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On the structure of quasi-modular spaces

by

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§ 1. Introduction. Let R be a *universally continuous semi-ordered linear space* (1) (i. e. a *conditionally complete vector-lattice* in Birkhoff's sense [1]) and ϱ be a functional satisfying the following conditions:

(ρ.1) $0 \leqslant \varrho(a) = \varrho(-a) \leqslant +\infty$ *for all* $a \in R$;

(ρ.2) $\varrho(a+b) = \varrho(a)+\varrho(b)$ *if* $a \perp b$ (2);

(ρ.3) *for any orthogonal system* $\{a_\lambda\}_{(\lambda\in\Lambda)}$ *with* $\sum_{\lambda\in\Lambda}\varrho(a_\lambda) < +\infty$, *there exists* $a_0 \in R$ *such that* $a_0 = \sum_{\lambda\in\Lambda} a_\lambda$ *and* $\varrho(a_0) = \sum_{\lambda\in\Lambda}\varrho(a_\lambda)$;

(ρ.4) $\overline{\lim_{\alpha\to 0}}\, \varrho(\alpha a) < +\infty$ *for all* $a \in R$.

R associated with ϱ (denoted by (R, ϱ) shortly) is called a *quasi-modular space* and ϱ is called a *quasi-modular*. The quasi-modular space was first defined in [3] and discussed in [3] and [4].

In [10] Nakano established the theory of *modular spaces* (3), where a *modular* $m(a)$ $(a \in R)$ is a functional on R satisfying (ρ.1), (ρ.2) and the additional conditions:

(i) $m(\xi a)$ *is a convex function of real* $\xi \geqslant 0$ *which is not identically zero but finite in a neighbourhood of* 0 (*depending on* a) *in* $[0, +\infty)$ *for each* $0 \neq a \in R$;

(ii) $|a| \leqslant |b|$ *implies* $m(a) \leqslant m(b)$;

(iii) $0 \leqslant a_\lambda \uparrow_{\lambda\in\Lambda} a$ *implies* $m(a) = \sup_{\lambda\in\Lambda} m(a_\lambda)$.

A modular m is called *monotone complete*, if $\sup_{\lambda\in\Lambda} m(a_\lambda) < +\infty$ and $0 \leqslant a_\lambda \uparrow_{\lambda\in\Lambda}$ imply $\bigcup_{\lambda\in\Lambda} a_\lambda \in R$ [10]. From above, it is easily seen that the

(1) This term is due to Nakano [10].

(2) $a \perp b$, $a, b \in R$, means that $|a| \frown |b| = 0$ and for any set $M \subset R$ we denote by $M^\perp$ the set $\{x : x \in R,\ x \perp y$ for all $y \in M\}$.

(3) Correctly it is called a *modulared semi-ordered linear space* in [10].

concept of a quasi-modular is a generalization of that of a monotone complete modular [4]. On a modular space (R, m) we can define a norm as

$$|||a||| = \inf_{m(\xi a)\leqslant 1} \frac{1}{|\xi|} \ ^{(5)} \qquad (a \in R),$$

and hence we can consider (R, m) as a normed space by this norm. If the modular m is monotone complete, the norm is complete ([10], Theorem 38.6).

We denote by $\bar{R}$ the *conjugate space* of R, i.e. the totality of all *universally continuous linear functionals* [6] on R. R is called *semi-regular*, if $\bar{R}$ is total [7] on R. R is called *non-atomic* if $0 \neq a \in R$ can be decomposed into $a = b+c$ with $b \perp c$ and $b, c \neq 0$.

In the earlier paper [3] we proved the following theorem:

THEOREM A. *If a quasi-modular space (R, ϱ) is semi-regular and non-atomic, R becomes a modular space (R, m_ϱ) with a (convex [8]) perfect [9] modular m_ϱ constructed by ϱ.*

A linear functional $\tilde{a}$ on R is called *bounded*, if $\sup_{|x|\leqslant a} |\tilde{a}(x)| < +\infty$ for all $0 \leqslant a \in R$, and the totality of all bounded linear functionals on R is called the *associated space* of R and denoted by $\tilde{R}$. From the definition, it is clear that $\bar{R} \subseteqq \tilde{R}$ and the equal sign does not hold in general.

The main aim of this paper is to *improve Theorem A by replacing the assumption that R is semi-regular by one that $\tilde{R}$ is total on R.*

First we prove that, if $\tilde{R}$ is total on $R = (R, \varrho)$, there exists the normal manifolds R_r and R_s such that $R = R_r \oplus R_s$ [10], where R_r is semi-regular and R_s is ϱ-singular, i.e. $\varrho(a) = 0$ or $+\infty$ for all $a \in R_s$ (Theorem 2.1). R_s is also decomposed into $R_s = R_\infty \oplus \mathfrak{N}$, where R_∞ has a strong unit and $\varrho(a) = 0$ for all $a \in \mathfrak{N}$ (Theorems 2.2 and 2.3). The space $\mathfrak{N}$ is extremely pathological, if it exists, and intrests us by itself. § 3 is devoted

(4) Recently the concept of a modular was also generalized and discussed by Musielak and Orlicz in [7] and [8].

(5) It is called in [10] the *second norm* by m.

(6) A linear functional f on R is said to be *universally continuous* if $\inf_{\lambda\in\Lambda}|f(a_\lambda)| = 0$ for any system of elements $\{a_\lambda\}_{\lambda\in\Lambda}$ with $a_\lambda \downarrow_{\lambda\in\Lambda} 0$.

(7) This means that if $\bar{a}(x) = 0$ for all $\bar{a} \in \bar{R}$, then $x = 0$.

(8) As is shown above, the essential difference between quasi-modulars and modulars is that the latter $m(\xi a)$ $(a \in R)$ are convex functions for real coefficient ξ. To emphasize convexity of a modular, we use the term "convex modular" in place of "modular".

(9) This means that every universally continuous linear functional is $|||\cdot|||$-bounded.

(10) A manifold $M \subset R$ is called *normal*, if $(M^\perp)^\perp = M$. $R = M \oplus N$ means that $N = (M^\perp)^\perp$ and each $a \in R$ is decomposed into $a = b+c$, $b \in M$, $c \in N$.

to discuss $\mathfrak{N}$ and it is proved that *every spectrum of b by a $(a \neq 0,\ a, b \in \mathfrak{N})$ is a point spectrum.*

In § 4 we prove the main theorem (Theorem 4.1) which improves Theorem A, as is stated above, and some remarks are given in this direction. The improvement of Theorem A enables us to discuss (R, ϱ) from the standpoint of linear topologies on R. In § 5 we paraphrase the results obtained in § 4, [3] and [4] in terms of linear topologies $\mathfrak{T}$ on (R, ϱ) and consequently show a generalization of a theorem due to Mazur and Orlicz ([6], 2.9). In fact we obtain that *if a quasi-modular space (R, ϱ) is topologized by a locally convex separated linear topology $\mathfrak{T}$ which is compatible and monotone complete, we may define a (convex) modular m_ϱ on R such that ϱ-convergence* [11] *coincides with that of the norm induced by m_ϱ.* It is to be noted that as for the Mazur-Orlicz's Theorem there is the nice and faithful generalizations to abstract semi-ordered linear spaces and function spaces by Itô [2].

§ 2. Decomposition theorems. Let (R, ϱ) be a quasi-modular space with a quasi-modular ϱ. We denote by $[p]$ $(p \in R)$ the projection operator defined by the set $\{p\}^{\perp\perp}$, that is, $[p]a = \bigcup_{n=1}^{\infty} (n|p| \frown a)$ for all $0 \leqslant a \in R$ and we call $[p]$ $(p \in R)$ a *projector by p*.

From the definition of ϱ we can easily prove that

$$(2.1) \qquad \varrho(0) = 0, \qquad \varrho(|a|) = \varrho(a)$$

and

$$(2.2) \qquad \varrho([p]a) = \sup_{\lambda\in\Lambda} \varrho([p_\lambda]a) \quad \text{for each} \quad a \in R \text{ and } [p_\lambda] \uparrow_{\lambda\in\Lambda} [p].$$

In [4] we proved the following theorem:

THEOREM B. *If R is a quasi-modular space with a quasi-modular ϱ, the functional ϱ' defined by the formula*

$$(2.3) \qquad \varrho'(a) = \sup_{|x|\leqslant|a|} \varrho(x) \qquad (a \in R)$$

is also a quasi-modular on R satisfying

(ρ.5) $|a| \leqslant |b|$, $a, b \in R$, *implies* $\varrho'(a) \leqslant \varrho'(b)$ ([4], Theorem 2.1).

In the argument below, it is convenient to utilize property (ρ.5) which is not fulfilled by general quasi-modulars. So far as investigation of the structure of (R, ϱ), however, we may assume that the quasi-modular ϱ satisfies (ρ.5) by itself without loss of generality in virtue of Theorem B. Thus we let *quasi-modulars ϱ satisfy* (ρ.5) *in § 2 and § 3* [12].

(11) A sequence of elements $\{a_\nu\}_{(\nu\geqslant 1)} \subset R$ is called *ϱ-convergent to a* if there is a fixed constant $K > 0$ such that $\overline{\lim_{\nu\to\infty}}\, \varrho(\xi(a_\nu - a)) \leqslant K$ for all $\xi > 0$ [3].

(12) The final result, however, comes to be free from (ρ.5) (Theorem 2.3).

An element $a \in R$ is called *ϱ-finite*, if $\varrho(\alpha a) < +\infty$ for every real $\alpha \geqslant 0$. Let F_0 be the totality of all ϱ-finite elements. Now we have

LEMMA 1. *F_0 is a semi-normal manifold (i. e. a linear manifold with the condition: $a \in F_0$ and $|a| \geqslant |b|$, $b \in R$ imply $b \in F_0$) of R.*

Proof. The facts that $|a| \geqslant |b|$ and $a \in F_0$ imply $b \in F_0$ and that $a \in F_0$ implies $2a \in F_0$ are easily verified from the definition of F_0. Since $\varrho(a \cup b) \leqslant \varrho(a) + \varrho(b)$ and $|a+b| \leqslant 2(|a| \cup |b|)$ hold for any $a, b \in R$, we can conclude that F_0 is a linear manifold, q. e. d.

For any $\tilde{a} \in \tilde{R}$ we denote by $\tilde{a}_{F_0}$ the functional $\tilde{a}$ restricted on F_0, i. e., $\tilde{a}_{F_0}(x) = \tilde{a}(x)$ for all $x \in F_0$.

LEMMA 2. *For any $\tilde{a} \in \tilde{R}$, $\tilde{a}_{F_0}$ is a continuous linear functional on F_0 (i. e. $\inf_{\nu \geqslant 1} |\tilde{a}_{F_0}([p_\nu]a)| = 0$ for any $[p_\nu] \downarrow_{\nu=1}^{\infty} 0$ and $a \in F_0$* [13]).

Proof. Since $\tilde{a}$ is written as $\tilde{a} = \tilde{a}^+ - \tilde{a}^-$, $\tilde{a}^+, \tilde{a}^- \geqslant 0$ [14] and $\tilde{a}^+ \perp \tilde{a}^-$, we may assume that $\tilde{a} \geqslant 0$ and $a \geqslant 0$ without loss of generality. Since $[p_\mu] \downarrow_{\mu=1}^{\infty} 0$ implies $\inf_{\mu \geqslant 1} \varrho([p_\mu]\nu a) = 0$ for each $\nu \geqslant 1$ in virtue of (ρ.2) and (2.2), we can find a subsequence $\{[p_{\mu_\nu}]\}_{(\nu \geqslant 1)}$ of $\{[p_\mu]\}_{(\mu \geqslant 1)}$ such that $\varrho([p_{\mu_\nu}]\nu a) \leqslant 1/2^\nu$ for all $\nu \geqslant 1$. We put $[q_\nu] = [p_{\mu_\nu}] - [p_{\mu_{\nu+1}}]$ and $b_\nu = \nu[q_\nu]a$ for all $\nu \geqslant 1$. Since $\{b_\nu\}_{(\nu \geqslant 1)}$ is an orthogonal sequence with $\sum_{\nu=1}^{\infty} \varrho(b_\nu) < +\infty$, there exists $0 \leqslant b_0 = \sum_{\nu=1}^{\infty} b_\nu \in R$ by (ρ.3). Then it follows from above

$$\tilde{a}([p_{\mu_\nu}]a) = \tilde{a}\Big(\sum_{\varrho=\nu}^{\infty} [q_\varrho]a\Big) = \tilde{a}\Big(\sum_{\varrho=\nu}^{\infty} \frac{1}{\varrho}[q_\varrho]\varrho a\Big)$$

$$\leqslant \frac{1}{\nu}\tilde{a}\Big(\sum_{\varrho=\nu}^{\infty} [q_\varrho]\varrho a\Big) \leqslant \frac{1}{\nu}\tilde{a}(b_0),$$

which yields $\inf_{\nu \geqslant 1} \tilde{a}([p_\nu]a) = 0$. Therefore $\tilde{a}_{F_0}$ is continuous, q. e. d.

R is said to be *superuniversally continuous*, if for any $\{a_\lambda\}_{\lambda \in \Lambda}$ with $a_\lambda \leqslant a$ there exists a sequence $\{\lambda_\nu\}_{\nu=1}^{\infty}$, $\lambda_\nu \in \Lambda$, such that $\bigcup_{\nu=1}^{\infty} a_{\lambda_\nu} = \bigcup_{\lambda \in \Lambda} a_\lambda$.

Remark 1. A continuous linear functional $\tilde{a}$ is not always universally continuous. On superuniversally continuous space R, however, every continuous linear functional on R is obviously universally continuous.

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[13] From this it follows that $\tilde{a}(a_\nu) \to 0$ as $\nu \to \infty$ for any $a_\nu \downarrow_{\nu=1}^{\infty} 0$.

[14] a^+ (a^-) is the *positive* (resp. *negative*) *part* of a, i. e. $a^+ = a \cup 0$ (resp. $a^- = -a \cup 0$).

The following lemma was proved by Nakano in the case of modular spaces ([10], Theorem 35.4). As the proof of Lemma 3 is obtained by the quite same way in virtue of (2.2), we omit it here.

LEMMA 3. *For any $a \in R$ with $0 < \varrho(a) < +\infty$, there exists $[p_0]$ ($0 \neq p_0 \in R$) such that $\varrho([p_0]a) = \varrho(a)$ and $\varrho([p]a) = 0$, $[p] \leqslant [p_0]$ implies $p = 0$. $[p_0]R = \{[p_0]x : x \in R\}$ is superuniversally continuous as a space.*

We denote by F the least normal manifold including F_0.

LEMMA 4. *If $\tilde{R}$ is total on R and $[p]F$ is superuniversally continuous, then $[p]F$ is semi-regular.*

Proof. For any $0 < a_0 \in [p]F$ there exists $a \in F_0$ such that $0 < a \leqslant a_0$. Since $\tilde{R}$ is total, there exists also $0 \leqslant \tilde{a} \in \tilde{R}$ such that $\tilde{a}(a) > 0$. Putting $\tilde{a}_0(x) = \sup_{0 \leqslant y \leqslant x,\, y \in F_0} \tilde{a}([p]y)$ for any $0 \leqslant x \in F$ and $\tilde{a}_0(x) = \tilde{a}_0(x^+) - \tilde{a}_0(x^-)$ for any $x \in F$, we obtain a linear functional $\tilde{a}_0$ on $[p]F$ and $\tilde{a}_0(x) = \tilde{a}(x)$ for all $x \in F_0$.

Since $\tilde{a}_{F_0}$ is continuous by Lemma 2 and $[p]F$ is superuniversally continuous, $\tilde{a}_{F_0}$ is a universally continuous linear functional on $[p]F_0$. Hence $\tilde{a}_0$ is also such a one on $[p]F$ by the definition of $\tilde{a}_0$, because, for any $x \geqslant x_\lambda \downarrow_{\lambda \in \Lambda} 0$, $x, x_\lambda \in [p]F$ ($\lambda \in \Lambda$) we have

$$\inf_{\lambda \in \Lambda} \tilde{a}_0(x_\lambda) = \inf_{\lambda \in \Lambda} \{\tilde{a}_0(x) - \tilde{a}_0(x - x_\lambda)\}$$

$$= \tilde{a}_0(x) - \sup_{\lambda \in \Lambda} \tilde{a}_0(x - x_\lambda) = \tilde{a}_0(x) - \sup_{\lambda \in \Lambda} \{\sup_{0 \leqslant y \leqslant x - x_\lambda,\, y \in F_0} \tilde{a}([p]y)\}$$

$$= \tilde{a}_0(x) - \sup_{0 \leqslant y \leqslant x,\, y \in F_0} \tilde{a}([p]y) = \tilde{a}_0(x) - \tilde{a}_0(x) = 0.$$

As $\tilde{a}_0(a_0) \geqslant \tilde{a}(a) > 0$ and $\tilde{a}_0 \in \overline{[p]F}$, $[p]F$ is semi-regular, q. e. d.

A manifold M of (R, ϱ) is called *ϱ-singular*, if $\varrho(a) = 0$ or $= +\infty$ for all $a \in M$. Now we obtain a decomposition theorem:

THEOREM 2.1. *Let (R, ϱ) be a quasi-modular space. If $\tilde{R}$ is total on R, R can be decomposed into $R = R_r \oplus R_s$, where R_r is semi-regular and R_s is ϱ-singular.*

Proof. Let R_r be the totality of all $a \in R$ such that $[a]R$ is semi-regular. Then it is clear that R_r is a normal manifold of R and $\overline{\{R_r^\perp\}} = \{0\}$, because, if $\bar{a}(x) \neq 0$ for some $\bar{a} \in \bar{R}$ and $x \in R_r^\perp$, then there exists $[p]$ ($p \in R_r^\perp$) such that $[p]R$ is semi-regular ([10], Theorem 24.1). Therefore it suffices to prove that $R_r^\perp = R_s$ is ϱ-singular.

Let $0 < \varrho(a) < +\infty$ hold for some $0 \leqslant a \in R_s$. Then, in virtue of Lemma 4, we may assume without loss of generality that $[a]R$ is superuniversally continuous and $\varrho([p]a) = 0$ implies $[p]a = 0$. Let E_ν ($\nu \geqslant 1$) be the totality of all $[p]$ ($p \in R$) such that $[p] \leqslant [a]$ and $\varrho(\nu[p]a) < +\infty$. Putting $[p_\nu] = \bigcup_{[p] \in E_\nu} [p]$, we have to consider the following two cases, that is,

(i) $[p_\nu] = [a]$ *holds for each* $\nu \geqslant 1$;

and

(ii) $[p_{\nu_0}] \lneqq [a]$ *holds for some* $\nu_0 \geqslant 1$.

If (i) holds, then for each $\nu \geqslant 1$ there exists a mutually orthogonal sequence of projectors: $\{[p_{\nu,\mu}]\}_{\mu=1}^{k_\nu}$ such that $[p_{\nu,\mu}] \in E_\nu$ for all $1 \leqslant \mu \leqslant k_\nu$ and

$$\varrho\Big(a - \sum_{\mu=1}^{k_\nu} [p_{\nu,\mu}]\,a\Big) \leqslant \frac{1}{2^{\nu+1}}\,\varrho(a)$$

in virtue of (2.2). We put also $\bigcup_{\mu=1}^{k_\nu} [p_{\nu,\mu}] = [q_\nu]$ for each $\nu \geqslant 1$ and $[p'] = \bigcup_{\nu=1}^{\infty} ([a] - [q_\nu])$. Since

$$\varrho\big(([a]-[q_\nu])\,a\big) \leqslant \frac{1}{2^{\nu+1}}\,\varrho(a)$$

holds for each $\nu \geqslant 1$, we have

$$\varrho([p']a) \leqslant \sum_{\nu=1}^{\infty} \varrho\big(([a]-[q_\nu])\,a\big) \leqslant \frac{1}{2}\,\varrho(a).$$

Therefore $\varrho\big((1-[p'])a\big) \geqslant \frac{1}{2}\varrho(a)$, hence $[p''] = [a]-[p'] \neq 0$ and $[p''] \leqslant [q_\nu]$ for all $\nu \geqslant 1$. Since $\varrho(\nu[p_{\nu,\mu}]a) < +\infty$ for each μ with $1 \leqslant \mu \leqslant k_\nu$ by the definition of E_ν and $[p'']\nu a \leqslant \sum_{\mu=1}^{k_\nu} [p_{\nu,\mu}]\nu a$, we have $\varrho(\nu[p'']a) < +\infty$ for each $\nu \geqslant 1$. This implies $[p'']a \in F_0$ and $[p'']F_0$ ($\subseteq [a]F_0$) is semi-regular. From Lemma 1 and Lemma 4 we conclude that $[p'']R$ is also semi-regular. This is a contradiction, because $[p'']R \subseteq [a]R \subseteq R_r^{\perp}$.

On the other hand, if (ii) holds, we can see that $[p_0]a$ is a *strong unit* in $[p_0]R$, where $[p_0] = [a]-[p_{\nu_0}]$ (i. e. for any $x \in [p_0]R$ there exists a real number $\xi_x \geqslant 0$ such that $|x| \leqslant \xi_x [p_0]a$ holds). Indeed, if such a ξ_x does not exist for some $x \in [p_0]R$, then $[q_\nu] = [(|x|-\nu a)^+][p_0] \neq 0$ for all $\nu \geqslant 1$. Now $[q_\nu]|x| \geqslant \nu[q_\nu]a$ and $\varrho(\nu[q_\nu]a) = +\infty$ for all $\nu > \nu_0$. This implies, for any $\mu \geqslant 1$ and for some σ with $(\mu+\sigma)/\mu > \nu_0$,

$$\varrho\left(\frac{1}{\mu}\,|x|\right) \geqslant \varrho\left(\frac{1}{\mu}\,[q_{\mu+\sigma}]x\right) \geqslant \varrho\left(\frac{\mu+\sigma}{\mu}\,[q_{\mu+\sigma}]\,a\right) = +\infty,$$

which contradicts (ϱ.4) too.

Now we put, for any $0 \leqslant y \in R$,

$$f_0(y) = \sup_{n\geqslant 1} \Big\{\sum_{\nu=1}^{n} \xi_\nu\,\varrho([p_\nu]a)\Big\} \tag{2.4}$$

$$\Big([p_0] = [p_1]+\ldots+[p_n],\ \sum_{\nu=1}^{n} \xi_\nu [p_\nu]a \leqslant y\Big)$$

and for any $y \in R$

$$f(y) = f_0(y^+) - f_0(y^-).$$

It is not difficult to verify that f is a universally continuous linear functional on R and $f([p_0]a) = \varrho([p_0]a) > 0$. This contradicts also $R_r^{\perp} = R_s$. Thus we have proved that R_s must be ϱ-singular, q. e. d.

Remark 2. Since a discrete space is semi-regular, R_s is always non-atomic.

As for the singular part R_s we have

THEOREM 2.2. *R_s can be decomposed into $R_s = R_\infty \oplus \mathcal{N}$, where $\mathcal{N} = \{x\colon x \in R_s,\ \varrho(\xi x) = 0$ for each $\xi\}$ and R_∞ is the normal manifold $\mathfrak{N}^{\perp}$. R_∞ has a strong unit e.*

Proof. It is clear that $\mathfrak{N}$ is a semi-normal manifold. Let $0 \leqslant a = \bigcup_{\lambda\in\Lambda} a_\lambda$, where $0 \leqslant a_\lambda \in \mathfrak{N}$ for each $\lambda \in \Lambda$. Then, putting $[p_\lambda] = [(2a_\lambda - a)^+]$ ($\lambda \in \Lambda$), we see that $2[p_\lambda]a_\lambda \geqslant [p_\lambda]a$ and $[p_\lambda]\uparrow_{\lambda\in\Lambda}[a]$, hence $0 \leqslant \varrho(a) = \sup_{\lambda\in\Lambda} \varrho([p_\lambda]a) \leqslant \sup_{\lambda\in\Lambda}(2[p_\lambda]a_\lambda) = 0$. From this it follows that $\mathfrak{N}$ is a normal manifold. Thus we obtain obviously that $R_s = \mathfrak{N} \oplus R_\infty$ because of $\mathfrak{N}^{\perp} = R_\infty$.

Now we denote by S the set $\{a\colon 0 \leqslant a \in R_\infty,\ \varrho(a) = 0\}$, and we see clearly that S is *directed* (with respect to the relation $\leqslant$). In the sequel we shall show that $\bigcup_{a\in S} a$ exists.

First we shall prove that for any $[p]$ ($p \in R_\infty$) we can find $0 \neq [q] \leqslant [p]$ such that the set $\{[q]a\}_{(a\in S)}$ is order-bounded.

Suppose the contrary case. Then, there exists an element $0 \leqslant p \in R_\infty$ such that the set $\{[p']a\}_{(a\in S)}$ is not order-bounded for any $0 \neq [p'] \leqslant [p]$. Putting $[p_a^n] = [(|a|-np)^+]$ ($a \in S$, $n = 1, 2, \ldots$), we have $\varrho(n[p_a^n]p) \leqslant \varrho([p_a^n]a) = 0$ and $[p_a^n]\uparrow_{a\in S}[p]$ for all $n \geqslant 1$, because $\bigcup_{a\in S}[p_a^n] = [p_0] \lneqq [p]$ for some $n \geqslant 1$ implies $n[p']p \geqslant [p']a$ for all $a \in S$, i. e. the set $\{[p']a\}_{(a\in S)}$ is order-bounded, where $[p'] = [p]-[p_0] \neq 0$. Now, since $\varrho(np) = \sup_{a\in S} \varrho([p_a^n]p)$ by (2.2), we have

$$\varrho(np) = 0 \quad (n = 1, 2, \ldots),$$

which contradicts that $p \in R_\infty$.

Secondly, let the set $\{[q]a\}_{(a\in S)}$ be order-bounded and $b = \bigcup_{a\in S}[q]a$. Since $[p_a] = [([q]a - \frac{1}{2}b)^+]\uparrow_{a\in S}[q]$ and $\varrho([q]a) = 0$ for all $a \in S$, we have $\varrho(\frac{1}{2}[p_a]b) \leqslant \varrho([p_a]a) = 0$ and $\varrho(\frac{1}{2}b) = 0$ on account of (2.2).

Now from above, we can find a mutually orthogonal system of elements $\{b_\gamma\}_{(\gamma\in\Gamma)}$ such that $b_\gamma = \bigcup_{a\in S}[b_\gamma]a$, $\varrho(\frac{1}{2}b_\gamma) = 0$ ($\gamma \in \Gamma$) and $\bigcup_{\gamma\in\Gamma}[b_\gamma] = [R_\infty]$. From this it follows that $\bigcup_{\gamma\in\Gamma}\frac{1}{2}b_\gamma \in R$, hence $\bigcup_{\gamma\in\Gamma} b_\gamma = \bigcup_{\gamma\in\Gamma}\bigcup_{a\in S}[b_\gamma]a = \bigcup_{a\in S}\bigcup_{\gamma\in\Gamma}[b_\gamma]a = \bigcup_{a\in S} a = e \in R$. This e is a strong unit in R_∞, because

$x \in R_\infty$ implies $\varrho(ax) = 0$ for some $a > 0$ and consequently $a|x| \leqslant e$ from the definition of e, q. e. d.

Remark 3. It may happen for this e that $\varrho(e) = +\infty$. $\varrho(a)$ $(a \in R_\infty)$ is not, therefore, a (convex) modular on R_∞ in general (cf. (iii) in § 1).

Now we can remove the additional condition (ρ.5) imposed on ϱ and obtain a general result:

THEOREM 2.3. *Let ϱ be an arbitrary quasi-modular on R (condition* (ρ.5) *is not assumed for ϱ) and $\bar{R}$ be total on R. Then we have $R = (R, \varrho) = R_\gamma \oplus R_\infty \oplus \mathfrak{N}$, where R_γ, R_∞ and $\mathfrak{N}$ are the same as in theorems 2.1 and 2.2.*

Proof. Suppose ϱ' be a quasi-modular defined by the formula (2.3) and ϱ. From theorems 2.1 and 2.2 it follows that $R = (R, \varrho') = R_\gamma \oplus R_\infty \oplus \mathfrak{N}$. As $\varrho \leqslant \varrho'$, $\varrho(a) = 0$ holds for each $a \in \mathfrak{N}$. Also $\varrho(a) = 0$ or $+\infty$ holds for every $a \in R_\infty$, since, in the contrary case, we may find $f \in \bar{R}$ such that $f(a) > 0$ for some $a \in R_\infty$, in the quite same manner as (2.4), q. e. d.

§ 3. A pathological space $\mathfrak{N}$. The normal manifold $\mathfrak{N}$ of R which appeared in the previous section is very pathological, if it exists. In fact, $\mathfrak{N}$ has the following properties:

(3.1) $\mathfrak{N}$ *is universally complete* (15);

(3.2) $\mathfrak{N}$ *is non-atomic*;

(3.3) $\tilde{\mathfrak{N}}$ *is total on* $\mathfrak{N}$ *and each* $\tilde{a} \in \tilde{\mathfrak{N}}$ *is continuous*;

(3.4) $\overline{\mathfrak{N}} = \{0\}$.

Indeed, (ρ.3) and the fact that $\varrho(a) = 0$ for each $a \in \mathfrak{N}$ imply (3.1). (3.2), (3.3) and (3.4) follow from the construction of $\mathfrak{N}$ immediately.

Remark 4. From (3.1) and (3.2) we see clearly that we can define no semi-continuous (16) semi-norm (or quasi-norm) on $\mathfrak{N}$ which is not identically zero.

It is interesting that $\mathfrak{N}$ *has the quite similar aspect as discrete spaces in spite of the fact that* $\mathfrak{N}$ *is non-atomic*, as is shown below.

Now let $\mathfrak{E}$ be the proper space of $\mathfrak{N}$, i. e. the topological space of all maximal ideals $\mathfrak{p}$ (17) of projectors $[p]$ on $\mathfrak{N}$ with a neighbourhood system $\{U_{[p]}\}$ for each $\mathfrak{p} \in \mathfrak{E}$, where $U_{[p]}$ is a set of all $\mathfrak{p}' \in \mathfrak{E}$ to which $[p]$ belongs (18).

(15) This means that for any mutually orthogonal system of elements $\{a_\lambda\}_{\lambda \in \Lambda}$, there exists $\bigcup_{\lambda \in \Lambda} a_\lambda \in R$.

(16) A semi-norm (or a quasi-norm) on R is called *semi-continuous*, if $\sup_{\lambda \in \Lambda} \|a_\lambda\| = \|a\|$ for any $0 \leqslant a_\lambda \uparrow_{\lambda \in \Lambda} a$.

(17) The set of projectors $\mathfrak{p}$ is called an *ideal*, if (i) $\mathfrak{p} \ni [p]$, $[p] \leqslant [q]$ implies $[q] \in \mathfrak{p}$; (ii) $\mathfrak{p} \ni [p], [q]$ implies $\mathfrak{p} \ni [p][q]$; (iii) $\mathfrak{p} \not\ni [0]$.

(18) $U_{[p]}$ is both open and compact.

An element $\mathfrak{p}_0 \in \mathfrak{E}$ is called *bounded point* of $\mathfrak{N}$ [9], if there exists an $a \in \mathfrak{N}$ such that the relative spectrum (19) of b by a, $(b/a, \mathfrak{p}_0)$, is finite for all $b \in \mathfrak{N}$, and an element $\mathfrak{p} \in \mathfrak{E}$ is called *transcendental*, if for any sequence of neighbourhoods $\{U_{[p_\nu]}\}_{(\nu \geqslant 1)}$ of $\mathfrak{p}$ $\bigcap_{\nu=1}^{\infty} U_{[p_\nu]}$ is a neighbourhood of $\mathfrak{p}$ too.

We denote by $C_{\tilde{a}}$ $(\tilde{a} \in \tilde{\mathfrak{N}})$ the *characteristic set* of $\tilde{a}$ in $\mathfrak{E}$, i. e. $C_{\tilde{a}} = (\bigcup_{\tilde{a}[p]=0} U_{[p]})^c$. Since $\mathfrak{N}$ is universally complete and each $\tilde{a} \in \tilde{\mathfrak{N}}$ is continuous, we infer that, for any $\tilde{a} \in \tilde{\mathfrak{N}}$, $C_{\tilde{a}}$ is composed only of a finite number of transcendental points $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k$, by applying the quite same argument given in Theorem 3.1 in [11]. On the other hand, if $\tilde{a} \in \tilde{\mathfrak{N}}$ and $C_{\tilde{a}} = \{\mathfrak{p}_0\}$, then $\mathfrak{p}_0$ is a bounded point of $\mathfrak{N}$ and

$$\tilde{a}(x) = \left(\frac{x}{a}, \mathfrak{p}_0\right) \qquad (x \in \mathfrak{N})$$

holds for some $a \in \mathfrak{N}$ with $U_{[a]} \ni \mathfrak{p}_0$ in virtue of Theorem 1 in [9]. Therefore we obtain (cf. Theorem 3.1 in [11])

THEOREM 3.1. *Let $\mathfrak{A}$ be the set $\bigcup_{\tilde{a} \in \tilde{\mathfrak{N}}} C_{\tilde{a}}$. Then we have*

(i) *C_a is composed only of a finite number of elements of $\mathfrak{E}$ which are both transcendental and bounded;*

(ii) *$\mathfrak{A}$ is dense in $\mathfrak{E}$, i. e. if $\mathfrak{p} \notin U_{[p]}$ for all $\mathfrak{p} \in \mathfrak{A}$, then $[p] = 0$ holds.*

Proof. (i) was already stated above. (ii) is a direct consequence of (3.3) because, in the contrary case, we can find $0 \neq p \in \mathfrak{N}$ such that $U_{[p]} \cap \mathfrak{A} = \emptyset$ and consequently $\tilde{a}(p) = \tilde{a}([p](p)) = 0$ for all $\tilde{a} \in \tilde{\mathfrak{N}}$, which yields that $\tilde{\mathfrak{N}}$ is not total on $\mathfrak{N}$, q. e. d.

It is proved by Nakano ([11], Theorem 1.3) that the cardinal number (20) of any fundamental neighbourhood system of transcendental point $\mathfrak{p}$ must be *singular*, when $\mathfrak{p}$ is not an isolated point. Therefore, as $\mathfrak{N}$ is non-atomic, we have from Theorem 3.1 obviously

Remark 5. If the cardinal number of $\mathfrak{E}$ is regular, $\mathfrak{N}$ vanishes.

From the theorem above we obtain the next theorem which shows the extreme resemblance of $\mathfrak{N}$ to discrete semi-ordered linear spaces.

THEOREM 3.2. *For any $0 \leqslant a, b \in \mathfrak{N}$ there exists a mutually orthogonal system of projectors $\{[p_\alpha]\}_{\alpha \in (-\infty, \infty)}$ such that*

$$[a]b = \sum_{\alpha \in (-\infty, \infty)} \alpha [p_\alpha] b,$$

i. e. every spectrum of b by a $(a, b \in \mathfrak{N})$ is a point spectrum.

(19) For the definition of a relative spectrum see [10], § 10.

(20) A cardinal number β is called *singular* [11], if (i) $\beta > \aleph_0$; (ii) $\beta > \gamma$ implies $\beta > 2^\gamma$; (iii) for any system of cardinal numbers $\gamma_\lambda < \beta$ $(\lambda \in \Lambda)$ with the density $< \beta$ we have $\sum_{\lambda \in \Lambda} \gamma_\lambda < \beta$. A cardinal number γ is called *regular*, if there is no singular cardinal number $\leqslant \gamma$.

Proof. Let $0 \lneq a, b \in \mathfrak{N}$ and $\mathfrak{p}_0 \ni \mathfrak{A} = \bigcup_{\tilde{a} \in \tilde{\mathfrak{N}}} C_{\tilde{a}}$ with $U_{[a]} \ni \mathfrak{p}_0$. As $\mathfrak{p}_0$ is a bounded point of $\mathfrak{N}$ in virtue of the above theorem, $(b/a, \mathfrak{p}_0) = \lambda_{\mathfrak{p}_0} < +\infty$. Since the set $\mathfrak{L}_\nu = \{\mathfrak{p}\colon \mathfrak{p} \in U_{[a]},\ |(b/a, \mathfrak{p}) - \lambda_{\mathfrak{p}_0}| < 1/\nu\}^-$ is open and compact for any $\nu \geqslant 1$, we can find a sequence of projectors $\{[p_\nu]\}_{(\nu \geqslant 1)}$ such that

$$U_{[p_\nu]} = \mathfrak{L}_\nu \quad (\nu = 1, 2, \ldots).$$

Since $\mathfrak{p}_0$ is transcendental, there exists $0 \neq p_0 \in \mathfrak{N}$ with $[p_0] \leqslant \bigcap_{\nu=1}^{\infty} [p_\nu]$ and $U_{[p_0]} \ni \mathfrak{p}_0$ which yields $(b/a, \mathfrak{p}) = \lambda_{\mathfrak{p}_0}$ for all $\mathfrak{p} \in U_{[p_0]}$, hence $[p_0]b = \lambda_{\mathfrak{p}_0}[p_0]a$. Therefore we see that for any $\mathfrak{p} \in U_{[a]} \frown \mathfrak{A}$ there exists a projector $0 \neq [p] = [p_\mathfrak{p}]$ such that $[p_\mathfrak{p}]b = (b/a, \mathfrak{p})[p_\mathfrak{p}]a$ holds.

Now we denote by D the set of all real numbers ξ for which $\xi = (b/a, \mathfrak{p}) = \lambda_\mathfrak{p}$ for some $\mathfrak{p} \in U_{[a]} \frown \mathfrak{A}$ holds. Then we put, for $\alpha \in (-\infty, \infty)$,

$$[p_\alpha] = \begin{cases} 0, & \text{if} \quad \alpha \notin D, \\ \bigcup_{\alpha = \lambda_\mathfrak{p}} [p_\mathfrak{p}], & \text{if} \quad \alpha \in D. \end{cases}$$

As $U_{[a]} \frown \mathfrak{A}$ is dense in $U_{[a]}$, $[a]b = \sum_{\alpha \in (-\infty, \infty)} \alpha [p_\alpha] a$ holds, q. e. d.

§ 4. The main theorem. In the sequel let ϱ be an arbitrary quasi-modular on R. By Theorem 2.3 we have

$$(R, \varrho) = R_r \oplus R_\infty \oplus \mathfrak{N},$$

provided that $\tilde{R}$ is total on R.

In order to exclude the pathological space $\mathfrak{N}$, we have to impose an additional condition on ϱ as follows:

(ρ.0) *for any $a \in R$, there exists $b \in R$ such that $b \in [a]R$ and $0 < \varrho(b)$.*

It is clear that under the condition (ρ.0) $\mathfrak{N}$ does not appear.

Now we obtain

THEOREM 4.1. *If $R = (R, \varrho)$ is non-atomic, $\tilde{R}$ is total on R and ϱ satisfies* (ρ.0), *then R becomes a quasi-modular space (R, m_ϱ) with a perfect (convex) modular m_ϱ constructed from ϱ.*

Proof. Let $R = (R, \varrho) = R_\gamma \oplus R_\infty$. As R_γ is semi-regular, we can define a perfect modular m_γ on R_γ in virtue of Theorem A in § 1 ([3], Theorem 3.1). Since R_∞ has strong unit e, we can also define a convex singular modular m_∞ such as

$$m_\infty(a) = \begin{cases} 0, & \text{if} \quad |a| \leqslant e; \\ +\infty, & \text{otherwise}. \end{cases}$$

Now putting

$$m_\varrho(a) = m_\gamma([R_\gamma]a) + m_\infty([R_\infty]a) \qquad (a \in R),$$

we obtain a (convex) modular on R. As $\bar{R} = \bar{R}_\gamma \oplus \bar{R}_\infty$ and $\bar{R}_\infty = \overline{R_\gamma^\perp} = \{0\}$, we have $\bar{R} = \bar{R}^{m_\gamma} = \bar{R}^{m_\varrho}$, whence m_ϱ is perfect, q. e. d.

Remark 6. m_ϱ is always monotone complete on R_∞, because, $m_\varrho(a) = m_\infty(a) < +\infty$ $(a \in R_\infty)$ if and only if $|a| \leqslant e$.

Theorem 3.2 and 3.4 in [3] hold valid too, if we replace the condition that R is semi-regular by that $\tilde{R}$ is total on R, as m_ϱ is monotone complete on R_∞ and $\mathfrak{N}$ vanishes in those cases on account of Remark 4. For instance, we have

THEOREM 4.2. *Let (R, ϱ) be a quasi-modular space and $\tilde{R}$ be total on R. In order that R be a Banach space with a semi-continuous norm $\|\cdot\|$, it is necessary and sufficient that we can define a monotone complete modular m_ϱ on R. In this case $\|\cdot\|$-convergence coincides with that of $|||\cdot|||$: the modular norm by m_ϱ.*

§ 5. (R, ϱ) with linear topologies. A linear topology $\mathfrak{T}$ on a semi-ordered linear space R is called *normal*, if it contains a fundamental neighbourhood system $\{U_\lambda\}_{\lambda \in \Lambda}$ of 0 satisfying the condition: $U_\lambda \ni a$, $|a| \geqslant |b|$ implies $b \in U_\lambda$ for each $\lambda \in \Lambda$. Also it is called o-*compatible* if it is normal and contains a fundamental system of neighbourhood of 0 composed of order-closed sets [12]. If $\tilde{R}$ is total on R, it is clear that the *weak absolute topology* $\mathfrak{T}_S^*(R, \tilde{R})$ (21) induced from $\tilde{R}$ is a locally convex separated linear topology which is normal. Conversely, let $\mathfrak{T}$ be a locally convex separated linear topology on R which is normal. From Hahn-Banach's Theorem it follows that for any $0 \neq a \in R$ there exists $f \in R'$ (the space of all $\mathfrak{T}$-continuous linear functionals on R) with $f(a) > 0$. As $\mathfrak{T}$ is normal and separated, we have $R' \subset \tilde{R}$ and $\tilde{R}$ comes to be total on R. Thus we obtain, recalling Theorem 4.1,

THEOREM 5.1. *Let a quasi-modular space (R, ϱ) be non-atomic. And let ϱ satisfy* (ρ.0) *and $\mathfrak{T}$ be a locally convex linear topology on R which is normal and separated. Then we may define a perfect (convex) modular m_ϱ on R.*

The modular m_ϱ in Theorem 5.1 may fail to be complete. In order to derive completeness of m_ϱ, we have to impose some additional conditions on linear topologies on R.

A linear topology $\mathfrak{T}$ on R is called *monotone complete* [12], if $0 \leqslant x_\lambda \uparrow_{\lambda \in \Lambda}$ and the set: $\{x_\lambda\}_{\lambda \in \Lambda}$ is topologically bounded, then $\bigcup_{\lambda \in \Lambda} x_\lambda \in R$. Since each o-closed convex neighbourhood V of 0 which is also $\mathfrak{T}$-closed determines a semi-continuous semi-norm $\|\cdot\|_{V}$ on R, the o-compatible locally convex linear topology $\mathfrak{T}$ is given completely by a system of semi-continuous semi-norms.

(21) The *weak absolute topology* $\mathfrak{T}_S^*(R, \tilde{R})$ is a linear topology generated by the sets: $V_{\tilde{a}} = \{x\colon |\tilde{a}|\,|x| \leqslant 1\}$ $(\tilde{a} \in \tilde{R})$ as a fundamental neighbourhood system of 0.

THEOREM 5.2. *Let (R, ϱ) be a quasi-modular space and a locally convex separated linear topology $\mathfrak{T}$ be o-compatible and monotone complete. Then (R, ϱ) becomes a monotone complete modular space (R, m_ϱ) with a (convex) modular m_ϱ and ϱ-convergence coincides with that of the norm induced by m_ϱ.*

Proof. From Theorem 2.1 and Remark 4 we have $R = R_\gamma \oplus R_\infty$. Since $\mathfrak{T}$ is monotone complete, $\overline{\overline{R}} = R_\gamma$ holds by virtue of Theorem of [5] stating that every semi-continuous semi-norm is reflexive ([22]). Hence m_ϱ is monotone complete on R_γ ([10], Theorem 39,5, or [3], Theorem 3.2). On the other hand, m_ϱ is a monotone complete modular on R_∞, whence m_ϱ is also such a one on the whole space R. The remainder of this theorem is obtained by the same manner as Theorem 3.2 in [3], q. e. d.

In [4] we proved that (R, ϱ) is decomposed into $R = R_0 \oplus R_1$, where R_0 is universally complete and R_1 has a semi-continuous quasi-norm $\|\cdot\|_0$ constructed from ϱ, and that the necessary and sufficient condition for the completeness of $\|\cdot\|_0$ on R_1 is that ϱ satisfies ([4], Theorem 3.2)

$$(\rho.4') \qquad \sup_{x \in R} \{\overline{\lim_{\alpha \to 0}}\, \varrho(\alpha x)\} < +\infty.$$

Since the topology $\mathfrak{T}_0$ induced by this quasi-norm $\|\cdot\|_0$ is o-compatible and monotone complete ([23]), we have on account of Theorem 5.2

THEOREM 5.3. *Let (R, ϱ) be a quasi-modular space which has no infinite dimensional universally complete normal manifold and let ϱ satisfy $(\rho.4')$. If the quasi-norm $\|\cdot\|_0$ of R by ϱ is locally convex, we can define a monotone complete (convex) modular m_ϱ on R which induces a norm $|||\cdot|||$ equivalent to $\|\cdot\|_0$ (hence R becomes a Banach space in this case).*

This theorem is considered as a generalization of Theorem ([6], 2.9) of Mazur and Orlicz on L_M and also those of Itô [2], because, in the cases of [6] or [2], we can see without difficulty that the assumptions on (R, ϱ) in Theorem 5.3 are satisfied by the condition setteled on M or ϱ previously.

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