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On the Picard property of lacunary power series

by

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§ 1. Introduction. We shall prove the following theorem:

THEOREM. *Let* $F(z) = \sum_{j=1}^{\infty} c_j z^{n_j}$ *be analytic when* $|z| < 1$ *and suppose* $\sum_{1}^{\infty} |c_j| = \infty$. *There exists a positive number* q *such that, if the powers* $n_1 < n_2 < \ldots < n_j < \ldots$ *satisfy* $n_{j+1}/n_j > q$ *for* $j = 1, 2, \ldots$, *then to each complex number* w *there correspond infinitely many numbers* z *in the interior of the unit circle satisfying* $F(z) = w$.

It is already known that lacunary power series exhibit this type of behavior on the boundary of the unit circle. More precisely, it has been shown, by purely real-variable methods (see [1] and [3]) that:

If $\sum_{j=1}^{\infty} c_j z^{n_j}$, $n_{j+1}/n_j > q > 1$ *satisfies the conditions* $c_j \to 0$ *and* $\sum_{1}^{\infty} |c_j| = \infty$ *and if* w *is a complex number then there exist infinitely many points* $\xi = e^{i\theta}$ *on the unit circle such that* $\sum_{j=1}^{\infty} c_j \xi^{n_j} = w$.

It follows easily from this that such a power series maps the interior of the unit circle *densely* into the complex plane. Thus, the theorem established in this paper, in particular, completes this result by showing that this *onto* property of lacunary power series holds in the interior of the unit circle as well. It should not be surprising that in order to obtain our result we shall need complex-variable methods in addition to real-variable methods.

There seem to be some essential differences between the two cases occurring when the coefficients of F are bounded and when they are unbounded. We have accordingly split the proof into two parts. In § 2 we consider the former while in § 3 the latter case.

We do not know what is the best lower bound of the q's for which the theorem is valid. The methods of this paper can be refined to show that a lower bound of about 100 can be used. Since we feel that this is not best-possible we shall consistently make simplifying (but very generous) assumptions on the size of q.

§ 2. The bounded coefficients case. We shall study power series of the form

$$F(z) = \sum_1^\infty c_j z^{n_j},$$

where $n_{j+1}/n_j \geqslant q$. We assume that $F(z)$ is analytic in the interior of the unit circle, but $\sum_1^\infty |c_j| = \infty$. Furthermore, in this section we shall assume that $\sup_{1\leqslant j<\infty} |c_j| = M < \infty$.

Let us fix an arbitrary complex number w. Our aim will be to show that there exist infinitely many points z in the interior of the unit circle for which $F(z) = w$. The following is a rough sketch of the idea of the proof.

Let $S_k(\theta)$ denote the partial sum $\sum_1^k c_j e^{in_j\theta}$. For appropriate choices of k, we shall show that $S_k(\theta)$, as θ ranges throughout the interval $[0, 2\pi)$, comes "close" to w. This, in turn, will imply that on a circle of radius r, where k and $r < 1$ are related, at least one of the values $F(re^{i\theta})$ comes within a "small" distance, d of w. We shall show, however, that certain neighbourhoods of each of the points ζ, $|\zeta| = r$, are mapped onto circles about $F(\zeta)$ of radii larger than d, thus implying that the value w is assumed by F. This will be achieved by obtaining appropriate estimates on the derivatives of F on the circle of radius r about the origin.

These three steps of the proof are formulated rigorously in the lemmas below. Before announcing them, however, it will be convenient to introduce the following notation for two expressions that we shall use repeatedly:

1) For technical reasons it will be more convenient at times to deal with the *modified partial sums*

$$S_k^*(\theta) = S_{k-1}(\theta) + \frac{1}{e} c_k e^{in_k\theta},$$

$k > 1$, than with the partial sums introduced above.

2) The expression

$$Q_m = \sum_{j=1}^m |c_j| q^{-m+j},$$

$m \geqslant 1$, $Q_0 = 0$, will arise naturally in two different contexts:

a) In estimating the modulus of the derivative

$$F'(z) = \frac{1}{z} \sum_1^\infty n_j c_j z^{n_j}$$

the lacunarity assumption will enable us to show that one term of this last series is dominant. Suppose that the term in question is $n_{m+1} c_{m+1} z^{n_{m+1}}$; then the modulus of the preceeding terms is certainly bounded by

$$\sum_{j=1}^m n_j |c_j| = n_m \sum_1^m \frac{n_j}{n_m} |c_j| \leqslant n_m \sum_1^m q^{-m+j} |c_j| = n_m Q_m. \tag{2.1}$$

b) A simplification of the idea of the proof that one of the partial sums $S_k(\theta)$ comes "close" to w is the following: Having approximated w with the partial sum $S_m(\theta)$ we then "aim" toward this point with the $(m+1)^{\rm st}$ term of our series and add it to $S_m(\theta)$. That is, we choose an angle Φ such that $n_{m+1}\Phi = \arg\{w - S_m(\theta)\}$ so that the distance between w and $S_m(\theta) + c_{m+1} e^{in_{m+1}\Phi}$ is at least $|c_{m+1}|$ less than the distance $|S_m(\theta) - w|$ (1). This would move us a step ahead in our proof if we could then show that $S_{m+1}(\Phi) = S_m(\Phi) + c_{m+1} e^{in_{m+1}\Phi}$ is "close" to $S_m(\theta) + c_{m+1} e^{in_{m+1}\Phi}$. Thus, we are led to estimate the modulus of the difference $S_m(\theta) - S_m(\Phi)$. Since the function $f(\xi) = e^{in_{m+1}\xi}$ has a full period in an interval of length $2\pi/n_{m+1}$, we can always choose

$$\Phi \in [\theta - \pi/n_{m+1}, \theta + \pi/n_{m+1}).$$

Thus, by the mean value theorem and the fact that $|\theta - \Phi| \leqslant \pi/n_{m+1}$, we obtain

$$\begin{aligned} & |S_m(\theta) - S_m(\Phi)| \\ & \leqslant \sum_{j=1}^m |c_j| \cdot |e^{in_j\theta} - e^{in_j\Phi}| \leqslant \sum_{j=1}^m |c_j| n_j |\theta - \Phi| \leqslant \pi \sum_{j=1}^m \frac{n_j}{n_{m+1}} |c_j| \\ & \leqslant \pi \sum_{j=1}^m q^{(-m+j)-1} |c_j| = \frac{\pi}{q} Q_m. \end{aligned} \tag{2.2}$$

Thus, we see that in both cases, a) and b), we shall be concerned with estimating Q_m from above.

The following lemma is the precise formulation of the first part of the argument sketched above:

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(1) Provided we have not "overshot" w; that is, provided $|S_m(\theta) - w| > |c_{m+1}|$. It is partly for this reason that we shall not always "aim" toward w, and will be forced to employ more complicated arguments in the proof of lemma (2.1).

LEMMA (2.1). *Suppose* $0 < \varepsilon < 1$ *and that* $16 < q\varepsilon^2$; *then there exist infinitely many positive integers* k *and corresponding* $\xi_k \in [0, 2\pi)$ *such that*

(a) $$|S_k^*(\xi_k) - w| \leqslant \varepsilon |c_k|;$$

(b) $$|c_j| \leqslant 2|c_k| \quad \textit{when} \quad j \geqslant k;$$

(c) $$\frac{\pi}{q} Q_{k-1} \leqslant \varepsilon |c_k|.$$

For each k of lemma (2.1) we define a radius $r = r(k)$ by the equation $r^{n_k} = 1/e$. Thus, $1/n_k = \log 1/r$ and, consequently, r tends to 1 as k tends to ∞ in such a way that

(2.3) $$\lim_{k\to\infty} n_k(1-r) = 1 = \lim_{k\to\infty} \frac{1}{(1-r)n_k}.$$

Thus, as is clear from (2.3), for k large enough,

(2.4) $$1 - \frac{\varepsilon}{2} < n_k(1-r)$$

and, moreover,

(2.5) $$n_k(1-r) < 1.$$

For the remainder of this section we shall assume that these inequalities hold for the k's under consideration. This means that we have excluded at most, only a finite number of the k's of lemma (2.1). Now, assuming this lemma for the moment, it is not hard to show

LEMMA (2.2). *For the* ε, k, ξ_k *and* $r = r(k)$ *just described we have*

$$|F(re^{i\xi_k}) - w| \leqslant 2\varepsilon |c_k|.$$

Proof. Since $r^{n_k} = 1/e$, we obtain, for any $\theta \in [0, 2\pi)$,

$$|F(re^{i\theta}) - S_k^*(\theta)| = \Big|\sum_{j=1}^{k-1} c_j(1-r^{n_j})e^{in_j\theta} - \sum_{k+1}^{\infty} c_j r^{n_j} e^{in_j\theta}\Big|$$
$$\leqslant \Big|\sum_{1}^{k-1} c_j(1-r^{n_j})e^{in_j\theta}\Big| + \Big|\sum_{k+1}^{\infty} c_j r^{n_j} e^{in_j\theta}\Big| = \mathrm{I} + \mathrm{II}.$$

But, because of inequality (2.5),

$$1 - r^{n_j} = (1-r)(1+r+r^2+\ldots+r^{n_j-1}) \leqslant (1-r)n_k \frac{n_j}{n_k}$$
$$< (1-r)n_k q^{-k+j} < q^{-k+j}.$$

Thus, by (c) of lemma (2.1),

$$\mathrm{I} = \Big|\sum_{1}^{k-1} c_j(1-r^{n_j})^{in_j\theta}\Big| < \sum_{1}^{k-1} |c_j| q^{-k+j} = \frac{1}{q} Q_{k-1} \leqslant \frac{\varepsilon}{2}|c_k|.$$

Now, using (b) of lemma (2.1) and the assumption $16 < q\varepsilon^2$, which certainly implies both the inequalities $q > 2$ and $2/(e^q - 1) < \varepsilon/2$, we obtain the estimate

$$\mathrm{II} \leqslant 2|c_k| \sum_{k+1}^{\infty} r^{n_j} = 2|c_k| \sum_{k+1}^{\infty} (r^{n_k})^{n_j/n_k} = 2|c_k| \sum_{j=1}^{\infty} \Big(\frac{1}{e}\Big)^{n_{k+j}/n_k}$$
$$\leqslant 2|c_k| \sum_{j=1}^{\infty} \Big(\frac{1}{e}\Big)^{q^j} \leqslant 2|c_k| \sum_{j=1}^{\infty} \Big(\frac{1}{e}\Big)^{qj} \leqslant 2|c_k| \frac{1}{e^q - 1} < \frac{\varepsilon}{2}|c_k|.$$

Thus, we have shown

$$|F(re^{i\theta}) - S_k^*(\theta)| < \varepsilon |c_k|$$

for all $\theta \in [0, 2\pi)$. Now, applying this inequality, when $\theta = \xi_k$, together with inequality (a) of lemma (2.1) we obtain

$$|F(re^{i\xi_k}) - w| \leqslant |F(re^{i\xi_k}) - S_k^*(\xi_k)| + |S_k^*(\xi_k) - w| \leqslant \varepsilon|c_k| + \varepsilon|c_k| = 2\varepsilon|c_k|$$

and the lemma is proved.

We shall now show that, if ε is sufficiently small, the disc of radius $\varrho = (1-r)/2$ about the point $\zeta = re^{i\xi_k}$ is mapped by F onto a region containing all points whose distance from $F(\zeta)$ is $\leqslant 2\varepsilon|c_k|$. Lemma (2.2) and this result clearly imply our theorem. To achieve this end we shall use the following result from the theory of functions ([2]):

LEMMA (2.3). *Suppose* $G(z)$ *is analytic in a region containing the circle* $C(\zeta; \varrho) = \{z; |z-\zeta| \leqslant \varrho\}$ *and that it satisfies the conditions*

(a) $|G'(z)|\varrho \leqslant A$ *for* $|z - \zeta| \leqslant \varrho$;

(b) $|G'(\zeta)|\varrho \geqslant B > 0$.

Then G *maps* $C(\zeta; \varrho)$ *onto a region containing the interior of the circle about* $G(\zeta)$ *of radius* $B^2/6A$.

Proof. Let $H(z) = G(z+\zeta) - G(\zeta)$, where $|z| \leqslant \varrho$. Then $H(z)$ is analytic in the closed disc of radius ϱ about 0 and $H(0) = 0$. Consequently,

(2.6) $$H(z) = a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

([2]) Lemma (2.3), in one form or other, can be found in several places (see, for example, § 12 of Chapter VII in [2]). We give a proof of it here only for the sake of completeness.

Because of inequality (a),

$$|H(z)| = |G(z+\zeta)-G(\zeta)| = \left|\int_{\zeta}^{z+\zeta} G'(v)\,dv\right| \leqslant \frac{A}{\varrho}|z| \leqslant A.$$

Thus, $|a_n| \leqslant A/\varrho^n$ for $n \geqslant 1$. Moreover, because of (b), $|a_1| = |H'(0)| = |G'(\zeta)| \geqslant B/\varrho$. Now, let $r = (B/4A)\varrho < \varrho$ (observe that we must have $B \leqslant A$) and suppose $|z| = r$. Using the estimates we have just obtained on the size of the coefficients of the series (2.6) we have

$$|H(z)| \geqslant |a_1 z| - \left|\sum_{n=2}^{\infty} a_n z^n\right| \geqslant \frac{B}{\varrho} r - \sum_{n=2}^{\infty} \frac{A}{\varrho^n} r^n = B\left(\frac{r}{\varrho}\right) - A\frac{(r/\varrho)^2}{1-(r/\varrho)}$$

$$= B\frac{B}{4A} - A\frac{(B/4A)^2}{1-(B/4A)} \geqslant B^2\left(\frac{1}{4A} - \frac{1/16A}{1-\frac{1}{4}}\right) = B^2\frac{1}{6A} > 0.$$

Thus, if $|w| < B^2/6A$, then, by Rouché's theorem, the function $H(z)-w$ has as many zeros within the circle of radius r about 0 as the function $H(z)$. Since $H(0) = 0$ we conclude that there exists a point z within this circle such that $H(z) = w$. That is, $G(z+\zeta) = G(\zeta)+w$. But the latter is an arbitrary point within the circle about $G(\zeta)$ of radius $B^2/6A$. Thus, the lemma is established.

Thus, in order to apply this lemma to our function F at the point $\zeta = re^{i\xi_k}$ we shall need appropriate estimates on the size of the derivative F' near this point. These are easily derived from lemma (2.1):

LEMMA (2.4). *For the k, r and ξ_k of the first two lemmas we have, for $\zeta = re^{i\xi_k}$,*

(a) $$|F'(z)|\varrho \leqslant 8|c_k| \quad \textit{when} \quad |z-\zeta| \leqslant \varrho = \frac{1-r}{2};$$

(b) $$|F'(\zeta)|\varrho \geqslant \tfrac{1}{12}|c_k|.$$

Before proving this lemma let us observe that our theorem, in case the coefficients are bounded, can now be obtained immediately. By lemma (2.2) we have $|F(\zeta)-w| \leqslant 2\varepsilon|c_k|$. On the other hand, by lemmas (2.3) and (2.4) we see that every point within $(|c_k|/12)^2/48|c_k| = |c_k|/4(12)^3$ of $F(\zeta)$ is assumed by F (the circle $C(\zeta;\varrho)$ is certainly interior to the unit circle). But w is such a point provided $2\varepsilon|c_k| < |c_k|/4(12)^3$. That is, provided

(2.7) $$\varepsilon < (24)^{-3}.$$

This bound on ε is far from best possible — even best possible by the methods we are using. In fact, no attempt is made by us, in this paper, to obtain such a best-possible bound (which is undoubtedly intimately connected with the lower bound on q discussed in the introduction).

Let us now turn to the proof of lemma (2.4). Suppose $|z-\zeta| \leqslant \varrho$; then (see comment after inequalities (2.4) and (2.5))

$$\frac{1}{2} < \frac{3r-1}{2} = r-\varrho \leqslant |z| \leqslant r+\varrho = \frac{r+1}{2} = R < 1.$$

Thus, by inequalities (2.1) and part (b) of lemma (2.1),

$$\frac{1}{2}|F'(z)| \leqslant |zF'(z)| = \left|\sum_{1}^{k-1} n_j c_j z^{n_j} + n_k c_k z^{n_k} + \sum_{k+1}^{\infty} n_j c_j z^{n_j}\right|$$

$$\leqslant \sum_{1}^{k-1} n_j|c_j| + n_k|c_k| + \sum_{k+1}^{\infty} n_j|c_j|R^{n_j}$$

$$\leqslant n_{k-1}Q_{k-1} + n_k|c_k| + 2|c_k|\sum_{k+1}^{\infty} n_j R^{n_j}.$$

Now, because of part (c) of lemma (2.1) and our lacunarity assumption $n_{k-1}/n_k < 1/q$, we have $n_{k-1}Q_{k-1} = (n_{k-1}/n_k)n_k Q_{k-1} < (\varepsilon/\pi)n_k|c_k|$. Furthermore, since our conditions on ε and q certainly imply $q/(q-1) < \frac{3}{2}$, we have

$$n_j = \frac{q}{q-1}n_j\left(1-\frac{1}{q}\right) < \frac{3}{2}n_j\left(1-\frac{n_{j-1}}{n_j}\right) = \frac{3}{2}(n_j - n_{j-1}).$$

Consequently,

$$\sum_{k+1}^{\infty} n_j R^{n_j} \leqslant n_1 R^{n_1} + \sum_{2}^{\infty} n_j R^{n_j} < \frac{3}{2}\left\{n_1 R^{n_1} + \sum_{2}^{\infty}(n_j - n_{j-1})R^{n_j}\right\}$$

$$< \frac{3}{2}\{(1+R+R^2+\ldots+R^{n_1}) + (R^{n_1+1}+R^{n_1+2}+\ldots+R^{n_2})+\ldots\}$$

$$= \frac{3}{2}\sum_{0}^{\infty} R^m = \frac{3}{2}\frac{1}{1-R} = \frac{3}{2}\varrho^{-1}.$$

These estimates, together with inequalities (2.5) and (2.7), give us

$$\frac{1}{2}|F'(z)| < \left(\frac{\varepsilon}{\pi}+1\right)n_k|c_k| + \frac{3}{\varrho}|c_k| < \left(\frac{\varepsilon}{\pi}+1\right)\frac{1}{2\varrho}|c_k| + \frac{3}{\varrho}|c_k|$$

$$\leqslant 2\frac{1}{2\varrho}|c_k| + \frac{3}{\varrho}|c_k| = \frac{4}{\varrho}|c_k|,$$

and inequality (a) of our lemma follows.

We now prove (b):

$$\zeta F'(\zeta) = \sum_{1}^{k-1} n_j c_j \zeta^{n_j} + n_k c_k r^{n_k} e^{in_k\xi_k} + \sum_{k+1}^{\infty} n_j c_j r^{n_j} e^{in_j\xi_k} = P + \frac{n_k c_k}{e}e^{in_k\xi_k} + Q.$$

We have just observed, however, that

$$|P| \leqslant \sum_{1}^{k-1} n_j|c_j| \leqslant \frac{\varepsilon}{\pi} n_k|c_k|.$$

Our assumption on ε certainly implies, therefore, that $|P| \leqslant (n_n/4e)|c_k|$. On the other hand, using (b) of lemma (2.1) and the inequality $4/(e^{q/2}-1) < 1/4e$ (obtainable from the inequalities (2.7) and $16 < q\varepsilon^2$), we see that

$$|Q| \leqslant \sum_{k+1}^{\infty} n_j|c_j|r^{n_j} \leqslant 2|c_k|n_k \sum_{k+1}^{\infty} \frac{n_j}{n_k} r^{n_j} = 2|c_k|n_k \sum_{k+1}^{\infty} \frac{n_j}{n_k} e^{-n_j/n_k}$$

$$\leqslant 2|c_k|n_k \sum_{j=1}^{\infty} 2e^{\frac{1}{2}(n_{k+j}/n_k)} e^{-n_{k+j}/n_k} \leqslant 4|c_k|n_k \sum_{j=1}^{\infty} e^{-\frac{1}{2}q^j}$$

$$\leqslant 4|c_k|n_k \sum_{j=1}^{\infty} e^{-\frac{1}{2}q^j} = \frac{4|c_k|n_k}{e^{\frac{1}{2}q}-1} \leqslant \frac{n_k}{4e}|c_k|.$$

Consequently, using these estimates on $|P|$ and $|Q|$ as well as inequality (2.4) we obtain

$$|F'(\zeta)| \geqslant |\zeta F'(\zeta)| \geqslant \frac{n_k}{e}|c_k| - (|P|+|Q|) \geqslant \frac{n_k}{e}|c_k| - \frac{n_k}{4e}|c_k| - \frac{n_k}{4e}|c_k|$$

$$= \frac{n_k}{2e}|c_k| > \frac{1}{2e}\left(1-\frac{\varepsilon}{2}\right)\frac{|c_k|}{1-r} = \frac{1}{4e\varrho}\left(1-\frac{\varepsilon}{2}\right)|c_k| > \frac{1}{12\varrho}|c_k|,$$

which proves (b).

It only remains for us to prove lemma (2.1). We shall break up the proof into a number of steps.

(i) *There exists a sequence* $\{\xi_m\} \subset [0, 2\pi)$ *such that*

$$|w - S_m(\xi_m)| \geqslant |w| + \left(1 - \frac{\pi}{q-1}\right) \sum_{j=1}^{m} |c_j|. \tag{2.8}$$

Proof. Since

$$\sum_{j=1}^{m-1} Q_j = |c_1|(1+q^{-1}+\ldots+q^{-m+2}) +$$

$$+|c_2|(1+q^{-1}+\ldots+q^{-m+3}) + \ldots + |c_{m-2}|(1+q^{-1}) + |c_{m-1}| \leqslant \frac{q}{q-1} \sum_{j=1}^{m-1} |c_j|,$$

inequality (2.8) will hold if we can construct a sequence $\{\xi_m\} \subset [0, 2\pi)$ such

that

$$|w - S_m(\xi_m)| \geqslant |w| + \sum_{j=1}^{m} |c_j| - \frac{\pi}{q} \sum_{j=1}^{m-1} Q_j. \tag{2.9}$$

We shall do this by an inductive process. Let $\xi_1 \in [0, 2\pi)$ satisfy $\arg\{c_1 e^{in_1\xi_1}\} = \arg\{-w\}$. Then $|w - S_1(\xi_1)| = |w| + |c_1|$ and (2.9) is satisfied when $m = 1$. Now assume we have a value ξ_m for which this inequality holds we shall then find ξ_{m+1} for which it is valid.

In fact, choose $\xi_{m+1} \in [\xi_m - \pi/n_{m+1}, \xi_m + \pi/n_{m+1}) = I$, so that $\arg\{c_{m+1} e^{in_{m+1}\xi_{m+1}}\} = \arg\{S_m(\xi_m) - w\}$. Such a number certainly exists since I contains a full period of the function $e^{in_{m+1}\theta}$. Thus,

$$|w - S_m(\xi_m) - c_{m+1}e^{in_{m+1}\xi_{m+1}}| = |w - S_m(\xi_m)| + |c_{m+1}|.$$

Consequently, because of this inequality, our induction assumption and inequality (2.2) (with $\theta = \xi_m$ and $\Phi = \xi_{m+1}$)

$$|w - S_{m+1}(\xi_{m+1})| = |w - S_m(\xi_{m+1}) - c_{m+1}e^{in_{m+1}\xi_{m+1}}|$$

$$= |\{w - S_m(\xi_m) - c_{m+1}e^{in_{m+1}\xi_{m+1}}\} - \{S_m(\xi_{m+1}) - S_m(\xi_m)\}|$$

$$\geqslant |w - S_m(\xi_m)| + |c_{m+1}| - |S_m(\xi_{m+1}) - S_m(\xi_m)|$$

$$\geqslant |w - S_m(\xi_m)| + |c_{m+1}| - \frac{\pi}{q}Q_m \geqslant |w| + \sum_{j=1}^{m} |c_j| + |c_{m+1}| - \frac{\pi}{q}Q_m$$

and (2.9) is valid for $m+1$.

(ii) *Whenever* $0 \leqslant j < r$

$$Q_j + Q_{j+1} + \ldots + Q_r \leqslant \frac{q}{q-1}Q_j + |c_{j+1}| + |c_{j+2}| + \ldots + |c_r|.$$

Proof. If $m \leqslant j$ the coefficient of $|c_m|$ in $\sum_{s=j}^{r} Q_s$ is

$$\sum_{s=j}^{r} q^{m-s} \leqslant \sum_{s=j}^{\infty} q^{m-s} = q^{m-j} \sum_{s=0}^{\infty} q^{-s} = q^{m-j}\frac{q}{q-1}.$$

Thus, the terms involving $|c_m|$, for $m \leqslant j$, contribute no more than

$$\frac{q}{q-1} \sum_{m=1}^{j} |c_m| q^{m-j} = \frac{q}{q-1} Q_j.$$

If $r \geqslant m > j$, then $|c_m|$ occurs only in $Q_m, Q_{m+1}, \ldots, Q_r$; in the term Q_s, $m \leqslant s \leqslant r$, its coefficient is q^{-s+m}. Thus, its coefficient in the sum $\sum_{s=j}^{r} Q_s$ is $\sum_{s=m}^{r} q^{-s+m} \leqslant q/(q-1)$. Proposition (ii) is now clear.

(iii) There exists a pair of integers (j, J), $0 \leqslant j < J$, and corresponding $\theta_j, \theta_{j+1}, \ldots, \theta_J \in [0, 2\pi)$ such that the numbers $E_m = |w - S_m(\theta_m)|$, $j \leqslant m \leqslant J$ (if $j = 0$ we set $E_0 = |w|$) satisfy

(1) *either*

$$E_m = E_{m-1} + \frac{\pi}{q} Q_{m-1} - \frac{|c_m|}{e}$$

or

$$M \leqslant E_m < E_{m-1} + \frac{\pi}{q} Q_{m-1} - \frac{|c_m|}{e}$$

for $j < m \leqslant J$;

(2) $E_m \geqslant \frac{1+\varepsilon}{e} |c_{m+1}|$ *when* $j \leqslant m < J$;

(3) $E_J < \frac{1+\varepsilon}{e} |c_{J+1}|$.

The pair (j, J) can be chosen so that j (and, hence, J) is arbitrarily large.

Proof. Let j be any integer satisfying

$$|w| + \left(1 - \frac{\pi}{q-1}\right) \sum_{s=1}^{j} |c_s| \geqslant M. \tag{2.10}$$

Such an integer exists since $\sum_{s=1}^{\infty} |c_s| = \infty$ and our conditions on q and ε clearly imply $1 - \pi/(q-1) > 0$. We shall now show how to obtain the integer J (since the only condition on j we need is that (2.10) be satisfied the last part of our proposition is thus obvious). By proposition (i) we can find $\theta_j \in [0, 2\pi)$ such that

$$E_j = |w - S_j(\theta_j)| \geqslant |w| + \left(1 - \frac{\pi}{q-1}\right) \sum_{s=1}^{j} |c_s|.$$

Thus, by (2.10), $M \leqslant E_j$. Since $|c_{j+1}| \leqslant M$ and $(1+\varepsilon)/e < 1$, this shows that (2) is satisfied for $m = j$. Note that we have shown, in this case, $|c_{j+1}| \leqslant E_j$.

Now let $\theta' \in [\theta_j - \pi/n_{j+1}, \theta_j + \pi/n_{j+1})$ satisfy $\arg\{c_{j+1} e^{in_{j+1}\theta'}\} = \arg\{w - S_j(\theta_j)\}$ (as before, we note that such a θ' must exist since we are choosing it in a period-interval of the function $e^{in_{j+1}x}$). Then $E_j - |c_{j+1}| = |w - S_j(\theta_j) - c_{j+1} e^{in_{j+1}\theta'}|$ and, hence (using again the reasoning that gave us inequality (2.2))

$$\begin{aligned} |w - S_{j+1}(\theta')| &= |\{w - S_j(\theta_j) - c_{j+1} e^{in_{j+1}\theta'}\} - \{S_j(\theta') - S_j(\theta_j)\}| \\ &\leqslant E_j - |c_{j+1}| + |S_j(\theta') - S_j(\theta_j)| \leqslant E_j - |c_{j+1}| + \frac{\pi}{q} Q_j \leqslant E_j - \frac{|c_{j+1}|}{e} + \frac{\pi}{q} Q_j. \end{aligned} \tag{2.11}$$

Suppose that there exists $\theta'' \in [0, 2\pi)$ such that

$$|w - S_{j+1}(\theta'')| \geqslant E_j - \frac{|c_{j+1}|}{e} + \frac{\pi}{q} Q_j. \tag{2.12}$$

Then, because of the continuity of $f(\theta) = w - S_{j+1}(\theta)$, the two inequalities (2.11) and (2.12) imply the existence of a number θ_{j+1}

$$E_{j+1} = |w - S_{j+1}(\theta_{j+1})| = E_j - \frac{|c_{j+1}|}{e} + \frac{\pi}{q} Q_j.$$

If, on the other hand, for all $\theta'' \in [0, 2\pi)$, $|w - S_{j+1}(\theta'')| < E_j - |c_{j+1}|/e + (\pi/q) Q_j$, we choose $\theta_{j+1} \in [0, 2\pi)$ such that $E_{j+1} = |w - S_{j+1}(\theta_{j+1})| \geqslant M$. That such a point θ_{j+1} exists follows from proposition (i) (see inequality (2.8)) and inequality (2.10). In either case part (1) of proposition (iii) is satisfied for $m = j+1$.

If, now, $E_{j+1} < \frac{1+\varepsilon}{e} |c_{j+2}|$ we let $J = j+1$ and (iii) is established. If, however, $E_{j+1} \geqslant \frac{1+\varepsilon}{e} |c_{j+2}|$ (that is, part (2) holds for $m = j+1$) we consider two cases: 1° $E_{j+1} \geqslant |c_{j+2}|$ and 2° $E_{j+1} < |c_{j+2}| \leqslant \frac{e}{1+\varepsilon} E_{j+1}$. In the former case, we apply the argument just used, with $(j+1)$ instead of j, to obtain a number θ_{j+2} for which $E_{j+2} = |w - S_{j+2}(\theta_{j+2})|$ satisfies (1) with $m = j+2$. In the second case we proceed as follows: Let $\xi \in [0, 2\pi)$ satisfy

$$|c_{j+2}| = |w - S_{j+1}(\xi)|$$

(such a ξ exists by continuity, our assumption $E_{j+1} = |w - S_{j+1}(\theta_{j+1})| < |c_{j+2}|$ and our choice of j which, together with proposition (i), guarantees the existence of ξ_{j+1} such that $|c_{j+2}| \leqslant M \leqslant |w - S_{j+1}(\xi_{j+1})|$). Now choose $\theta \in [\xi - \pi/n_{j+2}, \xi + \pi/n_{j+2})$ so that $\arg\{c_{j+2} e^{in_{j+2}\theta}\} = \arg\{w - S_{j+1}(\xi)\}$. Hence,

$$\begin{aligned} |w - S_{j+2}(\theta)| &= |\{w - S_{j+1}(\xi) - c_{j+2} e^{in_{j+2}\theta}\} - \{S_{j+1}(\theta) - S_{j+1}(\xi)\}| \\ &= |S_{j+1}(\theta) - S_{j+1}(\xi)| \leqslant \frac{\pi}{q} Q_{j+1} < \frac{\pi}{q} Q_{j+1} + E_{j+1} - \frac{|c_{j+2}|}{e}. \end{aligned}$$

But this last expression is dominated by (our assumption on q certainly implies $1/e-\pi/(q-1)>0$)

$$\frac{\pi}{q}\sum_{s=1}^{j+1}|c_s|q^{-(j+1)+s}+|c_{j+2}|-\frac{|c_{j+2}|}{e}\leqslant M\left(\frac{\pi}{q}\sum_{0}^{\infty}q^{-s}+1-\frac{1}{e}\right)$$
$$=M\left(\frac{\pi}{q-1}+1-\frac{1}{e}\right)\leqslant M.$$

On the other hand, by (i) and inequality (2.10), we can find $\xi_{j+2}\in[0,2\pi)$ such that $|w-S_{j+2}(\xi_{j+2})|\geqslant M$. Thus, by continuity, we can find $\theta_{j+2}\in[0,2\pi)$ such that

$$E_{j+2}=|w-S_{j+2}(\theta_{j+2})|=E_{j+1}+\frac{\pi}{q}Q_{j+1}-\frac{|c_{j+2}|}{e}$$

and (1) is satisfied when $m=j+2$.

We continue this process until we reach an integer $J>j$ for which (3) is true. That such an integer must exist can be seen as follows. If not, for all $r>j$ we would have (using the estimate of proposition (ii))

$$E_{r+1}\leqslant E_r+\frac{\pi}{q}Q_r-\frac{|c_{r+1}|}{e}$$
$$\leqslant\left\{E_{r-1}+\frac{\pi}{q}Q_{r-1}-\frac{|c_r|}{e}\right\}+\frac{\pi}{q}Q_r-\frac{|c_{r+1}|}{e}\leqslant\ldots$$
$$\ldots\leqslant E_j+\frac{\pi}{q}\sum_{s=j}^{r}Q_s-\frac{1}{e}\sum_{s=j+1}^{r+1}|c_s|\leqslant E_j+\frac{\pi}{q-1}Q_j+\frac{\pi}{q-1}\sum_{s=j+1}^{r}|c_s|-\frac{1}{e}\sum_{s=j+1}^{r+1}|c_s|$$
$$\leqslant E_j+\frac{\pi}{q-1}Q_j+\left(\frac{\pi}{q-1}-\frac{1}{e}\right)\sum_{s=j+1}^{r}|c_s|.$$

But, if r is large enough, the last expression is negative and, thus, the inequality is impossible.

(iv) *For each J of proposition* (iii) *we have* $|c_{J+1}|\geqslant\frac{1}{2}\varepsilon Q_J$.

Proof. Condition (3) of (iii) certainly implies $E_J<M$. Thus, by (1), $E_J=E_{J-1}-|c_J|/e+(\pi/q)Q_{J-1}$. Since, by (2), we know that $E_{J-1}\leqslant\frac{1+\varepsilon}{e}|c_J|$, it follows that

$$\frac{\varepsilon}{e}Q_J=\frac{\varepsilon}{e}\left(|c_J|+\frac{1}{q}Q_{J-1}\right)\leqslant\frac{\varepsilon}{e}|c_J|+\frac{\pi}{q}Q_{J-1}\leqslant E_{J-1}-\frac{|c_J|}{e}+\frac{\pi}{q}Q_{J-1}$$
$$=E_J<\frac{1+\varepsilon}{e}|c_{J+1}|.$$

Thus, $\frac{1}{2}\varepsilon Q_J<\frac{\varepsilon}{1+e}Q_J\leqslant|c_{J+1}|$.

(v) *For each J of proposition* (iii) *there exists $\theta_{J+1}\in[0,2\pi)$ such that*

$$|S^*_{J+1}(\theta_{J+1})-w|\leqslant\frac{\varepsilon}{e}|c_{J+1}|+\frac{\pi}{q}Q_J.$$

Proof. Because of inequality (3) of proposition (iii), proposition (i) and inequality (2.10) we are assured (by the usual continuity argument) of a number ξ such that $|S_J(\xi)-w|=\frac{1+\varepsilon}{e}|c_{J+1}|$. Now choose $\theta_{J+1}\in[0,2\pi)$ satisfying $\arg\{c_{J+1}^{in_{J+1}\theta_{J+1}}\}=\arg\{w-S_J(\theta_J)\}$. Then,

$$|S^*_{J+1}(\theta_{J+1})-w|=\left|S_J(\theta_{J+1})+\frac{1}{e}c_{J+1}e^{in_{J+1}\theta_{J+1}}-\omega\right|$$
$$=\left|\left\{S_J(\xi)-w+\frac{1}{e}c_{J+1}e^{in_{J+1}\theta_{J+1}}\right\}+\{S_J(\theta_{J+1})-S_J(\xi)\}\right|\leqslant\frac{\varepsilon}{e}|c_{J+1}|+\frac{\pi}{q}Q_J.$$

(vi) *When J and θ_{J+1} are as in proposition* (v), *then*

(1) $$\frac{\pi}{q}Q_J<\frac{\varepsilon}{2}|c_{J+1}|,$$

(2) $$|S^*_{J+1}(\theta_{J+1})-w|\leqslant\varepsilon|c_{J+1}|.$$

Proof. (1) follows immediately from (iv) and our condition on q, which certainly implies $4\pi/\varepsilon^2<q$; thus

$$\frac{\pi}{q}Q_J<\frac{\varepsilon^2}{4}Q_J=\frac{\varepsilon}{2}\left\{\frac{\varepsilon}{2}Q_J\right\}\leqslant\frac{\varepsilon}{2}|c_{J+1}|.$$

(2) now follows from this estimate and proposition (v):

$$|S^*_{J+1}(\theta_{J+1})-w|\leqslant\frac{\varepsilon}{e}|c_{J+1}|+\frac{\pi}{q}Q_J<\frac{\varepsilon}{2}|c_{J+1}|+\frac{\varepsilon}{2}|c_{J+1}|=\varepsilon|c_{J+1}|.$$

(vii) *If $k>J+1$ is such that $|c_r|\leqslant|c_k|$ for $J+1\leqslant r<k$, then there exists $\zeta\in[0,2\pi)$ such that*

(1) $$\frac{\pi}{q}Q_{k-1}\leqslant\varepsilon|c_k|,$$

(2) $$|S^*_k(\zeta)-w|\leqslant\varepsilon|c_k|.$$

Remark. Let us note that once (vii) is established so is lemma (2.1). If $|c_j|\leqslant2|c_{J+1}|$ when $j\geqslant J+1$ then, by proposition (vi), the conditions (a), (b), (c) of lemma (2.1) hold for $k=J+1$. If, on the other hand, there exists $j>J+1$ for which $|c_j|>2|c_{J+1}|$, let k be the first index larger than $J+1$ such that $2|c_k| \geqslant \sup_{s\geqslant J+1} |c_s|$. Then, clearly k satisfies the hypothesis of proposition (vii) and condition (b) of lemma (2.1) as well. This last fact together with inequalities (1) and (2) gives us the desired lemma.

Proof. Because of (3) of proposition (iii) $E_J = |S_J(\theta_J)-w| \leqslant |c_{J+1}|$. But, by (i) and (2.10) we know that there exists ξ_J such that $|c_{J+1}| \leqslant M \leqslant |S_J(\xi_J)-w|$. Thus, by the continuity argument employed repeatedly above, we conclude that there exists α_J such that $|S_J(\alpha_J)-w| = |c_{J+1}|$. Now let $\alpha_{J+1}\in[\alpha_J-\pi/n_{J+1},\ \alpha_J+\pi/n_{J+1})$ satisfy $\arg\{c_{J+1}e^{in_{J+1}\alpha_{J+1}}\} = \arg\{w-S_J(\alpha_J)\}$. Then $|S_{J+1}(\alpha_{J+1})-w| = |\{S_J(\alpha_J)+c_{J+1}e^{in_{J+1}\alpha_{J+1}}-w\}+ +\{S_J(\alpha_{J+1})-S_J(\alpha_J)\}| = |S_J(\alpha_{J+1})-S_J(\alpha_J)| \leqslant \frac{\pi}{q}Q_J$.

We now choose α_{J+2}. If $|c_{J+2}| < |S_{J+1}(\alpha_{J+1})-w|$ we choose $\alpha_{J+2} \in[\alpha_{J+1}-\pi/n_{J+2},\ \alpha_{J+1}+\pi/n_{J+2})$ satisfying

$$\arg\{c_{J+2}e^{in_{J+2}\alpha_{J+2}}\} = \arg\{w-S_{J+1}(\alpha_{J+1})\}.$$

Then, by the usual argument,

$$\begin{aligned}&|S_{J+2}(\alpha_{J+2})-w|\\ &\quad = |\{S_{J+1}(\alpha_{J+1})+c_{J+2}e^{in_{J+2}\alpha_{J+2}}-w\}+\{S_{J+1}(\alpha_{J+2})-S_{J+1}(\alpha_{J+1})\}|\\ &\quad \leqslant |S_{J+1}(\alpha_{J+1})-w|-|c_{J+2}|+\frac{\pi}{q}Q_{J+1} \leqslant \frac{\pi}{q}(Q_J+Q_{J+1})-|c_{J+2}|.\end{aligned}$$

If, on the other hand, $|c_{J+2}| \geqslant |S_{J+1}(\alpha_{J+1})-w|$ then, as before, we find a point ξ such that $|S_{J+1}(\xi)-w| = |c_{J+2}|$. We now choose $\alpha_{J+2} \in[\xi-\pi/n_{J+2},\ \xi+\pi/n_{J+2})$ satisfying

$$\arg\{c_{J+2}e^{in_{J+2}\alpha_{J+2}}\} = \arg\{w-S_{J+1}(\xi)\}.$$

Thus,

$$\begin{aligned}|S_{J+2}(\alpha_{J+2})-w| &= |\{S_{J+1}(\xi)+c_{J+2}e^{in_{J+2}\alpha_{J+2}}-w\}+\{S_{J+1}(\alpha_{J+2})-S_{J+1}(\xi)\}|\\ &= |S_{J+1}(\alpha_{J+2})-S_{J+1}(\xi)| \leqslant \frac{\pi}{q}Q_{J+1}.\end{aligned}$$

We continue this process until we have defined α_{k-1}. Let p be the largest integer satisfying $J+1\leqslant p\leqslant k-1$ and $|c_p| \geqslant |S_{p-1}(\alpha_{p-1})-w|$. The method we are employing for choosing the numbers α_m, $J+1\leqslant m\leqslant k-1$, implies, therefore, that

$$|S_p(\alpha_p)-w| \leqslant \frac{\pi}{q}Q_{p-1}.$$

Hence, using proposition (ii),

$$\begin{aligned}|S_{k-1}(\alpha_{k-1})-w| &\leqslant \frac{\pi}{q}(Q_{p-1}+\ldots+Q_{k-2})-(|c_{p+1}|+\ldots+|c_{k-1}|)\\ &\leqslant \frac{\pi}{q-1}Q_{p-1}+\frac{\pi}{q-1}|c_p|-\left(1-\frac{\pi}{q-1}\right)\sum_{j=p+1}^{k-1}|c_j| \leqslant \frac{\pi}{q-1}(Q_{p-1}+|c_p|).\end{aligned}$$

But, because of our assumption on $|c_k|$,

$$\begin{aligned}Q_{p-1} &= q^{-1}Q_{p-2}+|c_{p-1}| = q^{-2}Q_{p-3}+q^{-1}|c_{p-2}|+|c_{p-1}| = \ldots\\ &\ldots = q^{J+1-p}Q_J+\sum_{j=2}^{p-J}q^{J+j-p}|c_{J+j-1}| \leqslant Q_J+|c_k|\frac{q}{q-1}.\end{aligned}\tag{2.13}$$

Let us note that exactly the same reasoning gives us the inequality

$$Q_{k-1} \leqslant Q_J+|c_k|\frac{q}{q-1}.\tag{2.14}$$

Using proposition (iv) and our condition on q (which in particular, implies $\pi(q-1)^{-1}\cdot 2\varepsilon^{-1} < \frac{1}{2}\varepsilon$)

$$\begin{aligned}&|S_{k-1}(\alpha_{k-1})-w|\\ &\leqslant \frac{\pi}{q-1}\left(Q_J+\frac{q}{q-1}|c_k|+|c_p|\right) \leqslant \frac{\pi}{q-1}\left(\frac{2}{\varepsilon}|c_{J+1}|+\frac{q}{q-1}|c_k|+|c_p|\right)\\ &\leqslant \frac{\pi}{q-1}\left(\frac{2}{\varepsilon}|c_k|+\frac{q}{q-1}|c_k|+|c_k|\right) < \left(\frac{\varepsilon}{2}+\frac{\pi q}{(q-1)^2}+\frac{\pi}{q-1}\right)|c_k| \leqslant \frac{|c_k|}{e}.\end{aligned}$$

Now, by the usual continuity argument, we can find θ satisfying $|c_k|/e = |S_{k-1}(\theta)-w|$. Now we choose $\zeta\in[\theta-\pi/n_k,\ \theta+\pi/n_k)$ satisfying $\arg\{c_ke^{in_k\zeta}\} = \arg\{w-S_{k-1}(\theta)\}$ and we obtain, using (2.14), (iv) and our condition on q,

$$\begin{aligned}&|S_k^*(\zeta)-w|\\ &= \left|S_{k-1}(\zeta)+\frac{c_k}{e}e^{in_k\zeta}-w\right| = \left|\left\{S_{k-1}(\theta)-w+\frac{c_k}{e}e^{in_k\zeta}\right\}+\{S_{k-1}(\zeta)-S_{k-1}(\theta)\}\right|\\ &= |S_{k-1}(\zeta)-S_{k-1}(\theta)| \leqslant \frac{\pi}{q}Q_{k-1} \leqslant \frac{\pi}{q}Q_J+\frac{\pi}{q-1}|c_k|\\ &\leqslant \frac{\pi}{q}\cdot\frac{2}{\varepsilon}|c_{J+1}|+\frac{\pi}{q-1}|c_k| \leqslant \frac{2\pi\varepsilon}{16}|c_k|+\frac{\pi}{q-1}|c_k| < \varepsilon|c_k|.\end{aligned}$$

Hence, inequality (2) of (vii) is established. We note that the last part of this argument gives us inequality (1). Lemma (2.1) and, thus, our theorem in case the coefficients are bounded are proved.

§ 3. The unbounded coefficients case. Let us now assume that the coefficients of $F(z) = \sum_1^\infty c_j z^{n_j}$ satisfy $c_1 \neq 0$ and $\limsup_{j\to\infty} |c_j| = \infty$. Since F is assumed to be analytic in the interior of the unit circle, however, we do have $\sum_1^\infty |c_j| \varrho^{n_j} < \infty$ whenever $0 \leqslant \varrho < 1$. We shall assume that q and ε satisfy the same inequality as in § 2.

Again we shall show that on each of a sequence of increasing circles, whose radii tend to 1, we can find a point at which F approximates a fixed complex number w. We then shall obtain estimates on the derivative of F near this point which, as in the previous case, allow us to apply lemma (2.3) to show that w must be assumed by F.

The following lemma, which shall be applied to appropriate partial sums of F, can be considered to be an analog to lemma (2.1):

LEMMA (3.1). *Suppose* $P(\theta) = b_1 e^{in_1\theta} + b_2 e^{in_2\theta} + \ldots + b_N e^{in_N\theta}$ *satisfies* $n_{k+1}/n_k > q$, $1 \leqslant k \leqslant N-1$, *and* $M \leqslant \sum' |b_k|$, *where* $M = \max_{1\leqslant k\leqslant N} |b_k|$ *and the symbol* $\sum'$ *denotes summation over all terms except one at which this maximum is attained. Suppose, furthermore, that* $|w| \leqslant (1-\varepsilon) \sum_{k=1}^N |b_k|$. *Then there exists* θ_N *such that*

$$|P(\theta_N) - w| \leqslant \tfrac{3}{4}\varepsilon M.$$

Proof. We shall define $\theta_1, \theta_2, \ldots, \theta_N$ inductively in such a way that θ_N satisfies the conclusion of our lemma. First, $\theta_1 \in [0, 2\pi)$ is chosen so that $\arg\{b_1 e^{in_1\theta_1}\} = \arg\{w\}$ is safisfied. Now, suppose θ_k, $1 \leqslant k < N$, has been chosen; we then show how to obtain θ_{k+1}. In case $|b_{k+1}| \leqslant |S_k(\theta_k) - w|$, where $S_k(\theta)$ denotes the partial sum $\sum_{j=1}^k b_j e^{in_j\theta}$, we choose $\theta_{k+1} \in [\theta_k - \pi/n_{k+1}, \theta_k + \pi/n_{k+1})$ satisfying $\arg\{b_{k+1} e^{in_{k+1}\theta_{k+1}}\} = \arg\{w - S_k(\theta_k)\}$. If, on the other hand, $|b_{k+1}| > |S_k(\theta_k) - w|$ we consider two cases:

(a) There exists $\theta_k' \in [0, 2\pi)$ such that $|S_k(\theta_k') - w| \geqslant |b_{k+1}|$;

(b) for all $\theta \in [0, 2\pi)$, $|S_k(\theta) - w| < |b_{k+1}|$.

If case (a) holds, the continuity of S_k implies the existence of a θ such that $|S_k(\theta) - w| = |b_{k+1}|$. We then choose $\theta_{k+1} \in [\theta - \pi/n_{k+1}, \theta + \pi/n_{k+1})$ so that $\arg\{b_{k+1} e^{in_{k+1}\theta_{k+1}}\} = \arg\{w - S_k(\theta)\}$.

If case (b) holds, let θ satisfy $|w - S_k(\theta)| = \max_{\Phi\in[0,2\pi)} |w - S_k(\Phi)|$. We then choose $\theta_{k+1} \in [\theta - \pi/n_{k+1}, \theta + \pi/n_{k+1})$ again, so that

$$\arg\{b_{k+1} e^{in_{k+1}\theta_{k+1}}\} = \arg\{w - S_k(\theta)\}.$$

Let $S_0(\theta_0) = 0$. We claim that for at least one k, $0 \leqslant k < N$, we have

$$|b_{k+1}| > |S_k(\theta_k) - w|. \tag{3.1}$$

Suppose not. Then, by our choice of θ_1 and this hypothesis (when $k = 0$ it asserts $|b_1| \leqslant |w|$), we have $|S_1(\theta_1) - w| = |w| - |b_1|$. Consequently, since we assume $|b_2| \leqslant |S_1(\theta_1) - w|$,

$$\begin{aligned} &|S_2(\theta_2) - w| \\ &= |\{S_1(\theta_1) - w + b_2 e^{in_2\theta_2}\} + \{S_1(\theta_2) - S_1(\theta_1)\}| \leqslant |S_1(\theta_1) - w| - |b_2| + \frac{\pi}{q} Q_1 \\ &\leqslant |w| + \frac{\pi}{q} Q_1 - |b_1| - |b_2| \end{aligned}$$

(see inequality (2.2)).

Continuing in this way we obtain (using proposition (ii) of the last section)

$$\begin{aligned} &|S_{k+1}(\theta_{k+1}) - w| \\ &\leqslant |w| - \sum_{j=1}^{k+1} |b_j| + \frac{\pi}{q} \sum_{j=1}^{k} Q_j \leqslant |w| - \sum_{j=1}^{k+1} |b_j| + \frac{\pi}{q-1} Q_1 + \frac{\pi}{q-1} \sum_{j=2}^{k} |b_j| \\ &\leqslant |w| - \sum_{j=1}^{k+1} |b_j| + \frac{\pi}{q-1} \sum_{j=1}^{k+1} |b_j| = |w| - \left(1 - \frac{\pi}{q-1}\right) \sum_{j=1}^{k+1} |b_j|. \end{aligned}$$

Thus, when $k = N-1$, this would give us

$$\begin{aligned} |S_N(\theta_N) - w| &\leqslant |w| - \left(1 - \frac{\pi}{q-1}\right) \sum_{j=1}^{N} |b_j| \\ &\leqslant \left\{(1-\varepsilon) - \left(1 - \frac{\pi}{q-1}\right)\right\} \sum_{j=1}^{N} |b_j|. \end{aligned} \tag{3.2}$$

But our assumption on the values of q and ε certainly implies that

$$(1-\varepsilon) - \left(1 - \frac{\pi}{q-1}\right) = \frac{\pi}{q-1} - \varepsilon < 0$$

and, thus, inequality (3.2) is impossible. Thus (3.1) must hold for some k, $0 \leqslant k < N$.

Let j, $0 \leqslant j < N$, be the largest of the numbers k for which $|b_{k+1}| > |S_k(\theta_k) - w|$ holds. Then, for this j either (a) or (b) holds. Suppose, first, that (a) holds. Then, using the notation established above, with $j = k$,

$$|S_{j+1}(\theta_{j+1})-w| = |\{S_j(\theta)+b_{j+1}e^{in_{j+1}\theta_{j+1}}-w\}+\{S_j(\theta_{j+1})-S_j(\theta)\}|$$
$$= |S_j(\theta_{j+1})-S_j(\theta)| \leqslant \frac{\pi}{q}Q_j.$$

Since, by assumption, $|b_{r+1}| \leqslant |S_r(\theta_r)-w|$ whenever $j<r<N$ we must have

$$|S_{r+1}(\theta_{r+1})-w| = |\{S_r(\theta_r)+b_{r+1}e^{in_{r+1}\theta_{r+1}}-w\}+\{S_r(\theta_{r+1})-S_r(\theta_r)\}|$$
$$\leqslant |S_r(\theta_r)-w|-|b_{r+1}|+\frac{\pi}{q}Q_r.$$

Hence,

$$(3.3)\qquad |P_N(\theta_N)-w| = |S_N(\theta_N)-w| \leqslant |S_{N-1}(\theta_{N-1})-w|-|b_N|+\frac{\pi}{q}Q_{N-1}$$
$$\leqslant |S_{N-2}(\theta_{N-2})-w|-|b_{N-1}|+\frac{\pi}{q}Q_{N-2}-|b_N|+\frac{\pi}{q}Q_{N-1} \leqslant \ldots$$
$$\ldots \leqslant |S_{j+1}(\theta_{j+1})-w|+\frac{\pi}{q}\sum_{j+1}^{N-1}Q_r-\sum_{j+2}^{N}|b_r| \leqslant \frac{\pi}{q}\sum_{j}^{N-1}Q_r-\sum_{j+2}^{N}|b_r|.$$

But, by proposition (ii) of § 2, and the relation between ε and q this last expression is majorized by

$$\frac{\pi}{q-1}Q_j+\frac{\pi}{q-1}\sum_{j+1}^{N-1}|b_r-\sum_{j+2}^{N}|b_r|$$
$$\leqslant \frac{\pi}{q-1}Q_j+\frac{\pi}{q-1}|b_{j+1}|-\left(1-\frac{\pi}{q-1}\right)\sum_{j+2}^{N}|b_r| \leqslant \frac{\pi}{q-1}Q_j+\frac{\pi}{q-1}|b_{j+1}|$$
$$< \frac{\pi}{q-1}\sum_{r=1}^{j}Mq^{-j+r}+\frac{\pi}{q-1}M < M\left(\frac{\pi}{q-1}\,\frac{q}{q-1}+\frac{\pi}{q-1}\right) < \frac{3}{4}\varepsilon M$$

and, in this case, the lemma is proved.

Now suppose that (b) holds. Repeating the argument used to establish proposition (i) of the last section we can show that there exists θ_j' such that

$$|w-S_j(\theta_j')| \geqslant |w|+\left(1-\frac{\pi}{q-1}\right)\sum_{r=1}^{j}|b_r|.$$

But, since (b) holds,

$$|w-S_{j+1}(\theta_{j+1})| = |\{w-S_j(\theta)-b_{j+1}e^{in_{j+1}\theta_{j+1}}\}+\{S_j(\theta)-S_j(\theta_{j+1})\}|$$
$$< |b_{j+1}|-|w-S_j(\theta)|+\frac{\pi}{q}Q_j \leqslant |b_{j+1}|-|w-S_j(\theta_j')|+\frac{\pi}{q}Q_j.$$

Hence, using all but the last of the inequalities (3.3) and (ii) of § 2,

$$|P_N(\theta_N)-w| \leqslant |S_{j+1}(\theta_{j+1})-w|+\frac{\pi}{q}\sum_{j+1}^{N-1}Q_r-\sum_{j+2}^{N}|b_r|$$
$$\leqslant \left\{|b_{j+1}|-|w-S_j(\theta_j')|+\frac{\pi}{q}Q_j\right\}+\frac{\pi}{q}\sum_{j+1}^{N-1}Q_r-\sum_{j+2}^{N}|b_r|$$
$$= \frac{\pi}{q}\sum_{j}^{N-1}Q_r+|b_{j+1}|-\sum_{j+2}^{N}|b_r|-|w-S_j(\theta_j')|$$
$$\leqslant \frac{\pi}{q-1}Q_j+|b_{j+1}|\left(1+\frac{\pi}{q-1}\right)-\left(1-\frac{\pi}{q-1}\right)\sum_{j+2}^{N}|b_r|-|w-S_j(\theta_j')|$$
$$\leqslant \frac{\pi}{q-1}Q_j+|b_{j+1}|\left(1+\frac{\pi}{q-1}\right)-\left(1-\frac{\pi}{q-1}\right)\left(\sum_{r=1}^{j}|b_r|+\sum_{r=j+2}^{N}|b_r|\right)-|w|.$$

But $M \leqslant \sum'|b_r| \leqslant \sum_{r=1}^{j}|b_r|+\sum_{j+2}^{N}|b_r|$ and $Q_j \leqslant Mq/(q-1)$, thus the last expression is majorized by (using the relation between ε and q)

$$\frac{\pi}{q-1}M\frac{q}{q-1}+M\left(1+\frac{\pi}{q-1}\right)-\left(1-\frac{\pi}{q}\right)M = \left\{\frac{\pi q}{(q-1)^2}+\frac{\pi}{q-1}+\frac{\pi}{q}\right\}M$$
$$< \frac{4\pi}{q-1}M < \frac{3\varepsilon}{4}M$$

and the lemma is proved.

The next lemma will enable us to construct partial sums of F which, by an application of lemma (3.1), shall give us the desired approximation to w:

Lemma (3.2). *There exists an increasing sequence of radii, $\{r_j\}$, with $\lim_{j\to\infty} r_j = 1$, and an accompanying sequence of positive integers, $\{N_j\}$, with $\lim_{j\to\infty} N_j = \infty$, such that for each pair $(r, N) = (r_j, N_j)$*

(1) $\frac{3}{4}M_N \leqslant |c_N|r^{n_N}$, *where* $M_N = \max_{1\leqslant k\leqslant N}|c_k|r^{n_k}$;

(2) *If* $N'\in\{1,2,\dots,N\}$ *satisfies* $M_N = |c_{N'}|r^{n_{N'}}$, *then*

$$M_N \leqslant \sum_{\substack{k\neq N'\\ 1\leqslant k\leqslant N}} |c_k|\, r^{n_k};$$

(3) $|c_k|r^{n_k} \leqslant (\frac{3}{4})^{(n_k/n_N)}(\frac{4}{3})|c_N|r^{n_N}$ *if* $k > N$.

Proof. Let us choose a strictly increasing sequence of positive numbers $\{\varrho_j\}$ such that $\lim_{j\to\infty}\varrho_j = 1$. Given a fixed member of this sequence $\varrho = \varrho_j$ we define $P = P_j$ by the relation

$$(3.4)\qquad |c_P|\varrho^{n_P} = \max_{1\leqslant k<\infty} |c_k|\,\varrho^{n_k};$$

this maximum clearly exists since F is analytic in the interior of the unit circle and, thus, $\lim_{k\to\infty}|c_k|\,\varrho^{n_k} = 0$. We then choose $s = s_j < \varrho$ satisfying $(s/\varrho)^{n_P} = \frac{3}{4}$. Inequality (1) with (r, N) replaced by (s, P) is then easy to show: letting $P'\in\{1,2,\dots,P\}$ satisfy $|c_{P'}|s^{n_{P'}} = \max_{\leqslant k\leqslant P}|c_k|s^{n_k} = M_P$ we have

$$|c_P|s^{n_P} = |c_P|\,\varrho^{n_P}\left(\frac{s}{\varrho}\right)^{n_P} = \frac{3}{4}|c_P|\,\varrho^{n_P} \geqslant \frac{3}{4}|c_{P'}|\,\varrho^{n_{P'}} = \frac{3}{4}|c_{P'}|s^{n_{P'}}\left(\frac{\varrho}{s}\right)^{n_{P'}}$$

$$> \frac{3}{4}|c_{P'}|s^{n_{P'}} = \frac{3}{4}M_P.$$

Similarly, inequality (3) follows readily. If $k > P$, we then have

$$|c_k|s^{n_k} = |c_k|\,\varrho^{n_k}\left(\frac{s}{\varrho}\right)^{n_P\left(\frac{n_k}{n_P}\right)} = (\tfrac{3}{4})^{n_k/n_P}|c_k|\,\varrho^{n_k} \leqslant (\tfrac{3}{4})^{n_k/n_P}|c_P|\,\varrho^{n_P}$$

$$= \left(\frac{3}{4}\right)^{n_k/n_P}\cdot\frac{4}{3}|c_P|\,s^{n_P}.$$

Now, if part (2) is satisfied when (r, N, N') are replaced by (r, P, P') we let $(r, N) = (s, P)$. If not, we proceed as follows. Suppose that $P = P'$; then the negation of (2) becomes

$$M_P = |c_P|s^{n_P} > \sum_{1}^{P-1}|c_k|s^{n_k}.$$

Dividing by s^{n_P} and noting that $\sum_{1}^{P-1}|c_k|s^{n_k-n_P}$ increases to ∞ as s tends to 0 we see that there exists a unique r, $0 < r < s$, satisfying

$$(3.5)\qquad |c_P|r^{n_P} = \sum_{1}^{P-1}|c_k|\,r^{n_k}.$$

We then let $N = P$. Note that the fact $M_N = |c_N|r^{n_N}$ is an immediate consequence of equality (3.5). Parts (1) and (2) of our lemma are then obviously satisfied. Since decreasing s does not alter the sense of the inequality $|c_k|s^{n_k} \leqslant (\frac{3}{4})^{n_k/n_P}\frac{4}{3}|c_P|s^{n_P}$, $k > P$, we see that (3) is also satisfied.

We observe that in either of these two cases (provided these cases occur infinitely often) it is easy to see that $r = r_j \to 1$ and $N = N_j \to \infty$. For in both cases we have chosen the value $N_j = P$ satisfying equality (3.4). Since $\limsup_{j\to\infty}|c_j| = \infty$ and $\varrho = \varrho_j \to 1$ it is clear that $N_j \to \infty$. Now, in the first case

$$\tfrac{3}{4}\max_{1\leqslant k<\infty}|c_k|\,\varrho^{n_k} = \tfrac{3}{4}|c_N|\,\varrho^{n_N} = |c_N|r^{n_N}.$$

Since the left-hand side of this equality increases to ∞ as $j\to\infty$ (and, thus, $\varrho = \varrho_j \to 1$), must the right-hand side tend to ∞. But this can only happen if $r = r_j \to 1$ (since $\sum_{1}^{\infty}|c_k|r^{n_k}$ is bounded if, say, $r \leqslant r_0 < 1$, and thus, so is $|c_N|r^{n_N}$). In the second case equality (3.5) holds. This implies $|c_1| \leqslant |c_P|r^{n_P-n_1}$. If r did not tend to 1 then the fact that $z^{-n_1}F(z)$ is analytic implies that $|c_P|r^{n_P-n_1}$ is as small as we wish for some large enough values of P. Since c_1 is assumed to be different from 0, this is impossible.

Finally, suppose part (2) is not satisfied with our choice of s and P and that $P' < P$. We then let $N = P'$. For $k < N$, $|c_k|s^{n_k} < \frac{3}{4}M_P = \frac{3}{4}M_N$; otherwise the contribution of a term $|c_k|s^{n_k} \geqslant \frac{3}{4}M_P$ and $|c_P|s^{n_P} \geqslant \frac{3}{4}M_P$ in the sum in (2) would be at least $\frac{3}{2}M_P$ and (2) would be satisfied, contrary to hypothesis. Now let $0 < r < s$ satisfy $(r/s)^{n_N} = \frac{3}{4}$. Thus, if $k < N$,

$$|c_k|\,r^{n_k} = |c_k|\,s^{n_k}\left(\frac{r}{s}\right)^{n_k} \leqslant \frac{3}{4}M_N\left(\frac{r}{s}\right)^{n_k} = \frac{3}{4}|c_N|\,s^{n_N}\left(\frac{r}{s}\right)^{n_k}$$

$$= \frac{3}{4}|c_N|\,\frac{4}{3}r^{n_N}\left(\frac{r}{s}\right)^{n_k} \leqslant |c_N|\,r^{n_N}.$$

Furthermore, if $N < k < P$, $|c_k|r^{n_k} = (\frac{3}{4})^{n_k/n_N}|c_k|\,s^{n_k} \leqslant (\frac{3}{4})^{n_k/n_N}|c_N|\,s^{n_N} = (\frac{3}{4})^{n_k/n_N}\frac{4}{3}|c_N|\,r^{n_N}$; while if $k \geqslant P$, then

$$|c_k|\,r^{n_k} = \left(\frac{3}{4}\right)^{n_k/n_N}|c_k|\,s^{n_k}$$

$$= \left(\frac{3}{4}\right)^{n_k/n_N}|c_k|\,\varrho^{n_k}\left(\frac{s}{\varrho}\right)^{n_k} \leqslant \left(\frac{3}{4}\right)^{n_k/n_N}|c_P|\,\varrho^{n_P}\left(\frac{s}{\varrho}\right)^{n_k}$$

$$\leqslant \left(\frac{3}{4}\right)^{n_k/n_N}|c_P|\,\varrho^{n_P}\left(\frac{s}{\varrho}\right)^{n_P} = \left(\frac{3}{4}\right)^{n_k/n_N}|c_P|\,s^{n_P} \leqslant \left(\frac{3}{4}\right)^{n_k/n_N}|c_N|\,s^{n_N}$$

$$= \left(\frac{3}{4}\right)^{n_k/n_N}|c_N|\,\frac{4}{3}\,r^{n_N}.$$

In either case we obtain inequality (3). If (2) is still not satisfied we see that we now have the previous case. It remains to be shown that (if this last case occurs infinitely often) $r\to 1$ [3] and $N\to\infty$. We have $|c_N|r^{n_N} = |c_N|\frac{3}{4}s^{n_N} \geqslant |c_P|\frac{3}{4}s^{n_P} = |c_P|\frac{9}{16}\varrho^{n_P} = \frac{9}{16}\max_{1\leqslant k<\infty}|c_k|\varrho^{n_k}$. Since the latter tends to ∞ so must $|c_N|r^{n_N}$ and this clearly implies $N\to\infty$ and that $M_N\to\infty$. If r does not tend to 1 inequality (2) together with the absolute convergence of $F(z)$ in a disc smaller than the unit disc would contradict this last fact: $M_N\to\infty$.

One simple consequence of part (3) of lemma (3.2) is the following inequality:

$$\sum_{N+1}^{\infty}|c_k|r^{n_k}\leqslant \varepsilon|c_N|r^{n_N}. \tag{3.6}$$

For

$$\sum_{N+1}^{\infty}|c_k|r^{n_k}\leqslant \frac{4}{3}|c_N|r^{n_N}\sum_{k=N+1}^{\infty}\left(\frac{3}{4}\right)^{n_k/n_N}\leqslant \frac{4}{3}|c_N|r^{n_N}\sum_{k=N+1}^{\infty}\left(\frac{3}{4}\right)^{q^{k-N}}$$

$$=\left(\frac{4}{3}\right)|c_N|r^{n_N}\sum_{j=1}^{\infty}\left(\frac{3}{4}\right)^{q^j}\leqslant\frac{4}{3}|c_N|r^{n_N}\sum_{j=1}^{\infty}\left(\frac{3}{4}\right)^{qj}$$

$$=\frac{4}{3}|c_N|r^{n_N}\left(\frac{3}{4}\right)^{q}\frac{1}{1-\left(\frac{3}{4}\right)^{q}}<\left(\frac{4}{3}\right)|c_N|r^{n_N}\left(\frac{3}{4}\right)^{q}\cdot 4<\varepsilon|c_N|r^{n_N}.$$

LEMMA (3.3). *Except for a finite number of the pairs $(r, N) = (r_j, N_j)$ of lemma (3.2) we have*

$$|F(re^{i\theta})-w|\leqslant 2\varepsilon|c_N|r^{n_N} \tag{3.7}$$

for some $\theta=\theta_j\in[0,2\pi)$.

Proof. We shall show that (3.7) holds as long as $M_N = M_{N_j}\geqslant |w|/(1-\varepsilon)$. Since $r=r_j\to 1$, it follows that $M_N = M_{N_j}\to\infty$; we thus exclude at most finitely many pairs (r_j, N_j). Let $P(\Phi)=\sum_1^N c_k r^{n_k}e^{in_k\Phi}$. Then, clearly, P satisfies the hypotheses of lemma (3.1). Thus, there exists $\theta = \theta_j\in[0,2\pi)$ such that

$$|P(\theta)-w| = \left|\sum_1^N c_k r^{n_k}e^{in_k\theta}-w\right|\leqslant\tfrac{3}{4}\varepsilon M_N\leqslant\varepsilon|c_N|r^{n_N}.$$

Combining this inequality with inequality (3.6) we obtain our lemma.

By exactly the same reasoning we used in the previous section we now see that an application of lemma (2.3), the last lemma and the following estimates on the derivative of F give us our theorem:

LEMMA (3.4). *Except for a finite number of the pairs (r, N) of lemma (3.2) we have*

(a) $$|F'(z)|\leqslant\frac{|c_N|r^{n_N}}{4\varrho}=\frac{A}{\varrho}\quad\textit{when}\quad |z|\leqslant r+\varrho,\quad\textit{where}\quad \varrho=\frac{1}{16n_N},$$

(b) $$|F'(\zeta)|\geqslant\frac{|c_N|r^{n_N}}{32\varrho}=\frac{B}{\varrho}\quad\textit{when}\quad|\zeta|=r.$$

Proof. It follows from our argument which established lemma (3.2) that $r^{n_N}\leqslant\frac{3}{4}$; this inequality, in turn, implies that $r+1/16n_N = r+\varrho<1$. Thus, $F'(z)$ is well defined when $|z|\leqslant r+\varrho$. Using our assumption on the size of q, lemma (3.2) and r near enough 1 (say, close enough for $1/16r = (\varrho/r)n_N\leqslant\frac{1}{14}$; so that

$$\left(\frac{r+\varrho}{r}\right)^{n_N}=\left(1+\frac{\varrho}{r}\right)^{n_N}\leqslant e^{\varrho n_N/r}\leqslant 1+2\frac{\varrho}{r}\cdot n_N\leqslant\frac{8}{7},$$

we have, for $|z|=r+\varrho$,

$$|F'(z)|=\frac{8}{7}|zF'(z)|\leqslant\frac{8}{7}\sum_1^\infty|c_k|n_k(r+\varrho)^{n_k}=\frac{8}{7}\sum_1^\infty|c_k|n_k r^{n_k}\left(\frac{r+\varrho}{r}\right)^{n_k}$$

$$\leqslant\frac{4}{3}\cdot\frac{8}{7}|c_N|r^{n_N}n_N\left(\frac{r+\varrho}{r}\right)^{n_N}\sum_1^N\frac{n_k}{n_N}+$$

$$+\frac{8}{7}\cdot\frac{4}{3}|c_N|r^{n_N}n_N\sum_{N+1}^\infty\left(\frac{3}{4}\right)^{n_k/n_N}\frac{n_k}{n_N}\left(1+\frac{\varrho}{r}\right)^{n_k}=P+Q.$$

[3] If case (2) is still not satisfied we must, as we have shown before, choose a still smaller value of r, for which equality (3.5) is satisfied (with P replaced by N). We have shown that the fact that even these smaller r's tend to 1 follows once we show $N\to\infty$.

Now

$$P \leqslant \left(\frac{8}{7}\cdot\frac{4}{3} n_N \frac{8}{7}\cdot\frac{q}{q-1}\right)|c_N| r^{n_N} = \left(\frac{8\cdot 4\cdot 8q}{7\cdot 3\cdot 7(q-1)}\cdot\frac{1}{16}\right)\frac{1}{\varrho}|c_N| r^{n_N} < \frac{|c_N| r^{n_N}}{8\varrho},$$

$$Q \leqslant n_N \frac{8\cdot 4}{7\cdot 3}\left[\sum_{N+1}^{\infty}\left(\frac{3}{4}\right)^{n_k/n_N}\frac{n_k}{n_N}\left(1+\frac{\varrho}{r}\right)^{n_N(n_k/n_N)}\right]|c_N| r^{n_N}$$

$$\leqslant n_N \frac{8\cdot 4}{7\cdot 3}\left[\sum_{N+1}^{\infty}\left(\frac{3}{4}\right)^{n_k/n_N} n_k/n_N\left(\frac{8}{7}\right)^{n_k/n_N}\right]|c_N| r^{n_N}$$

$$\leqslant n_N \frac{8\cdot 4}{7\cdot 3}\left[\sum_{N+1}^{\infty}\left(\frac{6}{7}\right)^{n_k/n_N}\left(\frac{7}{6}\right)^{\frac{1}{2}(n_k/n_N)}\right]|c_N| r^{n_N}$$

$$\leqslant \frac{1}{\varrho}\cdot\frac{8\cdot 4}{16\cdot 7\cdot 3}\left[\sum_{N+1}^{\infty}\left(\sqrt{\frac{6}{7}}\right)^{q^{k-N}}\right]|c_N| r^{n_N} < \frac{|c_N| r^{n_N}}{8\varrho}.$$

Thus $P+Q \leqslant |c_N| r^{n_N}/4\varrho$ and part (a) is established.

We now prove (b). Let $\zeta = re^{i\Theta}$, then

$$\zeta F'(\zeta) = \sum_{1}^{N-1} n_k c_k r^{n_k} e^{in_k\theta} + n_N c_N r^{n_N} e^{in_N\theta} + \sum_{N+1}^{\infty} n_k c_k r^{n_k} e^{in_k\theta}.$$

By lemma (3.2), part (3), and the estimates on q,

$$\sum_{N+1}^{\infty} n_k |c_k| r^{n_k} \leqslant \frac{4}{3}|c_N| r^{n_N} n_N \sum_{N+1}^{\infty}\left(\frac{3}{4}\right)^{n_k/n_N}\frac{n_k}{n_N}$$

$$\leqslant \frac{4}{3}|c_N| r^{n_N} n_N \sum_{N+1}^{\infty}\left(\frac{4}{3}\right)^{\frac{1}{2}(n_k/n_N)}\left(\frac{3}{4}\right)^{n_k/n_N}$$

$$\leqslant \frac{4}{3}|c_N| r^{n_N} n_N \sum_{N+1}^{\infty}\left(\frac{3}{4}\right)^{\frac{1}{2}q^{k-N}} < \frac{1}{16\varrho} r^{n_N}|c_N|\cdot\frac{1}{4}.$$

Furthermore,

$$\sum_{1}^{N-1} n_k |c_k| r^{n_k} \leqslant \frac{4}{3}|c_N| r^{n_N} n_N \sum_{1}^{N-1}\frac{n_k}{n_N} < \frac{4}{3}\cdot\frac{1}{q-1}\cdot\frac{1}{16\varrho}|c_N| r^{n_N}$$

$$< \frac{1}{16\varrho} r^{n_N}|c_N|\cdot\frac{1}{4}.$$

Since $n_N|c_N| r^{n_N} = (1/16\varrho)|c_N| r^{n_N}$, it follows that

$$|\zeta F'(\zeta)| > \frac{1}{16\varrho}|c_N| r^{n_N} - \frac{1}{16\varrho} r^{n_N}\frac{|c_N|}{4} - \frac{1}{16\varrho} r^{n_N}\frac{|c_N|}{4} = \frac{1}{32\varrho}|c_N| r^{n_N}$$

and part (b) certainly holds for r close enough to one.

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