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On positive Banach halfalgebras without identity

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1. Introduction. In a recent number of this journal [1], we presented a theory of positive commutative Banach halfalgebras with identity. It is the purpose of this note to remove the assumption of an identity. We prove that *if H is a positive commutative Banach halfalgebra without identity, then the quotient halfring of H by a regular maximal ideal is the halffield of non-negative real numbers $\mathbf{R}^+$, and, there exists a homomorphism of H into the halfalgebra of all non-negative continuous functions on a locally compact space.*

2. Regular ideals. In our paper on the semiradical of a semiring [3], we introduced an important concept of semiring theory, the algebraic closure of an ideal:

Definition 1. The *algebraic closure* I^- of an ideal I is the ideal of all elements of the semiring which are congruent to 0 modulo I.

The algebraic closure consists of all elements which satisfy an equation $s+i=j$, for some i, j in I. Immediately, $I^= = I$, and $I \subseteq I^-$. If $I = I^-$, then I is said to be *closed*. Any congruence modulo I^- is equivalent to congruence modulo I [3].

Definition 2. An algebraically closed ideal I of a halfring H is called *regular* if H/I contains an identity.

Lemma. *If S is a commutative simple semiring with zero, then either $S^2 = 0$, or S is a semifield.*

Proof. Since S is commutative and simple, then either $xS = 0$, or $xS = S$, for any $x \neq 0$.

Corollary. *If η is a proper homomorphism of a commutative halfring H into a halffield F, then the kernel of η is a regular ideal.*

Proof. The kernel is a maximal and algebraically closed ideal, for F is simple. Since $\eta^a(x) \neq 0$, when $\eta(x) \neq 0$, the lemma implies that $H/\eta^{-1}(0)$ is a halffield. Thus the kernel is regular.

3. **Banach halfalgebras.** In [2] we introduced the concept of a Banach halfalgebra:

Definition 3. A set H of elements $s, t, \ldots$ is a *Banach halfalgebra* if and only if

(1) H is a halfalgebra over the halffield of non-negative reals $\boldsymbol{R}^+$.

(2) H is a semilinear space with invariant metric $d(s,t)$.

(3) $\|st\| \leqslant \|s\|\|t\|$, where $\|s\| = d(s, 0)$, for $s, t \in H$.

(4) H is complete with respect to this norm.

In [2] we proved that a Banach halfalgebra H is embeddable in the Banach algebra $\mathfrak{R}$ over the reals with norm

$$\|\nu(s_1, s_2)\| = \inf_{(u,v)\in\nu^{-1}\nu(s_1,s_2)} \|(u, v)\|.$$

Let $\mathfrak{M}_\mathfrak{R}$ be the set of regular ideals of $\mathfrak{R}$ and H be embedded in $\mathfrak{R}$. We denote the natural homomorphism of $\mathfrak{R}$ onto $\mathfrak{R}/M$, $M \in \mathfrak{M}_\mathfrak{R}$, by φ_M. If we hold s fixed and let M vary over $\mathfrak{M}_\mathfrak{R}$, we obtain a complexe valued function $f_s(M) = \varphi_M(s)$, defined on $\mathfrak{M}_\mathfrak{R}$.

Definition 4. A Banach halfalgebra is *positive* if $f_s(M) \neq -1$ for all $s \in H$ and $M \in \mathfrak{M}_\mathfrak{R}$.

Słowikowski and Zawadowski [4] gave a definition of a positive semiring which implies ours for a commutative Banach halfalgebra. As in our paper [2], we can prove for a positive commutative Banach halfalgebra $f_s(M)$ is a non-negative real number for $s \in H$ and $M \in \mathfrak{M}_\mathfrak{R}$. For each $M \in \mathfrak{M}_\mathfrak{R}$ the restriction of φ_M to H defines a proper homomorphism of H into the halffield $\boldsymbol{R}^+$ of non-negative reals. According to the corollary, this homomorphism defines a regular maximal ideal M^+ of H, such that $f_s(M^+) = f(M)$, for all $s \in H$. Let $\mathfrak{M}_H$ be a set of regular maximal ideals of H. As in [2] we set up a 1-1 correspondence between the sets $\mathfrak{M}_H$ and $\mathfrak{M}_\mathfrak{R}$ such that $f_s(M^+) = f_s(M)$ for any $s \in H$. Since $f_s(M)$ is a non-negative real number, we have

THEOREM 1. *If H is a positive commutative Banach halfalgebra without identity, the quotient halfring of H by a regular maximal ideal is the halffield of non-negative reals.*

Topologizing after Gelfand yields

THEOREM 2. *If H is a positive commutative Banach halfalgebra without identity, then there exists a homomorphism of H into the half-algebra $\boldsymbol{R}^+(\mathfrak{M}_H)$ of all non-negative continuous functions on a locally compact space.*

Example. Let H be a locally compact commutative halfgroup with property F [5]. Rothman [5] proved that H is embeddable in a locally compact commutative group G. Let $L_1^{\boldsymbol{R}}(G)$ be the group algebra of real-valued measurable functions on G and $L_1^{\boldsymbol{R}+}(G) = \{f: f \in L_1^{\boldsymbol{R}}(G) | f(g) = 0,\ g \in G \text{ and } g \notin H\}$. Then $L_1^{\boldsymbol{R}+}(G)$ is a halfalgebra without identity. (*added in proof, 21, 6. 1962*).

Bibliography

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