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Some analogies between the class of infinitely divisible distributions and the class $\mathcal{L}$ of distributions

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1. It is known that the class of infinitely divisible distributions may be characterized as follows (see Gnedenko and Kolmogorov [1], § 17, theorem 5):

THEOREM A. *The class of infinitely divisible distributions is equal to the class of compositions of a finite number of Poisson distributions and of their limits in the sense of weak convergence.*

In [3] I found the class $\mathcal{G}$ of distributions that play for the class $\mathcal{L}$ of distributions the same role as the class of Poisson distributions for the class of infinitely divisible distributions. Namely, the class $\mathcal{G}$ consists of all distributions with the Lévy-Khintchine function ([1]) of the form ([2])

$$(1)\qquad G(u)=\begin{cases}0 & \text{for}\quad u<A,\\ a\log\dfrac{1+A^2}{1+u^2} & \text{for}\quad A\leqslant u<0,\qquad (a\geqslant 0)\\ a\log(1+A)^2 & \text{for}\quad u>0,\end{cases}$$

([1]) The logarithm of the characteristic function $\varphi(t)$ of any distribution from the class $\mathcal{L}$ can be written in the Lévy-Khintchine form

$$\log\varphi(t)=i\gamma t+\int_{-\infty}^{+\infty}\left(e^{itu}-1-\frac{itu}{1+u^2}\right)\frac{1+u^2}{u^2}\,dG(u),$$

where γ is a real number and $G(u)$ is a non-decreasing bounded function ($G(-\infty)=0$) having the right derivative and the left derivative, denoted indifferently by $G'(u)$, at every point $u\neq 0$, and such that $\dfrac{1+u^2}{u}G'(u)$ is for $u<0$ and for $u>0$ non-increasing function. The function $G(u)$ will be called the *Lévy-Khintchine function.*

([2]) In [3] the distributions with the Lévy-Khintchine function of the form

$$G(u)=\begin{cases}0 & \text{for}\quad u\leqslant 0,\\ c & \text{for}\quad u>0,\end{cases}\qquad (c\geqslant 0)$$

were also included into the class $\mathcal{G}$. However, this is not necessary, because such distributions are — as easily seen — limits of distributions of the form (1) or (2).

or

$$(2)\qquad G(u)=\begin{cases}0 & \text{for}\quad u<0,\\ b\log(1+u^2) & \text{for}\quad 0\leqslant u<B,\qquad (b\geqslant 0)\\ b\log(1+B)^2 & \text{for}\quad u>B.\end{cases}$$

The following theorem (see [3]) corresponds to theorem A:

THEOREM B. *The class $\mathcal{L}$ of distributions is equal to the class of compositions of a finite number of distributions from the class $\mathcal{G}$ and of their limits in the sense of weak convergence.*

In [3] I also indicated a further analogy between Poisson distributions and the distributions from the class $\mathcal{G}$: It is known that every Poisson distribution is the limiting distribution of sums

$$\xi_{n1}+\xi_{n2}+\ldots+\xi_{nk_n}-A_n,\qquad A_n=\text{const},$$

of suitably chosen two-valued random variables ξ_{nk}. Similarly every distribution from the class $\mathcal{G}$ is the limiting distribution of sums

$$\frac{\xi_1+\xi_2+\ldots+\xi_n}{B_n}-A_n,\qquad A_n, B_n=\text{const},\ B_n>0,$$

of suitably chosen two-valued random variables ξ_k.

In the present paper I should like to indicate some other analogies between Poisson distributions and the class of infinitely divisible distributions on the one hand and the classes $\mathcal{G}$ and $\mathcal{L}$ on the other.

2. First of all let us consider the following analogy. Every infinitely divisible distribution can be characterized by a non-decreasing function $G(u)$ — the Lévy-Khintchine function. The Poisson distribution is characterized by the fact that all the increase of the function $G(u)$ takes place at one point. Beyond that point, the function $G(u)$ is constant. Every distribution from the class $\mathcal{L}$ can be characterized by a non-decreasing function $G(u)$ (Lévy-Khintchine function) having the right derivative and the left derivative at every point $u\neq 0$, and such that $\frac{1+u^2}{u}\times$ $\times G'(u)$ (3) is for $u<0$ and for $u>0$ a non-increasing function. Now the distribution from the class $\mathcal{G}$ is characterized by the fact that all the decrease of the function $\frac{1+u^2}{u}G'(u)$ on the half-line $(-\infty,0)$ or $(0,+\infty)$ takes place at one point. Beyond that point, the function $\frac{1+u^2}{u}G'(u)$ is, for $u<0$ and for $u>0$, constant.

(3) $G'(u)$ denotes in all this paper the right derivative or the letf derivative or in some points the right and in other points the left derivative.

3. In connection with theorems A and B the following question arises: are the class of Poisson distributions and the class $\mathcal{G}$ the only classes such that compositions of distributions from these classes and their limits form the whole class of infinitely divisible distributions and the whole class $\mathcal{L}$, respectively? It is easy to notice that those are not the only classes of distributions with such a property. In fact, instead of the class of Poisson distributions we may take the class of distributions approximating the Poisson distributions, for instance the class of distributions with Lévy-Khintchine functions of the form

$$G(u)=\begin{cases}0 & \text{for}\quad u<a-\varepsilon,\\ \dfrac{b}{2\varepsilon}(u-a+\varepsilon) & \text{for}\quad a-\varepsilon\leqslant u\leqslant a+\varepsilon,\\ b & \text{for}\quad u>a+\varepsilon.\end{cases}$$

With $\varepsilon\to 0$ we get the Poisson distribution. Instead of the class $\mathcal{G}$ we may take the class of distributions approximating the distributions from the class $\mathcal{G}$, for instance the class of distributions with Lévy-Khintchine functions of the form

$$G(u)=\begin{cases}0 & \text{for}\quad u<A,\\ a\log\dfrac{1+\delta A^2}{1+\delta u^2} & \text{for}\quad A\leqslant u<0,\qquad(\delta\geqslant 1)\\ a\log(1+\delta A^2) & \text{for}\quad u>0,\end{cases}$$

or

$$G(u)=\begin{cases}0 & \text{for}\quad u<0,\\ b\log(1+\delta u^2) & \text{for}\quad 0\leqslant u<B,\qquad(\delta\geqslant 1)\\ b\log(1+\delta B^2) & \text{for}\quad u>B.\end{cases}$$

With $\delta\to 1$ we get the distribution from the class $\mathcal{G}$.

However, the class of Poisson distributions as well as the class $\mathcal{G}$ are in a sense the narrowest classes with the property mentioned above.

If a class $\mathcal{T}$ of distributions (a subclass of the class of infinitely divisible distributions) has that property that compositions of those distributions and their limits form the whole class of infinitely divisible distributions, then the class $\mathcal{T}$ contains distributions which are "arbitrarily near" Poisson distributions. Namely, the following theorem holds:

THEOREM 1. *If $\mathcal{T}$ is a class of distributions (a subclass of the class of infinitely divisible distributions) such that the class of compositions of* *those distributions and of the limits of those compositions is equal to the class of infinitely divisible distributions, then for every a, every $\varepsilon > 0$ and every $\eta > 0$ there exists a distribution from the class $\mathscr{T}$ with the Lévy-Khintchine function $G(u)$ such that for all $u < a-\varepsilon$ we have $G(u) < \eta$ and for all $u > a+\varepsilon$ we have $G(+\infty)-G(u) < \eta$.*

Proof. Let us consider the Poisson distribution with Lévy-Khintchine function

$$G_0(u) = \begin{cases} 0 & \text{for} \quad u \leqslant a, \\ b & \text{for} \quad u > a. \end{cases}$$

From the assumption of our theorem follows the existence of distributions from the class $\mathscr{T}$ with Lévy-Khintchine functions $G_{ni}(u)$ $(i = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots)$ such that

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} G_{ni}(u) = G_0(u) \quad (u \neq a).$$

Since functions $G_{ni}(u)$ are non-decreasing and non-negative, we obtain our theorem immediately. For sufficiently large n, each of the functions $G_{n1}(u), G_{n2}(u), \ldots, G_{nk_n}(u)$ has the property of the function $G(u)$ mentioned in theorem 1.

This theorem says that the class $\mathscr{T}$ constains the distributions with Lévy-Khintchine functions $G(u)$ such that the main part of the increase of $G(u)$ takes place on a very small interval. Beyond that interval the function $G(u)$ increases very slowly.

Now we shall prove a theorem which says in what sense the class $\mathscr{G}$ is the narrowest class of distributions having the property that compositions of those distributions and their limits form all the class $\mathscr{L}$. That theorem corresponds to theorem 1.

THEOREM 2. *If $\mathscr{S} \subset \mathscr{L}$ is a class of distributions such that the class of compositions of those distributions and of the limits of those compositions is equal to the class $\mathscr{L}$, then for every $a > 0$, every $\varepsilon > 0$, and every $\eta > 0$, there exists a distribution from the class $\mathscr{S}$ with Lévy-Khintchine function $G(u)$ such that for all $u > a+\varepsilon$ we have*

$$\frac{1+u^2}{u} G'(u) < \eta, \tag{3}$$

and for all u $(\varepsilon < u < a-\varepsilon)$ we have

$$\frac{1+\varepsilon^2}{\varepsilon} G'(\varepsilon) - \frac{1+u^2}{u} G'(u) < \eta. \tag{4}$$

Similarly, for every $a < 0$, every $\varepsilon > 0$, and every $\eta > 0$, there exists a distribution from the class $\mathscr{S}$ with Lévy-Khintchine function $G(u)$ such that for all $u < a-\varepsilon$ we have

$$\left|\frac{1+u^2}{u} G'(u)\right| < \eta, \tag{5}$$

and for all u $(a+\varepsilon < u < -\varepsilon)$ we have

$$\frac{1+u^2}{u} G'(u) - \frac{1+\varepsilon^2}{-\varepsilon} G'(-\varepsilon) < \eta. \tag{6}$$

For the proof of this theorem we shall need the following simple lemma (see [2]):

LEMMA 1. *Let $f(x)$ and $h(x)$ be continuous functions defined in the interval $\langle a, b\rangle$. Let the right derivatives $f'_+(x)$, $h'_+(x)$ and the left derivatives $f'_-(x)$, $h'_-(x)$ exist at every point x of this interval. If $h(a) \neq h(b)$, then there exists a number c $(a < c < b)$ such that*

$$\left[\frac{f(b)-f(a)}{h(b)-h(a)} h'_+(c) - f'_+(c)\right]\left[\frac{f(b)-f(a)}{h(b)-h(a)} h'_-(c) - f'_-(c)\right] \leqslant 0. \tag{7}$$

If (7) holds, then either

$$\frac{f(b)-f(a)}{h(b)-h(a)} h'_+(c) - f'_+(c) \leqslant 0, \quad \frac{f(b)-f(a)}{h(b)-h(a)} h'_-(c) - f'_-(c) \geqslant 0, \tag{7'}$$

or

$$\frac{f(b)-f(a)}{h(b)-h(a)} h'_-(c) - f'_-(c) \leqslant 0, \quad \frac{f(b)-f(a)}{h(b)-h(a)} h'_+(c) - f'_+(c) \geqslant 0. \tag{7''}$$

In the sequel we shall use the following notation:

$$_*f'(c) = \begin{cases} f'_+(c) & \text{if} \quad (7') \text{ holds}, \\ f'_-(c) & \text{if} \quad (7'') \text{ holds}, \end{cases}$$

$$_{**}f'(c) = \begin{cases} f'_-(c) & \text{if} \quad (7') \text{ holds}, \\ f'_+(c) & \text{if} \quad (7'') \text{ holds}. \end{cases}$$

Proof of theorem 2. Let us take arbitrary numbers $a, \varepsilon, \eta > 0$. Let us consider the distribution from the class $\mathscr{L}$ with Lévy-Khintchine function

$$G_0(u) = \begin{cases} 0 & \text{for} \quad u \leqslant 0, \\ \log(1+u^2) & \text{for} \quad 0 < u \leqslant a, \\ \log(1+a^2) & \text{for} \quad u > a. \end{cases}$$

From the assumption of our theorem follows the existence of distributions from the class $\mathcal{S}$ with Lévy-Khintchine functions $G_{ni}(u)$ $(i = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots)$ such that

$$\lim_{n\to\infty}\sum_{i=1}^{k_n} G_{ni}(u) = G_0(u) \qquad (-\infty \leqslant u \leqslant +\infty).$$

Let us observe that in our case the convergence is uniform. For every $\delta > 0$ there exists then an N such that for $n \geqslant N$ and every u

$$|G_n(u) - G_0(u)| < \delta,$$

where

$$G_n(u) = \sum_{i=1}^{k_n} G_{ni}(u). \tag{8}$$

Hence

$$\frac{G_n(a)-G_n(a-\varepsilon)}{\log(1+a^2)-\log\big(1+(a-\varepsilon)^2\big)} \geqslant \frac{\log(1+a^2)-\delta-\log\big(1+(a-\varepsilon)^2\big)-\delta}{\log(1+a^2)-\log\big(1+(a-\varepsilon)^2\big)}$$
$$= 1-\frac{2\delta}{\log(1+a^2)-\log\big(1+(a-\varepsilon)^2\big)}$$

and

$$\frac{G_n(\varepsilon)-G_n(+0)}{\log(1+\varepsilon^2)} \leqslant \frac{\log(1+\varepsilon^2)+\delta}{\log(1+\varepsilon^2)} = 1+\frac{\delta}{\log(1+\varepsilon^2)}.$$

Let us take

$$\delta = \min\left\{\frac{\log(1+a^2)-\log\big(1+(a-\varepsilon)^2\big)}{8}\eta,\ \frac{\eta}{4}\log(1+\varepsilon^2)\right\}.$$

There exists a number N (dependent upon δ and therefore also upon η and ε) such that for $n \geqslant N$

$$\frac{G_n(a)-G_n(a-\varepsilon)}{\log(1+a^2)-\log\big(1+(a-\varepsilon)^2\big)} \geqslant 1-\frac{\eta}{4}, \qquad \frac{G_n(\varepsilon)-G_n(+0)}{\log(1+\varepsilon^2)} \leqslant 1+\frac{\eta}{4}.$$

Next — on account of lemma 1 — there exist numbers c_1 and c_2 such that $0 < c_1 < \varepsilon < a-\varepsilon < c_2 < a$, and

$$1-\frac{\eta}{4} \leqslant \frac{G_n(a)-G_n(a-\varepsilon)}{\log(1+a^2)-\log\big(1+(a-\varepsilon)^2\big)} \leqslant {}_{*}G_n'(c_2)\frac{1+c_2^2}{2c_2} \leqslant {}_{**}G_n'(c_1)\frac{1+c_1^2}{2c_1}$$
$$\leqslant \frac{G_n(\varepsilon)-G_n(+0)}{\log(1+\varepsilon^2)} \leqslant 1+\frac{\eta}{4}.$$

Since for all u $(c_1 \leqslant u \leqslant c_2)$

$$G_n'(c_2)\frac{1+c_2^2}{2c_2} \leqslant G_n'(u)\frac{1+u^2}{2u} \leqslant G_n'(c_1)\frac{1+c_1^2}{2c_1},$$

therefore

$$1-\frac{\eta}{4} \leqslant G_n'(u)\frac{1+u^2}{2u} \leqslant 1+\frac{\eta}{4},$$

i. e.

$$1-\frac{\eta}{4} \leqslant \sum_{i=1}^{k_n} G_{ni}'(u)\frac{1+u^2}{2u} \leqslant 1+\frac{\eta}{4}.$$

All the summands are non-negative and non-increasing functions. None of them can then decrease on the interval (c_1, c_2) more than $\eta/2$. Therefore, for every u $(\varepsilon < u < a-\varepsilon)$,

$$\frac{1+\varepsilon^2}{2\varepsilon}G_{ni}'(\varepsilon) - \frac{1+u^2}{2u}G_{ni}'(u) \leqslant \frac{\eta}{2} \qquad (i = 1, 2, \ldots, k_n).$$

Inequality (4) is thus proved.

We proceed to the proof of inequality (3). Suppose it is not true. This means that there exist $a > 0$, $\varepsilon > 0$, and $\eta > 0$ such that for every distribution from the class $\mathcal{S}$ with Lévy-Khintchine function $G(u)$ there exists a number $u_0 > a+\varepsilon$ such that

$$\frac{1+u_0^2}{u_0}G'(u_0) \geqslant \eta, \tag{9}$$

and, moreover,

$$\frac{1+(a+\varepsilon)^2}{a+\varepsilon}G'(a+\varepsilon) \geqslant \eta. \tag{10}$$

From the definition of the class $\mathcal{S}$ follows the existence of distributions from that class with Lévy-Khintchine functions $G_{n1}(u), G_{n2}(u), \ldots, G_{nk_n}(u)$ such that

$$\lim_{n\to\infty}\sum_{i=1}^{k_n} G_{ni}(u) = G_0(u).$$

In our case the convergence is uniform. For every $\delta > 0$ there exists then a number N such that, for $n \geqslant N$ and every u,

$$\Big|\sum_{i=1}^{k_n} G_{ni}(u) - G_0(u)\Big| < \delta,$$

whence

$$\left|\sum_{i=1}^{k_n} G_{ni}(a+\varepsilon) - \sum_{i=1}^{k_n} G_{ni}(a)\right| < 2\delta.$$

Functions $G_{ni}(u)$ are non-decreasing and so

$$\sum_{i=1}^{k_n} \{G_{ni}(a+\varepsilon) - G_{ni}(a)\} < 2\delta,$$

whence

(11) $$G_{ni}(a+\varepsilon) - G_{ni}(a) < 2\delta \quad (i = 1, 2, \ldots, k_n).$$

Let us take for example $G_{N1}(u)$. From (10) we have

$$\frac{1+(a+\varepsilon)^2}{a+\varepsilon} G'_{N1}(a+\varepsilon) \geqslant \eta.$$

$\frac{1+u^2}{u} G'_{N1}(u)$ does not increase. We thus have

$$\frac{1+u^2}{u} G'_{N1}(u) \geqslant \eta \quad \text{for} \quad a \leqslant u \leqslant a+\varepsilon.$$

Hence and from lemma 1 follows the existence of such c ($a < c < a+\varepsilon$) that

$$\frac{G_{N1}(a+\varepsilon) - G_{N1}(a)}{\log\left(1+(a+\varepsilon)^2\right) - \log(1+a^2)} \geqslant {}_{**}G'_{N1}(c)\frac{1+c^2}{2c} \geqslant \eta.$$

Therefore

$$G_{N1}(a+\varepsilon) - G_{N1}(a) \geqslant \eta\left\{\log\left(1+(a+\varepsilon)^2\right) - \log(1+a^2)\right\},$$

which contradicts (11) if we take

$$\delta = \left\{\log\left(1+(a+\varepsilon)^2\right) - \log(1+a^2)\right\}\eta/2.$$

(9) cannot then be true and so inequality (3) holds.

The proof of inequalities (5) and (6) is quite analogous. Theorem 2 is thus proved.

The class $\mathscr{S}$ contains then the distribution with such Lévy-Khintchine functions $G(u)$ that the main part of the decrease of $\frac{1+u^2}{u} G'(u)$ on the half-line $(-\infty, -\varepsilon)$ or $(\varepsilon, +\infty)$ takes place on the very small interval. Beyond that interval, the function $\frac{1+u^2}{u} G'(u)$ decreases on the given half-line very slowly. The class $\mathscr{S}$ contains then the distributions which are in a sense "arbitrarily near" distributions from the class $\mathscr{G}$.

4. In section 2 we indicated an analogy between Poisson distribution and the distribution from the class $\mathscr{G}$, consisting in that Lévy-Khintchine function of Poisson distribution increases in the same way as the function $\frac{1+u^2}{u} G'(u)$ decreases, where $G(u)$ is Lévy-Khintchine function of the distribution from the class $\mathscr{G}$. The question arises, if it is possible to establish an isomorphism between the class of infinitely divisible distributions and the class $\mathscr{L}$ consisting in the same, i. e. in that Lévy-Khintchine function $G(u)$ of an infinitely divisible distribution increases in the same way as the function $\frac{1+u^2}{u}\bar{G}'(u)$ decreases, where $\bar{G}(u)$ is Lévy-Khintchine function of the distribution from the class $\mathscr{L}$ which corresponds to the infinitely divisible distribution with Lévy-Khintchine function $G(u)$. Now, there exists such an isomorphism between the class of infinitely divisible distributions and the subclass of the class $\mathscr{L}$ consisting of such distributions that

(12) $$\lim_{u\to-0} \frac{1+u^2}{u}\bar{G}'(u) > -\infty,$$

(13) $$\lim_{u\to+0} \frac{1+u^2}{u}\bar{G}'(u) < +\infty.$$

This isomorphism is established by the following formulae:

(14) $$\frac{1+u^2}{u}\bar{G}'(u) = -G(u) \quad (u < 0),$$

(15) $$G(+0) - G(-0) = \bar{G}(+0) - \bar{G}(-0),$$

(16) $$\frac{1+u^2}{u}\bar{G}'(u) = G(+\infty) - G(u) \quad (u > 0).$$

In order to prove that formulae (14)-(16) establish the isomorphism mentioned above we shall need the following lemma:

LEMMA 2. *If $G(u)$ is Lévy-Khintchine function of a distribution from the class $\mathscr{L}$, then*

(17) $$\lim_{u\to\infty} \frac{1+u^2}{u} G'(u) = \lim_{u\to-\infty} \frac{1+u^2}{u} G'(u) = 0.$$

Proof. First of all let us observe that $G'(u)$ is for $\varepsilon \leqslant u < +\infty$ bounded, where ε is an arbitrary positive number. For, if $\lim_{u\to u_0} G'(u) = +\infty$ for a u_0 ($\varepsilon \leqslant u_0 < \infty$) were true,

$$\lim_{u\to u_0} \frac{1+u^2}{u} G'(u) = +\infty,$$

would also hold, and since $\frac{1+u^2}{u}G'(u)$ is for $u>0$ non-increasing, we should have $G'(u)=+\infty$ for $0<u<u_0$, which is impossible.

On account of lemma 1 we have for all u_1 and u_2 such that $\varepsilon<u_1<<u_2$

$$\frac{G(u_2)-G(u_1)}{u_2-u_1}\leqslant {}_*G'(c) \qquad (u_1<c<u_2).$$

The function $G(u)$ satisfies then for $u>\varepsilon$ the Lipschitz condition and thus is for $u>\varepsilon$ absolutely continuous ([4], p. 404), whence it follows that

$$G(u)=G(\varepsilon)+\int_{\varepsilon}^{u}G'(u)\,du \qquad (u>\varepsilon). \tag{18}$$

Suppose now that formula (17) does not hold. There exists then such $\delta\geqslant 0$ that for $u>0$

$$\frac{1+u^2}{u}G'(u)\geqslant\delta,$$

whence it follows that

$$G'(u)=\delta\frac{u}{1+u^2}+\varphi(u),$$

where $\varphi(u)\geqslant 0$. Hence and from (18) we have

$$G(u)=G(\varepsilon)+\int_{\varepsilon}^{u}\delta\frac{u}{1+u^2}du+\int_{\varepsilon}^{u}\varphi(u)\,du\geqslant\frac{\delta}{2}\log\frac{1+u^2}{1+\varepsilon^2}\xrightarrow[u\to\infty]{}\infty,$$

which is impossible. There must then be

$$\lim_{u\to\infty}\frac{1+u^2}{u}G'(u)=0.$$

Similarly we prove the second part of formula (17). Lemma 2 is thus proved.

We shall now prove that formulae (14)-(16) establish an isomorphism between the infinitely divisible distributions with Lévy-Khintchine functions $G(u)$ and the distributions from the class $\mathscr{L}$ with Lévy-Kintchine functions satisfying (12) and (13).

Let us take an arbitrary distribution from the class $\mathscr{L}$ with Lévy-Khintchine function $\bar{G}(u)$ satisfying (12) and (13). Formula (14) determines uniquely a non-negative, non-decreasing, continuous from the left, function $G(u)$ ($G(-\infty)=0$ on account of lemma 2) for $u<0$. Formulae (14) and (15) give

$$G(+0)=\bar{G}(+0)-\bar{G}(-0)+\lim_{u\to-0}\frac{1+u^2}{-u}\bar{G}'(u),$$

and formula (16) gives

$$G(u)=\lim_{u\to+0}\frac{1+u^2}{u}\bar{G}'(u)+G(+0)-\frac{1+u^2}{u}\bar{G}'(u)\qquad(u>0).$$

We have obtained thus Lévy-Khintchine function $G(u)$ (and only one). To every function $\bar{G}(u)$ satisfying (12) and (13) there corresponds then one and only one Lévy-Khintchine function $G(u)$.

Let us now observe that the derivative $\bar{G}'(u)$ of arbitrary Lévy-Khintchine function $\bar{G}(u)$ satisfying (12) and (13) is for all u bounded. From the proof of lemma 2 it follows that $\bar{G}'(u)$ is bounded for all u from the half-lines $-\infty<u\leqslant-\varepsilon$ and $\varepsilon\leqslant u<+\infty$, with arbitrary $\varepsilon>0$. From (12) and (13) it follows that

$$\lim_{u\to-0}\bar{G}'(u)=\lim_{u\to+0}\bar{G}'(u)=0.$$

Therefore $\bar{G}'(u)$ is bounded for all u. $\bar{G}(u)$ satisfies then for $u<0$ and for $u>0$ the Lipschitz condition, and thus it is absolutely continuous for $u<0$ and $u>0$. We have thus

$$\bar{G}(u)=\bar{G}(-0)+\int_{-0}^{u}\bar{G}'(u)\,du\qquad(u<0), \tag{19}$$

$$\bar{G}(u)=\bar{G}(+0)+\int_{+0}^{u}\bar{G}'(u)\,du\qquad(u>0). \tag{20}$$

Let us now take an arbitrary Lévy-Khintchine function $G(u)$. From (19) and (14) we have

$$\bar{G}(u)=\bar{G}(-0)+\int_{u}^{-0}G(u)\frac{u}{1+u^2}du\qquad(u<0) \tag{21}$$

and from (20) and (16) we have

$$\bar{G}(u)=\bar{G}(+0)+\int_{+0}^{u}\{G(+\infty)-G(u)\}\frac{u}{1+u^2}du\qquad(u>0). \tag{22}$$

From (21) we obtain

$$0=\bar{G}(-\infty)=\bar{G}(-0)+\int_{-\infty}^{-0}G(u)\frac{u}{1+u^2}du. \tag{23}$$

Formula (23) together with (21) determines uniquely the function $\bar{G}(u)$ for $u<0$. From (15) and (23) we have

$$(24)\qquad \bar{G}(+0) = G(+0)-G(-0)-\int_{-\infty}^{-0} G(u)\frac{u}{1+u^2}\,du.$$

Formulae (22) and (24) determine uniquely the function $\bar{G}(u)$ for $u>0$. To every Lévy-Khintchine function $G(u)$ there corresponds then one and only one Lévy-Khintchine function $\bar{G}(u)$ satisfying (as easily seen) (12) and (13). If $\bar{G}_1(u)$ and $\bar{G}_2(u)$ correspond to $G_1(u)$ and $G_2(u)$ respectively, then $\bar{G}_1(u)+\bar{G}_2(u)$ corresponds to $G_1(u)+G_2(u)$. Therefore the correspondence (14), (15), and (16) preserves the composition of distributions.

The isomorphism in question is thus proved.

It is obvious that to a Poisson distribution corresponds the distribution from the class $\mathscr{G}$. It is easy to verify that to a distribution from the class $\mathscr{G}$ with Lévy-Khintchine function

$$G(u) = \begin{cases} 0 & \text{for} \quad u\leqslant 0, \\ b\log(1+u^2) & \text{for} \quad 0<u\leqslant B, \quad (b\geqslant 0) \\ b\log(1+B^2) & \text{for} \quad u>B, \end{cases}$$

treated as infinitely divisible distribution, corresponds the distribution from the class $\mathscr{L}$ with Lévy-Khintchine function

$$\bar{G}(u) = \begin{cases} 0 & \text{for} \quad u\leqslant 0, \\ \frac{b}{2}\log(1+B^2)\cdot\log(1+u^2)-\frac{b}{4}\log^2(1+u^2) & \text{for} \quad 0<u\leqslant B, \\ \frac{b}{4}\log^2(1+B^2) & \text{for} \quad u>B. \end{cases}$$

To the distribution from the class $\mathscr{L}$ with Lévy-Khintchine function given by the formula

$$(25)\qquad \bar{G}(u) = \begin{cases} \dfrac{1}{\sqrt{C(1+u^2)}} & \text{for} \quad u<0, \\ A-\dfrac{1}{\sqrt{B(1+u^2)}} & \text{for} \quad u>0, \end{cases}$$

$$A, B, C>0, \quad A\geqslant \frac{1}{\sqrt{B}}+\frac{1}{\sqrt{C}},$$

satisfying (12) and (13) corresponds the infinitely divisible distribution with Lévy-Khintchine function $G(u)=\bar{G}(u)$. The distributions with Lévy-Khintchine function (25) correspond then to themselves. It is obvious that the normal distribution ($G(u)\equiv 0$) also corresponds to itself. The normal distribution and the distributions (25) are the only distributions with this property.

References

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