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Intrinsic description of the Sz.-Nagy-Brehmer unitary dilation

by

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To the memory
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I. Introduction

1. In this paper, generalizing a theorem of [2] for a single contraction, we shall describe by "intrinsic" properties the Sz.-Nagy-Brehmer minimal unitary dilation, and also the Sz.-Nagy-Brehmer minimal isometric dilation, of an arbitrary family of commuting contractions on a Hilbert space (the scalars may be real, complex or quaternionic).

As a corollary we obtain, in a new way, necessary and sufficient conditions on commuting contractions, in order that a Sz.-Nagy-Brehmer unitary dilation should exist, conditions which were given previously by the writer in [3], and earlier, in less precise form, by Brehmer in [1].

2. We use the following terminology. J will denote an arbitrary set of indices a, and $\tilde{J}$ will denote the set of those integer valued functions $m \equiv m(a)$, $-\infty < m(a) < \infty$, for which $\tilde{m} \equiv \{a \mid m(a) \neq 0\}$ is a *finite* subset of J. We write $m \geqslant 0$ if $m(a) \geqslant 0$ for all a; we call m, n *positive-disjoint* if: $m \geqslant 0$, $n \geqslant 0$, and $\tilde{m} \cap \tilde{n} = \emptyset$ (empty set). We write $n = -m$ if $n(a) = -m(a)$ for all a.

If E_a, $a \epsilon J$, are commuting operators on any Hilbert space and $m \epsilon \tilde{J}$ with $m \geqslant 0$ we define $E(m)$ to mean $E_1^{m(1)} \ldots E_r^{m(r)}$ where $\tilde{m}$ has been denoted by $\{1, \ldots, r\}$ for convenience; we define $E(\emptyset)$ to be 1; we define $E(-m)$ to be $\big(E(m)\big)^*$. I, M, N will always denote *finite* subsets (possibly empty) of J. Any such finite set I will be denoted $\{1, \ldots, r\}$ for convenience.

Throughout this paper T_a, $a \epsilon J$, will denote a fixed family of commuting contractions on a fixed Hilbert space H. Operators V_a, $a \epsilon J$, acting on a Hilbert space $K \supset H$ will be called an *isometric dilation* of $\{T_a\}$ if $\{V_a\}$ are commuting isometric operators on K and for all $x \epsilon H$:

$$(1) \qquad T(m)x = P_H V(m)x \quad \text{for} \quad m \epsilon \tilde{J},\ m \geqslant 0.$$

P_H denotes the projection (orthogonal) onto H.

The dilation will be called a *Sz.-Nagy-Brehmer isometric dilation* if (stronger than (1)), for all $x \in H$,

(2) $T(-n)T(m)x = P_H V(-n)V(m)x$ *for* m, n *positive-disjoint.*

The dilation will be called a *unitary* one (in place of an isometric one) if the V_a are all unitary on K.

An isometric, respectively unitary, dilation $\{V_a\}$ acting on K will be called *minimal isometric*, respectively *minimal unitary* if $K = [\{V(m)H \mid m \in \tilde{J},\ m \geqslant 0\}]$, respectively $K = [\{V(m)H \mid m \in \tilde{J}\}]$ (we write $[A, B, \ldots]$ to denote the subspace spanned by $A, B, \ldots$).

It was shown in [2], sharpening previous results of Sz.-Nagy-Brehmer [4] and Brehmer [1], that *commuting* contractions $\{T_a\}$ possess a Sz.-Nagy-Brehmer minimal unitary dilation if and only if the following condition is satisfied:

(3) *For every finite subset $I \subset J$ the operator P_I is positive definite on H.*

Here we write I as $\{1, \ldots, r\}$ for convenience and we define $P_I = P_r$, and P_j, $0 \leqslant j \leqslant r$, by induction on j as follows:

$$P_0 = 1, \quad P_{j+1} = P_j - T^*_{j+1} P_j T_{j+1}.$$

We adopt the convention: $P_\emptyset = 1$.

Note that if (3) holds then $1 = P_0 \geqslant P_1 \geqslant P_2 \geqslant \ldots \geqslant P_r$.

3. If $\{U_a\}$ acting on K is a unitary dilation of $\{T_a\}$ we set $K^+ = [\{U(m)H \mid m \in \tilde{J},\ m \geqslant 0\}]$. Then clearly, $U_a K^+ \subset K^+$ for all $a \in J$ and the restrictions of the U_a to K^+ will be a minimal isometric dilation of $\{T_a\}$ (even a Sz.-Nagy-Brehmer one if $\{U_a\}$ is a Sz.-Nagy-Brehmer unitary dilation).

Thus the problem of constructing a Sz.-Nagy-Brehmer minimal unitary dilation $\{U_a\}$ breaks into two parts, (i) the construction of a Sz.-Nagy-Brehmer minimal isometric dilation $\{V_a\}$ of given commuting contractions $\{T_a\}$, and (ii) the construction of $\{U_a\}$ to be a minimal unitary dilation of given commuting isometries $\{V_a\}$.

It was shown by Brehmer [1] that for real or complex scalars (i) is possible under certain additional restrictions on $\{T_a\}$. The writer showed in [2] by another method (valid for real, complex or quaternionic scalars) that (3) is necessary and sufficient. In the present paper we obtain a description of $\{V_a\}$ in terms of $\{T_a\}$ which yields this condition (3) anew and throws light on its geometric significance.

As for (ii), this is always possible. This was shown by Brehmer [1] for real or complex scalars, by the writer [2] for real, complex or quaterionic scalars. The description of $\{U_a\}$ in terms of $\{V_a\}$ which we obtain in Section IV of this paper reveals at once the existence and uniqueness of $\{U_a\}$.

If J is finite the actual construction of $\{U_a\}$ in terms of $\{V_a\}$ is straightforward; but the construction when J is infinite seems to require either (i) a process of "identifications" which obscures the final result, or (ii) the use of transfinite induction (equivalently, the axiom of choice).

By combining (i) and (ii) a description of the Sz.-Nagy-Brehmer minimal unitary dilation $\{U_a\}$ in terms of the given commuting contractions $\{T_a\}$ (assuming (3) is satisfied) can be obtained. We shall not give the detailed description here. But we note that if $\{T_a\}$ are *doubly commuting* (this means: $T_a T_\beta = T_\beta T_a$ and $T^*_a T_\beta = T_\beta T^*_a$ for all $a \neq \beta$; this was the stronger hypothesis used by Sz.-Nagy in his original discovery [4] of the existence of a Sz.-Nagy-Brehmer minimal unitary dilation) then our description of $\{U_a\}$ becomes more transparent.

This paper does not assume familiarity with previous work on dilations.

II. Analysis of the Sz.-Nagy-Brehmer isometric dilation

4. In section II we shall assume that $\{V_a\}$ acting on K exists as a *Sz.-Nagy-Brehmer minimal isometric* dilation of $\{T_a\}$. We shall prove that (3) holds and we shall describe the behaviour of the V in terms of the given T_a.

For this purpose we define the operators D_I, $\bar{D}_I$ for each *finite* subset I of J as follows: Write $I = \{1, \ldots, r\}$ for convenience, let $D_0 = \bar{D}_0 = 1$; for each j let $\bar{D}_{j+1} = V_{j+1}\bar{D}_j - \bar{D}_j T_{j+1}$; let $D_j = V^*_j \ldots V^*_1 \bar{D}_j$; and let $\bar{D}_I = \bar{D}_r$, $D_I = D_r$. We let H_I denote the subspace $[\bar{D}_I H]$ of K, with the convention: $H_\emptyset = H$.

We note that if $a \in I$ then $\bar{D}_I$ can be expanded into a sum of addends each of the form $V(a)(V_a - T_a)T(b)$.

We shall first prove:

(4) *The subspaces $\{V(m)H_I \mid m \in \tilde{I},\ m \geqslant 0,\ I \subset J,\ I$ finite$\}$ are mutually orthogonal and if $\bar{K}$ denotes their orthogonal sum, then $\bar{K} = K$.*

To prove (4) we first show that $V(m)H_I \perp V(n)H_I$ if $m \in \tilde{I}$, $m \geqslant 0$, $n \in \tilde{I}$, $n \geqslant 0$, and $m \neq n$. It is sufficient to show that, for $x \in H$, $y \in H$,

$$E \equiv \big(V(m)\bar{D}_I x \mid V(n)\bar{D}_I y\big) = 0.$$

We must have for some $\beta \in I$:

$$m(\beta) > n(\beta) \geqslant 0 \quad \text{or} \quad n(\beta) > m(\beta) \geqslant 0.$$

By symmetry we may suppose the former holds. Then, since the V_a are commuting isometries, we may even suppose that $m(\beta) > n(\beta) = 0$.

In the above expression for E, expand $\bar{D}_I$ on the left so as to retain the factor $(V_\beta - T_\beta)$ and expand $\bar{D}_I$ on the right completely. Then E becomes a sum of addends, each of the form $\big(V(-b)V(a)(V_\beta - T_\beta)\bar{x}\,|\,\bar{y}\big)$ with $\bar{x}\in H$, $\bar{y}\in H$, a, b positive-disjoint, $a(\beta)\geqslant 0$, and $b(\beta)=0$. Since $\{V_a\}$ is assumed to be a Sz.-Nagy-Brehmer dilation, (2) shows that such an addend has value $\big(V(-b)V(a)(T_\beta - T_\beta)\bar{x}\,|\,\bar{y}\big)=0$. Hence $E=0$ as required.

Next we prove that $V(m)H_M \perp V(n)H_N$ if $m\in\tilde{M}$, $m\geqslant 0$, $n\in\tilde{N}$, $n\geqslant 0$ and $M\neq N$. We may assume that for some β: $\beta\in M$, $\beta\notin N$, and we need only show that, for all $x\in H$, $y\in H$,

$$E\equiv \big(V(m)\bar{D}_M x\,|\,V(n)\bar{D}_N y\big)=0\,.$$

If we expand $\bar{D}_M$ so as to retain the factor $(V_\beta - T_\beta)$ and expand $\bar{D}_N$ completely, we express E as a sum of addends each of the form $\big(V(-b)V(a)(V_\beta - T_\beta)\bar{x}\,|\,\bar{y}\big)$ with a, b positive-disjoint, $a(\beta)\geqslant 0$ and $b(\beta)=0$. By (2), each such addend has value $\big(V(-b)V(a)(T_\beta - T_\beta)\bar{x}\,|\,\bar{y}\big)=0$, so $E=0$, as required.

To complete the proof of (4) we need only show that $K\subset\bar{K}$. It is sufficient to show that for $m\in\tilde{J}$, $m\geqslant 0$, and $x\in H$ the element $U(m)x$ is in $\bar{K}$.

Write $I=\tilde{m}=\{1,\dots,r\}$. We shall prove by induction on r that $V(m)x\in\bar{K}$ for each $x\in H$. Clearly, if $r=0$, $V(m)x=V(0)x=1x=x\in\bar{K}$ since $\bar{K}\supset H_\emptyset = H$.

Assume now that $r>0$ and that $V(n)x\in\bar{K}$ whenever $\tilde{n}$ has less than r indices.

We shall use the identity

$$V^iW=\sum_{j=1}^{i}V^{i-j}(VW-WT)T^{j-1}+WT^i. \tag{5}$$

Let m_1 be defined by $m_1(1)=0$, $m_1(a)=m(a)$ for $a\neq 1$ and apply (5) with $V=V_1$, $T=T_1$, $W=V(m_1)$ and $i=m(1)$. We obtain

$$V(m)x=\sum_{j=1}^{i}V_1^{i-j}\big(V_1V(m_1)-V(m_1)T_1\big)T_1^{j-1}x+V(m_1)T_1^{m(1)}x.$$

By the inductive assumption, $V(m_1)y\in\bar{K}$ when $y=T_1^{m(1)}x\in H$, so it is sufficient to show that $V_1^{s_1}\big(V_1V(m_1)-V(m_1)T_1\big)x\in\bar{K}$ for all $s_1\geqslant 0$ and all $x\in H$.

Now for each $u=1,\dots,r$ let $m_u(a)=0$ if $a=1,\dots,u$, and let $m_u(a)=m(a)$ otherwise. By induction on u, we need only show for a *single* u $(1\leqslant u\leqslant r)$:

$$V_1^{s_1}\dots V_u^{s_u}\big(V(m_u)\bar{D}_u x\big)\in\bar{K}\ \textit{for all}\ x\in H\ \textit{and}\ s_i\geqslant 0,\ 1\leqslant i\leqslant u.$$

But for $u=r$, $V(m_r)=V(0)=1$ and $\bar{D}_r x=\bar{D}_I x\in H_I$ so $V_1^{s_1}\dots V_r^{s_r}\bar{D}_I x\in\bar{K}$. This completes the proof of (4).

5. Next we shall prove that for each finite subset I of J (write $I=\{1,\dots,r\}$ for convenience) and for each $x\in H$

(6) (i) *$V_1\dots V_r x$ is the orthogonal sum of its projections onto the subspaces H_M, M varying over all subsets of I; its projection onto $\bar{H}_I$ is $\bar{D}_I x$;*

(ii) *$(P_I x\,|\,x)=\|$ projection of $V_1\dots V_r x$ onto $H_I\|^2$;*

(iii) *The relation $W_IP_I^{1/2}x=$ projection of $V_1\dots V_r x$ onto H_I determines a linear isometric mapping W_I of [range of P_I] onto H_I;*

(iv) *$V_1\dots V_r x=\sum\oplus W_MP_M^{1/2}T(I-M)x$, where $T(I-M)$ denotes the product of the T_a for which $a\in I-M$ (by convention: $T(\emptyset)=1$).*

Proof of (i). We first prove identity

$$V_1\dots V_r=\sum\big(\bar{D}_MT(I-M)\,|\,M\subset I\big) \tag{7}$$

(I is denoted $\{1,\dots,r\}$ for convenience). If $r=0$, $I=\emptyset$, and (7) holds trivially; if (7) holds for some r, then

$$\begin{aligned}V_1\dots V_{r+1}&=\sum\big(V_{r+1}\bar{D}_MT(I_r-M)\,|\,M\subset I_r\big)\\&=\sum\big(V_{r+1}\bar{D}_M-\bar{D}_MT_{r+1}\big)T(I_r-M)\,|\,M\subset I_r\big)+\\&\qquad+\sum\big(\bar{D}_MT_{r+1}T(I_r-M)\,|\,M\subset I_r\big)\\&=\sum\big(\bar{D}_MT(I_{r+1}-M)\,|\,M\subset I_{r+1}\big).\end{aligned}$$

Thus, by induction, (7) holds for all finite I. But for $x\in H$, $\bar{D}_MT(I-M)x\in H_M$ and, by (4), the subspaces H_M, $M\subset I$, are mutually orthogonal; (i) now follows.

Proof of (ii). From (i) we deduce:

$$E\equiv\|\,\text{Projection of } V_1\dots V_r x \text{ onto } H_I\|^2=(\bar{D}_Ix\,|\,V_1\dots V_rx)=(D_Ix\,|\,x).$$

Since $\{V_a\}$ is a Sz.-Nagy-Brehmer isometric dilation, $(D_Ix\,|\,x)$ may be evaluated by expanding D_I and then replacing each V_a^* by T_a^*. This will show that $E=(P_Ix\,|\,x)$. This proves (ii) and *this establishes the necessity of condition* (3).

Proof of (iii). By (i) and (ii), for arbitrary $x\in H$, $\bar{D}_Ix$ is the projection of $V_1\dots V_rx$ onto H_I and $\|\bar{D}_Ix\|=\|P_I^{1/2}x\|$. Since $[P_I^{1/2}H]=[\text{range of } P_I]$ and $[\bar{D}_IH]=H_I$, (iii) follows.

Proof of (iv). This follows from (7), because of (iii).

6. We can now describe the Sz.-Nagy-Brehmer minimal isometric dilation $\{V_a\}$ acting on K in terms of the given $\{T_a\}$ as follows:

(i) K itself is the orthogonal sum of subspaces

$$K=\sum\oplus\big(H_{I,m}\,|\,m\in\tilde{I},\ m\geqslant 0,\ I\subset J,\ I\ \text{finite}\big), \tag{8}$$

where each $H_{I,v}$ is the map by an isometry $W_{I,v} = V(v)W_I$ defined on $[P_I H]$.

(ii) The behaviour of each V_β on K can be described as follows: If $\beta \in I$, then V_β on $H_{I,v}$ coincides with $W_{I,v'}(W_{I,v})^{-1}$ where $v'(\alpha) = v(\alpha)$ if $\alpha \neq \beta$ and $v'(\beta) = v(\beta)+1$.

But if $\beta \notin I$ then write I' for $\{\beta, \alpha | \alpha \in I\}$. If $y \in H_{I,v}$ and y is of the form $W_{I,v}P_I^{1/2}x$ with $x \in H$ (such y are dense in $H_{I,v}$), then $V_\beta y = W_{I',v'}P_{I'}^{1/2}x + W_{I,v}P_I^{1/2}T_\beta x$, where $v'(\alpha) = v(\alpha)$ if $\alpha \in I$ and $v'(\beta) = 0$. By continuity, these relations determine $V_\beta y$ for all $y \in H_{I,v}$.

We note: in the preceding paragraph, V_α is determined *uniquely*, that is, if $P_I^{1/2}x_1 = P_I^{1/2}x_2$, then $P_{I'}^{1/2}x_1 = P_{I'}^{1/2}x_2$ and $P_I^{1/2}T_\beta x_1 = P_I^{1/2}T_\beta x_2$. For (setting $x = x_1 - x_2$) we have in turn:

$$P_I^{1/2}x = 0, \quad (P_I x|x) = \|P_I^{1/2}x\|^2 = 0,$$

$$0 \leqslant (P_{I'}x|x) = (P_I x|x) - (T_\beta^* P_I T_\beta x|x) = -(P_I T_\beta x | T_\beta x).$$

Since $P_I \geqslant 0$, it follows that $P_I T_\beta x = 0$, hence $P_{I'}x = 0$ and $\|P_I^{1/2}T_\beta x\|^2 = (T_\beta^* P_I T_\beta x|x) = 0$, $P_I^{1/2}T_\beta x = 0$.

III. Existence and uniqueness of the Sz.-Nagy-Brehmer minimal isometric dilation

7. It is clear from sections 5, 6 that if $\{T_\alpha\}$ possess a Sz.-Nagy-Brehmer minimal isometric dilation $\{V_\alpha\}$ acting on some $K \supset H$, then the condition (3) holds and K and $\{V_\alpha\}$ are determined uniquely (to within a unitary isomorphism).

8. On the other hand, section 6 indicates how K and $\{V_\alpha\}$ can be *constructed* if (3) holds. Simply choose K to be the orthogonal sum of subspaces $K = \sum \oplus (H_{I,v} | v \in \tilde{I}, v \geqslant 0, I \subset J, I$ finite) with each $H_{I,v}$ a copy of $[P_I H]$, all subspaces $H_{\emptyset,v}$ to be interpreted by convention, to be the single space $H_\emptyset = H$.

Then for each finite, non-empty $I \subset J$ and for each $v \in \tilde{I}$, $v \geqslant 0$, choose a fixed, but arbitrary, isometric mapping $W_{I,v}$ of $[P_I H]$ onto $H_{I,v}$ (we adopt the convention: $W_{\emptyset,v} = 1$).

Finally, for each $\beta \in J$ we define an operator V_β on K as follows. Let I be an arbitrary finite subset of J, possibly empty, and let $v \in \tilde{I}$ with $v \geqslant 0$.

If $\beta \in I$ (then $I \neq \emptyset$), define V_β on $H_{I,v}$ to coincide with the isometry $W_{I,v'}(W_{I,v})^{-1}$ where $v'(\alpha) = v(\alpha)$ if $\alpha \neq \beta$ and $v'(\beta) = v(\beta)+1$.

If $\beta \notin I$ (then possibly $I = \emptyset$), write I' for $\{\beta, \alpha | \alpha \in I\}$. Now if $y \in H_{I,v}$ and if $y = W_{I,v}P_I^{1/2}x$ for some $x \in H$ (such y are dense in $H_{I,v}$) then set

$$V_\beta y = W_{I,v}P_I^{1/2}T_\beta x + W_{I',v'}P_{I'}^{1/2}x,$$

where $v'(\alpha) = v(\alpha)$ for $\alpha \in I$ and $v'(\beta) = 0$. Then by continuity define $V_\beta y$ for all $y \in H_{I,v}$, and then by linearity and continuity define $V_\beta y$ for all $y \in K$.

It remains to verify that these relations actually determine each V_β uniquely on K, that $\{V_\alpha\}$ are commuting isometric operators on K and they are a Sz.-Nagy-Brehmer minimal isometric dilation of $\{T_\alpha\}$.

But if one uses the arguments used in sections 5, 6 this verification offers no difficulties.

IV. The minimal unitary dilation of commuting isometries

9. In section IV we suppose $\{V_\alpha\}$ is a given family of commuting isometries on H (so $V_\alpha^* V_\alpha = 1$ for all α). For each $m \in \tilde{J}$ with $m \geqslant 0$ set $A(m) = H \ominus V(m)H$.

10. Now suppose $\{U_\alpha\}$ acting on $K \supset H$ is a minimal unitary dilation of $\{V_\alpha\}$. Set $B(m) = U(-m)A(m)$ and let B denote the subspace of K spanned by $\{B(m) | m \in \tilde{J}, m \geqslant 0\}$. We shall now prove:

(9) *If* $n \geqslant m$ *then* $B(n) \supset B(m)$.

(10) $B \perp H$ *and* $K = B \oplus H$.

Proof of (9) and (10). Let $p \in \tilde{J}$ be defined by $p(\alpha) = n(\alpha) - m(\alpha)$ for all α. Then, since $U_\alpha x = V_\alpha x$ for $x \in H$,

$$\begin{aligned} B(m) &= U(-m)\big(H \ominus V(m)H\big) = U(-m)H \ominus H \\ &= U(-n)V(p)H \ominus H \subset U(-n)H \ominus H \\ &= U(-n)\big(H - V(n)H\big) = U(-n)A(n) = B(n). \end{aligned}$$

This proves (9) and since $B(m) = U(-m)H \ominus H$, so $B(m) \perp H$ for all $m \in \tilde{J}$, $m \geqslant 0$. Hence $B \perp H$.

Now if $m \in \tilde{J}$, $U(m) = U(-b)U(a)$ for some $a \in \tilde{J}$, $b \in \tilde{J}$ with $a \geqslant 0$, $b \geqslant 0$. Then $U(m)H = U(-b)V(a)H \subset U(-b)H = B(b)+H \subset B+H$. So $B+H \supset U(m)H$ for all $m \in \tilde{J}$ and since $B+H$ is closed, so $B+H \supset [\{U(m)H | m \in \tilde{J}\}] = K$. This implies, $B+H = K$ and proves (10).

11. We can now describe K and $\{U_\alpha\}$ as follows (using identifications).

(i) $K \ominus H$ is spanned by subspaces $\{B(m) | m \in \tilde{J}, m \geqslant 0\}$ where each $B(m)$ is mapped onto $A(m)$ by an isometric mapping $W_m = U(m)$ extended to an isometric mapping of $B(m) \oplus H$ onto H by defining $W_m x = V(m)x$ for $x \in H$.

$x_1 \in B(m)+H$ is to be identified with $x_2 \in B(n)+H$ if and only if $V(n)W_m x_1 = V(m)W_n x_2$.

(ii) For each $\beta \in J$, and $y \in B(m)+H$, $U_\beta y$ coincides with $W_m^{-1}(V_\beta W_m y)$. By continuity, this determines $U_\beta y$ for all $y \in B+H$.

Note that in (ii), $U_\beta y$ is determined uniquely. For if $y = x_1 \epsilon B(m)+H$ and $y = x_2 \epsilon B(m)+H$, then $V(n)W_m x_1 = V(m)W_n x_2$, so $V_\beta V(n)W_m x_1 = V_\beta V(m)W_n x_2$ and hence $V(n)V_\beta W_m x_1 = V(m)V_\beta W_n x_2$, which implies that $W_m^{-1}V_\beta W_m x_1$ is identified with $W_n^{-1}V_\beta W_n x_2$.

12. The relations established in section 11 show that K and $\{U_\alpha\}$ are determined uniquely by $\{V_\alpha\}$. But they also indicate how to show the existence of K and $\{U_\alpha\}$ by actual construction.

Simply choose, for each $m \epsilon \tilde{J}$ with $m \geqslant 0$ a copy $B(m)$ of $A(m)$ with $B(m) \perp H$ and an isometric mapping W_m of $B(m)$ onto $A(m)$. Extend W_m to an isometric mapping of $B(m)+H$ onto H by defining $W_m x = V(m)x$ for $x \epsilon H$.

If $y_1 \epsilon H$, $y_2 \epsilon H$, identify $W_m^{-1}y_1$ with $W_n^{-1}y_2$ if and only if $V(n)y_1 = V(m)y_2$. After such identifications the set union of H and all $B(m)$ form a (possibly incomplete) inner-product space $K' \supset H$. On K' define the operator U_β: if $y \epsilon B(m)+H$ so $y = W_m^{-1}x$ for some $x \epsilon H$, define $U_\beta y$ to be $W_m^{-1}V_\beta x$.

There is no difficulty in proving that the extensions of $\{U_\alpha\}$ to the completion of K' form a minimal unitary dilation of $\{V_\alpha\}$.

13. The use of identifications in sections 11 and 12 can be avoided if J is finite and, by use of suitable (transfinite) induction, if J is infinite, as follows.

Assume now that $J = J_\Omega$ consists of all ordinal numbers $\alpha < \Omega$ for some Ω (finite or infinite) and let J_γ (for $\gamma \leqslant \Omega$) consist of all $\alpha < \gamma$.

If $\{U_\alpha\}$ acting on K is a minimal unitary dilation of $\{V_\alpha\}$, let $K_\gamma = [\{U(m)H \mid m \epsilon \tilde{J}_\gamma\}]$. Then $U_\beta K_\gamma \subset K_\gamma$ for all $\beta \epsilon J$ and each U_β is *isometric* on K_γ. Moreover, U_β is unitary on K_γ for $\beta < \gamma$ and the restrictions to K_γ or $\{U_\beta \mid \beta < \gamma\}$ are a minimal unitary dilation of $\{V_\beta \mid \beta < \gamma\}$. Finally, $K_\beta \subset K_\gamma$ if $\beta \leqslant \gamma$.

So $\{U_\alpha\}$ on K can be obtained by step-by-step extension of *all* $\{U_\alpha \mid \alpha < \Omega\}$ from K_β to $K_{\beta+1}$.

Thus K_1 is of the form $H \oplus \sum_{i=1}^{\infty} \oplus E_i$ where each E_i is a copy of $H \ominus T_1 H$; there exists an isometric mapping $W_i = U_1^i$ which maps E_i onto $H \ominus T_1 H$; U_1 is unitary on K_1 under the relations $U_1 x = V_1 x$ if $x \epsilon H$ and $U_1 W_i^{-1} x = W_{i-1}^{-1} x$ for $x \epsilon H \ominus T_1 H$ with the convention $W_0 = W_0^{-1} = 1$. On K_1, the extended V_β $(\beta \epsilon J)$ satisfy the relations $V_\beta W_i^{-1} x = W_i^{-1} V_\beta x$ where W_i^{-1} is to be extended to all $x \epsilon H$ by the relation $W_i^{-1} = W_i^{-1} P_{H \ominus T_1 H} + W_{i-1}^{-1} P_{H \ominus T_1 H} V_1^* + \ldots + W_1^{-1} P_{H \ominus T_1 H} (V_1^*)^{i-1} + (V_1^*)^i$.

It is clear how to use this procedure to show the existence of K and $\{U_\alpha\}$ by actual construction.

References

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