

An example in pursuit theory

by

A. ZIĘBA (Wrocław)

1. The pursuit and evasion is a game between two players in which the pursuer wants to catch the evader or to approach him as near as possible and the evader has the opposite tendency. One can imagine for example a ship which wants to overtake another ship, or, more generally, such a war between two fleets. We consider here only the case in which each player has complete information on the moves and future possibilities of his antagonist. If one wants to formalize these intuitions in terms of the theory of games, it is clear that there will be then obtained a construction analogous to the games in an extensive form with perfect information⁽¹⁾. Therefore in most cases such games have values.

To perform this formalization three mathematical tools giving three different theories have been used:

- 1° differential equations;
- 2° difference equations;
- 3° approximation by discrete positional games.

The first theory will be described in section 3 of this paper. The second is described in [3]⁽²⁾. The third, in which continuous moves of ships and fleets are represented by sequences of points such that consecutive points are near, will be published-by J. Mycielski (cf. [3])⁽³⁾.

As has been mentioned, every theory which aims at describing a real game of pursuit and evasion must give a game with a value (see sections 4 and 5).

It is the purpose of this paper to show that the first theory is essentially unsuccessful since it often gives games without values. We give two examples of such games. The first, called Γ (an exact definition is given in section 6), is the following:

⁽¹⁾ For the definition and the theory of these games cf. [2].

⁽²⁾ I have been informed by J. Mycielski that he is preparing a paper containing some corrections to [3] and a further development.

⁽³⁾ Cf. [1] (other references can be found there).

There are two ships, the pursuer p and the evader q . The sea S is a circular disc. In the initial moment $t = 0$ of the play p is situated in the middle and q on the border of S . Then p and q move freely in S , possibly also on its border. The velocity of p is bounded by a constant c . The strategies of p and q , denoted by the letters a and b respectively, are rules defining the direction and velocity of the motion of ships at each moment of the play according to the motion and position of the antagonist up to this moment. It is supposed that all available b are continuous in that sense that a small modification of the route of p produces a small modification of the route of q determined by b . The result of the play is the distance between p and q at the moment $t = 1$. It is proved (section 7) that such a game Γ has no value.

In the second example (section 8) there are three ships of the pursuer and one ship of the evader and the sea is the whole plane. The result is analogous.

In spite of our examples there are such games of pursuit and evasion in which 1^o gives a good theory and the games have values (see section 8). In this case the first theory has a great advantage over the remaining two theories since it gives the possibilities of effective solutions of the games.

A general theory of pursuit and evasion can be obtained by means of 2^o and 3^o. These theories give a large class of games with values (see [3]).

I am indebted to Jan Mycielski for the wording of this paper and some simplifying modifications of my original idea.

2. A 0-sum 2-person game is a triple $\langle A, B, v \rangle$, where A and B are two sets and $v = v(a, b)$, is a real-valued function defined for $a \in A$ and $b \in B$. The elements of A and B are called *strategies*.

(One plays in the following way: there are two players \mathfrak{U} and \mathfrak{B} ; \mathfrak{U} chooses a strategy $a \in A$, independently and without information regarding the choice of \mathfrak{U} player \mathfrak{B} chooses a strategy $b \in B$ and \mathfrak{U} pays to \mathfrak{B} the value $v(a, b)$.)

Let us put

$$M = \inf_{a \in A} \sup_{b \in B} v(a, b) \quad \text{and} \quad m = \sup_{b \in B} \inf_{a \in A} v(a, b);$$

of course $M \geq m$.

$\langle A, B, v \rangle$ is said to *have a value* if $M = m$ and this number is called the *value* of $\langle A, B, v \rangle$ (⁴).

(The game-theoretical meaning of M and m is well known — see [2]).

(⁴) Some authors reserve this terminology for the more restrictive situation in which there exist such $a_0 \in A$ and $b_0 \in B$ that $\sup_{b \in B} v(a_0, b) = \inf_{a \in A} v(a, b_0)$. Such a_0 and b_0 are called *optimal strategies*. We do not exclude the possibility that the value of the game is $+\infty$ or $-\infty$.

3. Now we give a brief description of a theory of pursuit and evasion suggested in section 1. This will be a game $\langle A, B, v \rangle$ defined as follows:

P and Q are two subsets of the m -dimensional and n -dimensional vector-spaces R^m and R^n respectively. Two points, $p_0 \in P$ and $q_0 \in Q$, are given. P_0 and Q_0 are certain spaces of functions defined over the interval $T = \langle 0, 1 \rangle$ (⁵), with values in P and Q respectively, satisfying the Lipschitz condition with certain fixed Lipschitz constants, c_1 for P_0 and c_2 for Q_0 , and such that $p(0) = p_0$ and $q(0) = q_0$ for any $p \in P_0$ and $q \in Q_0$.

A and B are two sets of three argument continuous functions $a(r, s, t)$ and $b(r, s, t)$ defined for $r \in P$, $s \in Q$, and $t \in T$, with values in R^m and R^n respectively. We suppose that the functions $a(r, s, t)$ and $b(r, s, t)$ satisfy the following fundamental condition:

(1) For every pair $a \in A$ and $b \in B$ the system of differential equations

$$p'(t) = a(p(t), q(t), t),$$

$$q'(t) = b(p(t), q(t), t),$$

with the condition $p \in P_0$ and $q \in Q_0$, has a unique solution p, q .

Let $f(p, q)$ be a real-valued function defined for $p \in P_0$ and $q \in Q_0$, and $v(a, b) = f(p, q)$, where p, q is the solution from (1).

We give the following interpretation of this involved definition of $\langle A, B, v \rangle$. T is the time interval. The points of P indicate the possible positions of the fleet of \mathfrak{U} and those of Q of the fleet of \mathfrak{B} . At the initial moment these positions are fixed as p_0 and q_0 . Then using their chosen strategies a and b , the players fix their velocity vectors at every moment according to the information about their own position, the position of their antagonists and the time. Their maximal velocities are c_1 and c_2 respectively.

4. The above game has not much in common with a game with perfect information. In fact perfect information would require larger sets of strategies — the existence of strategies which use not only the present position of the players during the play, but also the whole history of the play.

There are, however, conditions for a finite game in an extensive form which imply the existence of optimal strategies of a special kind, in which the actual choice depends only on the actual position. In our situation (where the moves of one player have no impact on the set of possible moves of the other player) a natural counterpart of these conditions (in a strong form) is the following:

(2) The value of $f(p, q)$ depends only on $p(1)$ and $q(1)$.

(⁵) It will be seen in the sequel that the choice of the interval $\langle 0, 1 \rangle$ is not essential.

5. Supposition (1) implies in a known way the following fact:

(3) \mathbf{p} and \mathbf{q} depend continuously on the strategies a and b , with the topology of uniform convergence in \mathbf{P}_0 , \mathbf{Q}_0 , A and B .

It is the purpose of this paper to show (sections 6-7) that (3) is the reason why the game may be without value, which is inconsistent with our intuitions on pursuit games satisfying (2). Therefore a general theory of pursuit cannot be based on the theory of differential equations.

6. Now we define the game Γ , which gives the example promised above:

$P = Q = \{z: |z| \leq 1\}$ (z is a complex variable), $p_0 = 0$, $q_0 = 1$ and $T = \langle 0, 1 \rangle$ (a closed interval).

\mathbf{P}_T is any class of functions $\mathbf{p}(t)$ defined for $t \in T$ with values in P and satisfying the following conditions:

$$\mathbf{p}(0) = 0, \quad |\mathbf{p}(t_1) - \mathbf{p}(t_2)| \leq |t_1 - t_2| \quad \text{for } \mathbf{p} \in \mathbf{P}_T.$$

Moreover, we suppose that $\mathbf{p}_z \in \mathbf{P}_T$ for any $z \in P$, where $\mathbf{p}_z(t) = tz$ for any $t \in T$.

\mathbf{Q}_T is any class of functions $\mathbf{q}(t)$ defined for $t \in T$ with values in Q and satisfying the following condition:

$$\mathbf{q}(0) = 1 \quad \text{for } \mathbf{q} \in \mathbf{Q}_T.$$

Moreover, we suppose that

$$\mathbf{q}_1, \mathbf{q}_2 \in \mathbf{Q}_T,$$

where

$$\mathbf{q}_1(t) = e^{ict}, \quad \mathbf{q}_2(t) = \begin{cases} e^{ict} & \text{for } 0 \leq t \leq \frac{1}{2}, \\ e^{ic(1-t)} & \text{for } \frac{1}{2} \leq t \leq 1, \end{cases}$$

where c is a constant with $0 < c \leq 2\pi/3$.

A and B are any classes of functionals $a(\mathbf{q}, t)$ resp. $b(\mathbf{p}, t)$ defined for $\mathbf{p} \in \mathbf{P}_T$ resp. $\mathbf{q} \in \mathbf{Q}_T$ and $t \in T$, with values in P , satisfying the following conditions:

1° $a(\mathbf{q}_1, t) = a(\mathbf{q}_2, t)$ for any $a \in A$ and $t \leq 1/2$;

2° for every $a \in A$ and $b \in B$ there exists a unique solution \mathbf{p}, \mathbf{q} of the equations

$$\mathbf{p}(t) = a(\mathbf{q}, t), \quad \mathbf{q}(t) = b(\mathbf{p}, t), \quad \mathbf{p} \in \mathbf{P}_T, \mathbf{q} \in \mathbf{Q}_T;$$

3° $a_z \in A$ for every $z \in P$, where $a_z(\mathbf{q}, t) = \mathbf{p}_z(t)$ for every $\mathbf{q} \in \mathbf{Q}_T, t \in T$ and $z \in P$;

4° $b_1, b_2 \in B$, where $b_1(\mathbf{p}, t) = \mathbf{q}_1(t)$ and $b_2(\mathbf{p}, t) = \mathbf{q}_2(t)$ for every $\mathbf{p} \in \mathbf{P}_T$ and $t \in T$;

5° for every $b \in B$ the point $\mathbf{q}(1)$ determined by

$$\mathbf{q}(t) = b(\mathbf{p}_z, t),$$

i. e. the point $b(\mathbf{p}_z, 1)$, is a continuous function of z ;

$f(\mathbf{p}, \mathbf{q}) = |\mathbf{p}(1) - \mathbf{q}(1)|$; $v(a, b) = f(\mathbf{p}, \mathbf{q})$, where \mathbf{p}, \mathbf{q} is given by 2°, and Γ denotes the game $\langle A, B, v \rangle$ defined above.

At some points Γ is more general and at some points more specialised than the games of section 3. P, Q, p_0 and q_0 are fixed, but the sets $\mathbf{P}_T, \mathbf{Q}_T, A$ and B are very undetermined. First it should be noticed that the conditions for these sets are consistent since they are fulfilled if we put $\mathbf{P}_T = \{\mathbf{p}_z: z \in P\}$, $\mathbf{Q}_T = \{\mathbf{q}_1, \mathbf{q}_2\}$, $A = \{a_z: z \in P\}$, $B = \{b_1, b_2\}$. It is more interesting that these sets can also be very large. Condition 1° is satisfied if the strategies of \mathcal{U} do not use any information regarding the future during the play (since $\mathbf{q}_1(t) = \mathbf{q}_2(t)$ for $t \leq 1/2$). Stronger conditions regarding the strategies of \mathcal{U} and \mathcal{B} will be necessary if one wants to have a game of pursuit and evasion, but this is not needed in the reasoning below! Condition 2° corresponds to (1) and condition 5° to (3). Of course (2) is satisfied in Γ .

The fact that \mathbf{q}_2 is not differentiable in $t = 1/2$ is not essential in our example since "small" modifications of \mathbf{q}_2 are permitted (see the proof of the theorem below). For other remarks see section 8.

7. Now we prove our chief result.

THEOREM. *The game Γ has no value.*

Proof. First note that $\mathbf{q}_1(1) = e^{ic}$ and $\mathbf{q}_2(1) = 1$. Since $0 < c \leq 2\pi/3$ it is evident that the functional $a^*(\mathbf{q}, t)$ which is best for minimizing the number

$$\max_{j=1,2} v(a^*, b_j) = \max_{j=1,2} |a^*(\mathbf{q}_j, 1) - \mathbf{q}_j(1)|$$

and can belong to A (therefore such that $a^*(\mathbf{q}_j, t)$ satisfies 1°) satisfies the relations

$$a^*(\mathbf{q}, t) = \begin{cases} te^{ic/2} & \text{for } 0 \leq t \leq 1/2 \text{ and } \mathbf{q} = \mathbf{q}_1 \text{ or } \mathbf{q} = \mathbf{q}_2, \\ \frac{1}{2}e^{ic/2} + \left(t - \frac{1}{2}\right) \frac{z}{|z|} & \text{for } 1/2 \leq t \leq 1 \text{ and } \mathbf{q} = \mathbf{q}_1, \\ \frac{1}{2}e^{ic/2} + \left(t - \frac{1}{2}\right) \frac{\lambda}{|\lambda|} & \text{for } 1/2 \leq t \leq 1 \text{ and } \mathbf{q} = \mathbf{q}_2, \end{cases}$$

where

$$z = e^{ic} - \frac{1}{2}e^{ic/2}, \quad \lambda = 1 - \frac{1}{2}e^{ic/2}.$$

Since $|z| = |\lambda| > 1/2$, we have

$$|a^*(\mathbf{q}_j, 1) - \mathbf{q}_j(1)| = |e^{ic/2} - \frac{1}{2}| |1 - 1/(2|z|)| > 0 \quad \text{for } j = 1, 2.$$

Clearly it is not supposed that such an a^* belongs to A but of course (see section 2) the following inequality holds:

$$M \geq \max_{j=1,2} v(a^*, b_j) > 0.$$

By 5° the function $b(p_x, 1)$ is a continuous mapping of the closed disc P into itself; thus by the fixed point theorem there exists such a $z(b) \in P$ that

$$b(p_{z(b)}, 1) = z(b).$$

Since $p_x(1) = z$, it is clear that

$$0 \leq m \leq \sup_{b \in B} \inf_{a \in A} v(a, b) \leq \sup_{b \in B} v(a_{z(b)}, b) = |p_{z(b)}(1) - b(p_{z(b)}, 1)| = 0$$

and $m = 0$.

Therefore $M > m$, q. e. d.

8. A game of pursuit and evasion of two points in the space R^n has a value since the optimal strategies (in the sense of section 3) are

$$p'(t) = c_1 \frac{q(t) - p(t)}{\|q(t) - p(t)\|}, \quad q'(t) = c_2 \frac{p(t) - q(t)}{\|p(t) - q(t)\|},$$

$$p(0) = p_0, \quad q(0) = q_0.$$

Therefore it is essential in our example that P and Q should not be whole plane. It is not so if the number of moving points of the two players is greater. Consider the game described briefly as follows. The pursuer \mathcal{U} has three moving points in the plane, situated at the moment 0 in the angles of an equilateral triangle with side 1. The evader \mathcal{Z} has one point situated at the moment 0 at the central point of this triangle. $P = Q = R^2$. The velocity of all points is at most $1/\sqrt{3}$. Again, under some very general assumptions regarding the classes of strategies (\mathcal{U} must have some straight line strategies, the strategies of \mathcal{Z} must be continuous on the set of straight line movements of \mathcal{U} , and \mathcal{Z} must have some appropriate finite set of strategies) this game has no value. The idea of the proof is the same as that of section 7.

References

- [1] L. D. Berkowitz, W. H. Fleming and H. E. Scarf, *Contributions to the theory of games III*, Part V, *Annals of Math. Studies* 39 (1957), p. 393-435.
 [2] J. C. C. McKinsey, *Introduction to the theory of games*, New York 1952.
 [3] Jan Mycielski, S. Świerczkowski and A. Zięba, *On infinite games*, *Bull. Acad. Pol. Sci., Cl. III*, 3 (1955), p. 133-136.

Reçu par la Rédaction le 13. 12. 1960

Analytic functions of polynomial growth

by

J. MIKUSIŃSKI (Katowice)

In this paper we give simple proofs of some theorems (of type Paley-Wiener) on the representation of analytic functions in the form

$$\Phi(z) = \int_{-\infty}^{\infty} e^{-zt} f(t) dt,$$

where $f(t)$ is a distribution. Usual proofs are based on similar theorems of classical Analysis [1], [2], [3], [4]. This way is circuitous for distributions, for in this case it suffices to start from a class of very regular functions, which makes proofs easier and shorter, the tools of the theory of the Lebesgue integral being then superfluous. This idea is followed in this paper.

The proof of the fundamental theorem is essentially similar to that in [5], but some further simplifications are introduced.

A function $\Phi(z)$ is said of *polynomial growth* in a set G if there exists a polynomial P such that

$$|\Phi(z)| < P(r) \text{ in } G \quad (r = |z|).$$

The aim of this paper is to prove the following

THEOREM. *If a function $\Phi(z)$, analytic in $\text{Re } z > 0$, is of polynomial growth, then it can be represented in the form*

$$(1) \quad \Phi(z) = \int_{-\infty}^{\infty} e^{-zt} f(t) dt^{(1)},$$

where $f(t)$ is a distribution, tempered for $t > 0$ and vanishing for $t < 0$.

Here, by a distribution *tempered* for $t > 0$ we understand every distribution which is a derivative of some order of a continuous function of polynomial growth for $t > 0$.

⁽¹⁾ The meaning of this integral may be understood in the sense of [6]. See also [7].