_0(\xi_1, \ldots, \xi_d)} \begin{vmatrix} \varphi_{1,1} & \varphi_{1,2} & \cdots \\ \cdots & \cdots & \cdots \\ \varphi_{d,1} & \varphi_{d,2} & \cdots \\ a_{1,1} & a_{1,2} & \cdots \\ \cdots & \cdots & \cdots \\ a_{j-1,1} & a_{j-1,2} & \cdots \\ \psi_1 & \psi_2 & \cdots \\ a_{j+1,1} & a_{j+1,2} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} \quad (j = 1, 2, \ldots).$$

References

[1] F. W. Atkinson, *Normal solubility of linear equations in normed spaces* [in Russian], Mat. Sb. 28 (1951), p. 3-14.

[2] A. Grothendieck, *La théorie de Fredholm*, Bull. Soc. Math. France 84 (1956), p. 319-384.

[3] T. Leżański, *The Fredholm theory of linear equations in Banach spaces*, Studia Math. 13 (1953), p. 244-276.

[4] — *Sur les fonctionnelles multiplicatives*, ibidem 14 (1953), p. 13-23.

[5] D. Przeworska-Rolewicz, *Sur les équations involutives et leurs applications*, ibidem 20 (1961), p. 95-117.

[6] A. F. Ruston, *On the Fredholm theory of integral equations for operators belonging to the trace class of a general Banach space*, Proceedings of the London Mathematical Society (2) 53 (1951), p. 109-124.

[7] — *Direct product of Banach spaces and linear functional equations*, ibidem (3) 1 (1951), p. 327-384.

[8] R. Sikorski, *On Leżański's determinants of linear equations in Banach spaces*, Studia Math. 14 (1953), p. 24-48.

[9] — *On determinants of Leżański and Ruston*, ibidem 16 (1957), p. 99-112.

[10] — *Determinant systems*, ibidem 18 (1959), p. 161-186.

[11] — *On Leżański endomorphisms*, ibidem 18 (1959), p. 187-188.

[12] — *Remarks on Leżański's determinants*, ibidem 20 (1961), p. 145-161.

[13] — *On the Carlemann determinants*, ibidem 20 (1961), p. 327-346.

[14] — *The determinant theory in Banach Spaces*, Colloquium Mathematicum 8 (1961), p. 141-198.

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