il existerait une fonction $\Phi(t)$ mesurable et essentiellement bornée telle que

$$\int_0^T f(t)\Phi(t)\,dt \neq 0 \tag{1}$$

et

$$\int_0^T (g*k)(t)\,\Phi(t)\,dt = 0, \tag{2}$$

quelle que soit la fonction continue k. En posant $\Psi(t) = \Phi(T-t)$, $\int_0^T k(t)(g*\Psi)(T-t)\,dt = k*(g*\Psi)(T) = \big((k*g)*\Psi\big)(T) = \int_0^T (g*k)(t)\,\Phi(t)\,dt = 0$, d'ou l'on déduit que $g*\Psi = 0$ presque partout. En vertu du théorème de Titchmarsh et de l'hypothèse faite sur g on a $\Psi = 0$ presque partout, ce qui contredit (1).

Nous pouvons maintenant démontrer notre

THÉORÈME. *Pour tout opérateur* $\frac{f}{g} \in \mathfrak{M}$ *tel que* g *n'est pas identiquement nulle au voisinage de l'origine, il existe une suite* $\{k_n\}$, $k_n \in C[0,\infty)$, *telle que* $k_n \to \frac{f}{g}$ *dans* $\mathfrak{M}$.

Démonstration. D'après le Lemme, il existe des fonctions k_n continues sur $[0,n]$ telles que

$$\int_0^n |(g*k_n)(t) - f(t)|\,dt < \frac{1}{n}. \tag{3}$$

Ceci entraîne $|1*g*k_n - 1*f| \leqslant 1/n$ pour $0 \leqslant t \leqslant n$. On peut supposer que $k_n \in C[0,\infty)$, ce qui ne restreint pas la généralité. Cela étant, on a $1*g*k_n \to 1*f$ dans $C[0,\infty)$, donc

$$k_n = \frac{1*g*k_n}{1*g} \to \frac{1*f}{1*g} = \frac{f}{g} \quad \text{dans} \quad \mathfrak{M},$$

la division étant ici entendue comme l'opération inverse de la convolution.

Ajouté pendant la correction. Le résultat ci-dessus a été implicitement obtenu par M. I. Fenyő (qui nous a obligeamment attiré l'attention) dans son ouvrage *Á Mikusiński-féle operátorfogalom és disztribúció fogalma közti kapcsolatról,* Á Magyar Tud. Akad. Mat. és Fiz. Tud. Osztályának Közleményei 8 (1958), p. 385-392.

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Weak* bases in conjugate Banach spaces

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Introduction. The notion of a basis introduced by J. Schauder [8] has a natural extension to topological linear spaces [2]. A basis in a topological linear space U is such a sequence $\{x_n\} \subset U$ that to every x in U there corresponds a unique sequence $\{a_n\}$ of scalars for which the following equation holds:

$$x = \sum_{n=0}^{\infty} a_n x_n. \tag{1}$$

Convergence of the series is that of the topology on U. Here the coefficients are obviously additive and homogeneous functionals on U:

$$a_n = \varphi_n(x), \quad n = 1, 2, \ldots \tag{2}$$

When all these coefficient functionals are continuous on U, the basis $\{x_n\}$ is called [2] a *Schauder basis.*

In the present paper we shall examine the particular case when U is the conjugate space E^* of a Banach space E (over the real or complex field), endowed with its weak topology $\sigma(E^*, E)$; this space is locally convex. In this case we shall use for the bases and the Schauder bases of U, the terms: weak* basis and weak* Schauder basis of E^*, respectively. We shall also use the notation $\sum_{n=1}^{\infty}{}^{*}$ for the series in the weak topology $\sigma(E^*, E)$; thus,

$$\sum_{n=1}^{\infty}{}^{*} g_n = g \quad (g_n, g \in E^*)$$

means that

$$\sum_{n=1}^{\infty} g_n(x) = g(x) \quad \text{for all} \quad x \in E.$$

In § 1 we show an example of a weak* basis which is not a weak* Schauder basis. § 2 contains our main result which is the construction of a $\sigma(E^*, E)$-separable conjugate Banach space E^* in which there exists no weak* Schauder basis. Finally, in § 3 we prove that the existence of a weak* Schauder basis in the conjugate space E^* of a separable Banach space E, is equivalent to the existence of a basis in E.

§ 1. A weak* basis which is not a weak* Schauder basis. It is known (see [2]) that several classes of topological linear spaces U, such as Fréchet spaces and strict inductive limits of Fréchet spaces, do not tolerate non-Schauder bases (i. e. every basis in U is a Schauder basis). On the other hand, in [2], § 6, there is also given an example of a basis in a topological linear space U, which is not a Schauder basis; however, the space U in that example is not locally convex.

In this paragraph we shall give such an example in a locally convex space U. We shall prove that *the non retro-basis $\{f_n\}$ of $E^* = l$ ($E = c_0$) constructed by B. R. Gelbaum in* [6] *is a weak* basis, but not a weak* Schauder basis.*

In fact, the sequence $\{f_n\}$ is defined [6] by

(3) $$f_n(x) = (-1)^{n+1}\xi_1 + \xi_n \quad \text{for all} \quad x = \{\xi_n\} \in c_0 \quad (n = 1, 2, \ldots).$$

Let us show that $\{f_n\}$ is a weak* basis of l. Since $\{f_n\}$ is a basis of l [6], every $f \in l$ admits a unique representation

$$f = \sum_{n=1}^{\infty} a_n f_n,$$

the convergence of the series being the strong (i. e. the norm) convergence on l. Consequently, every $f \in l$ admits a representation

(4) $$f = \sum_{n=1}^{\infty}{}^{*} a_n f_n,$$

and it remains to show that this representation is also unique. Assume that

$$\sum_{n=1}^{\infty}{}^{*} b_n f_n = 0,$$

i. e. that

(5) $$\lim_{k\to\infty} \sum_{n=0}^{k} b_n f_n(x) = 0 \quad \text{for all} \quad x \in c_0.$$

For $x_j = \{0, \ldots, 0, 1, 0, \ldots\} \in c_0$ (1 in j-th place) we have, by (3),

$$f_n(x_1) = (-1)^{n+1}, \quad n = 1, 2, 3, \ldots$$

and

$$f_n(x_j) = \delta_{nj}, \quad n = 1, 2, 3, \ldots; \quad j = 2, 3, \ldots$$

Consequently, applying (5) for $x = x_j$ ($j = 1, 2, \ldots$), we obtain

$$b_1 - b_2 + b_3 - \ldots + (-1)^{n+1} b_n + \ldots = 0,$$

$$b_2 = b_3 = \ldots = b_n = \ldots = 0,$$

whence

$$b_n = 0, \quad n = 1, 2, \ldots,$$

which proves the uniqueness of the representation (4). Thus $\{f_n\}$ is a weak* basis in l.

On the other hand, it is easy to see that the first coefficient functional φ_1 is given by

(6) $$\varphi_1(f) = \eta_1 + \eta_2 - \eta_3 + \eta_4 - \eta_5 + \ldots \quad \text{for all} \quad f = \{\eta_n\} \in l.$$

In fact, this may be obtained either as a consequence of the above uniqueness of the representation (4), together with [6] ([1]), or from (3), by a simple direct computation. Now, since $\varphi_1 \notin c_0$, it is not $\sigma(l, c_0)$—continuous on l, and thus $\{f_n\}$ is not a weak* Schauder basis. This concludes the proof of our assertion.

§ 2. The Schauder basis problem for separable locally convex spaces and its solution. A locally convex space U which possesses a Schauder basis $\{x_n\}$ is clearly separable: the set of all linear combinations of the form

$$\sum_{n=1}^{m} r_n x_n,$$

where the r_n are rational numbers and where m is a positive integer, constitutes a countable set everywhere dense in U. It is natural to ask whether or not the converse is also true, i. e.:

THE SCHAUDER BASIS PROBLEM. *Does every separable locally convex space* ([2]) *possess a Schauder basis?*

In this paragraph we shall prove that the answer to this problem is negative, by exhibiting a $\sigma(E^*, E)$-separable conjugate Banach space E^* in which there exists no weak* Schauder basis.

THEOREM 1. *If the conjugate space E^* of a Banach space E possesses a weak* Schauder basis $\{f_n\}$, then the space E has a basis.*

([1]) In this case we must take into account that in the first line of [6], p. 189, there is a misprint: instead of $f_1 = \{1, -1, 1, \ldots\}$ should be $f_1 = \{1, 1, -1, 1, -1, \ldots\}$.

([2]) It would not be reasonable to formulate this problem for more general topological linear spaces. E. g. the spaces L^p for $0 < p < 1$ admit no continuous linear functional ([4], Theorem 1) and hence no Schauder basis is admitted.

Proof. Since $\{f_n\}$ is a weak* Schauder basis in E^*, the coefficient functionals

$$a_n = \varphi_n(f), \quad n = 1, 2, \ldots,$$

are all $\sigma(E^*, E)$-continuous, and consequently, there exists a sequence $\{x_n\} \subset E$ such that

$$\varphi_n(f) = f(x_n) \quad \text{for all} \quad f \in E^* \quad (n = 1, 2, \ldots).$$

Then, from $\varphi_i(f_j) = \delta_{ij}$ we infer that

$$f_j(x_i) = \delta_{ij}, \quad i, j = 1, 2, \ldots,$$

i. e. that (x_n, f_n) is a biorthogonal system. Since $\{f_n\}$ is a weak* basis in E^*, we have, in the sense of the weak topology $\sigma(E^*, E)$,

$$\lim_{m\to\infty} \sum_{n=1}^{m} f(x_n) f_n = \lim_{m\to\infty} \sum_{n=1}^{m} \varphi_n(f) f_n = f \quad \text{for all} \quad f \in E^*.$$

Consequently, by a theoerem of S. Karlin ([7], Theorem 2), $\{x_n\}$ is a basis of the space E.

COROLLARY 1. *If the conjugate space E^* of a Banach space E possesses a weak* Schauder basis $\{f_n\}$, then the space E is separable.*

COROLLARY 2. *Let E be a non-separable Banach space, such that its conjugate space E^* is separable for the weak topology $\sigma(E^*, E)$. Then E^* has no weak* Schauder basis.*

Now we shall show that such a Banach space E *exists*, i. e. that the converse of [3], p. 124, Theorem 4 ([3]), is not valid. This will also yield the negative answer to the Schauder basis problem.

Let us denote by Q_0 a compact ([4]) non-metrizable topological space, containing a countable everywhere dense set. Such a space exists, see e. g. [1], Remark 4 of the first complement to Chapter VII. Denote by $C(Q_0)$ the space of all scalar-valued continuous functions defined on Q_0, endowed with the usual vector operations and with the usual norm $\|x(\cdot)\| = \max_{q \in Q_0} |x(q)|$.

THEOREM 2. *The Banach space $E = C(Q_0)$ constructed above is non-separable, but its conjugate space E^* is separable for the weak topology $\sigma(E^*, E)$. Consequently, E^* has no weak* Schauder basis.*

Proof. Since Q_0 is non metrizable, the space $E = C(Q_0)$ is non-separable (see e. g. [5], p. 437, Exercise 17).

([3]) Let us recall this theorem of Banach: *If a Banach space E is separable, then its conjugate space E^* is $\sigma(E^*, E)$-separable.*

([4]) We use the term "compact" in the sense of Bourbaki (i. e. bicompact Hausdorff).

On the other hand, let $T = \{t_k\} \subset Q_0$ be a sequence, everywhere dense in Q_0. We shall show that the countable set of all finite linear combinations of the form

$$\sum_{i=1}^{m} r_i f_{t_{k_i}},$$

where $t_{k_1}, \ldots, t_{k_m} \in T$, $r_1, \ldots, r_m$ are rational scalars, and where the f_q $(q \in T)$ are defined by

$$f_q[x(\cdot)] = x(q) \quad \text{for all} \quad x(\cdot) \in C(Q_0), \tag{7}$$

is everywhere dense in E^* for the weak topology $\sigma(E^*, E)$. In fact, let us recall (see e. g. [5], p. 441, Lemma 6), that every extremal point of the unit cell S^* of E^* is of the form αf_q, where $q \in Q_0$, α is a scalar with $|\alpha| = 1$, and where f_q is defined by (7). Let us also recall that by the Krein-Milman theorem ([5], p. 440, Theorem 4), S^* is the $\sigma(E^*, E)$-closed convex hull of the set of its extreme points. Consequently, if $f_0 \in S^*$ and if

$$V_{x_1, \ldots, x_n, \varepsilon}(f_0) = \{f \in E^* \mid |f(x_j) - f_0(x_j)| < \varepsilon, j = 1, \ldots, n\}$$

is a given $\sigma(E^*, E)$-neighbourhood of f_0, then there exists a finite linear combination ([5])

$$\sum_{i=1}^{m} \lambda_i f_{q_i} \in V_{x_1, \ldots, x_n, \varepsilon}(f_0) \quad \Big(\sum_{i=1}^{m} |\lambda_i| = 1\Big). \tag{8}$$

Now, since $T = \{t_k\}$ is dense in Q_0 and since the mapping $q \to f_q$ $(q \in Q_0)$ is a homeomorphism if we endow E^* with the weak topology $\sigma(E^*, E)$, (see e. g. [5], p. 442, Lemma 7), it follows that for every $\delta > 0$ there exist m elements $t_{k_1}, \ldots, t_{k_m} \in T$ such that

$$f_{t_{k_i}} \in V_{x_1, \ldots, x_n, \delta}(f_{q_i}), \quad i = 1, \ldots, m,$$

i. e. that

$$|f_{q_i}(x_j) - f_{t_{k_i}}(x_j)| < \delta, \quad j = 1, \ldots, n;\ i = 1, \ldots, m.$$

Furthermore, for every $\gamma > 0$ we can choose m rational scalars $r_1, \ldots, r_m$ so that

$$|\lambda_i - r_i| < \gamma, \quad i = 1, \ldots, m.$$

Then we shall have, for suitable δ and γ,

$$\Big|\sum_{i=1}^{m} \lambda_i f_{q_i}(x_j) - \sum_{i=1}^{m} r_i f_{t_{k_i}}(x_j)\Big| \leqslant \sum_{i=1}^{m} |\lambda_i f_{q_i}(x_j) - r_i f_{t_{k_i}}(x_j)| < \varepsilon, \quad j = 1, \ldots, n,$$

([5]) The coefficients α_i with $|\alpha_i| = 1$ we have enclosed in the λ_i's.

whence, taking into account (8),

$$\Big|\sum_{i=1}^{m} r_i f_{t_{k_i}}(x_j) - f_0(x_j)\Big| < 2\varepsilon,$$

i. e.

$$\sum_{i=1}^{m} r_i f_{t_{k_i}} \in V_{x_1,\ldots,x_n,2\varepsilon}(f_0).$$

Consequently S^*, and hence also E^*, is $\sigma(E^*, E)$-separable.

Finally, Corollary 2 shows that E^* has no weak* Schauder basis, and the proof of Theorem 2 is complete.

§ 3. The restricted weak* Schauder basis problem in conjugate Banach spaces. It is natural to ask whether or not the converse of Corollary 1 is also true, i. e.:

THE RESTRICTED WEAK* SCHAUDER BASIS PROBLEM. *Does the conjugate space E^* of an arbitrary separable Banach space E possess a weak* Schauder basis?*

We have seen in the preceding paragraph that the *general* weak* Schauder basis problem (i. e. does every $\sigma(E^*, E)$-separable conjugate Banach space possess a weak* Schauder basis?) has a negative answer. Now we shall prove that the *restricted* weak* Schauder basis problem, formulated above, is equivalent to the classical basis problem (6) (which is till now unsolved). For this purpose we shall prove that the converse of theorem 1 is also true, i. e. that we have

THEOREM 3. *The conjugate space E^* of a Banach space E possesses a weak* Schauder basis $\{f_n\}$ if and only if the space E has a basis.*

Proof. The necessity part is nothing else but theorem 1.

Conversely, assume that the space E has a basis $\{x_n\}$. Let $\{f_n\} \subset E^*$ be the sequence of the coefficient functionals corresponding to the basis $\{x_n\}$. We shall prove that $\{f_n\}$ is a weak* Schauder basis of E^*. It is known ([3], p. 106, Theorem 1) that every $f \in E^*$ admits a representation

$$f = \sum_{n=1}^{\infty}{}^{*} f(x_n) f_n. \tag{9}$$

Now, assume that

$$\sum_{n=1}^{\infty}{}^{*} a_n f_n = 0,$$

i. e. that

$$\lim_{k\to\infty} \sum_{n=1}^{k} a_n f_n(x) = 0 \quad \text{for all} \quad x \in E.$$

(6) I. e. to the following problem: does every separable Banach space possess a basis?

Then, for $x = x_j$, $j = 1, 2, \ldots$, taking into account the biorthogonality relations $f_n(x_j) = \delta_{nj}$, $n, j = 1, 2, \ldots$, we obtain

$$a_j = 0, \quad j = 1, 2, \ldots,$$

which proves the uniqueness of the coefficients of the representation (9). Thus $\{f_n\}$ is a weak* basis of E^*.

Finally, the coefficient functionals

$$\varphi_n(f) = f(x_n) \quad (f \in E^*, \quad n = 1, 2, \ldots)$$

of the representation (9) are obviously $\sigma(E^*, E)$-continuous, and thus $\{f_n\}$ is a weak* Schauder basis of E^*. This completes the proof of Theorem 3.

COROLLARY 3. *The restricted weak* Schauder basis problem is equivalent to the classical basis problem.*

Added in proof. By using theorem 4 of the paper of J. Dixmier, *Sur un théorème de Banach*, Duke Math. Journ. 15 (1948), p. 1057-1071, one can give a more simple proof of theorem 2 above and one can also prove the following theorem:

THEOREM 2′. *The conjugate space m^* (of the non separable Banach space m) is separable for the weak topology $\sigma(m^*, m)$. Consequently, m^* has no weak* Schauder basis.*

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