Banch spaces of Lipschitz functions

by

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§ 1. Introduction. If \( 0 < a < 1 \), \( \text{Lip}_a \) is the space of all complex valued continuous functions on the real line \( R \) of period 1 with

\[
\sup_{\tau \in R} |f(\sigma + \tau) - f(\sigma)| = O(|\tau|^a) \quad \text{as} \quad \tau \to 0.
\]

\( \text{lip}_a \) is the subset of \( \text{Lip}_a \) consisting of those \( f \) with

\[
\sup_{\tau \in R} |f(\sigma + \tau) - f(\sigma)| = o(|\tau|^a) \quad \text{as} \quad \tau \to 0.
\]

Supplied with the norm \( \| \cdot \|_a \), defined by

\[
\|f\|_a = \sup_{\tau, \sigma \in R} \frac{|f(\sigma + \tau) - f(\sigma)|}{|\tau|^a},
\]

\( \text{Lip}_a \) is a Banach space and \( \text{lip}_a \) is a closed linear subspace (1).

We show in § 2 that the Banach space \( \text{Lip}_a \) is canonically isomorphic and isometric to the second dual space of the Banach space \( \text{lip}_a \). In § 3 we identify the extreme points of the unit sphere of the dual of \( \text{lip}_a \) and obtain as a consequence in § 4 the fact that \( \text{lip}_a \) has no isometries in addition to the expected ones.

§ 2. \( \text{Lip}_a \) is the second dual of \( \text{lip}_a \). Two definitions are necessary before we are able to state the main result of this section. For each \( \sigma \) in \( R \), we define the functional \( \Phi_\sigma \) in the dual space \( (\text{lip}_a)^* \) of \( \text{lip}_a \) by

\[
\Phi_\sigma(f) = f(\sigma), \quad f \in \text{lip}_a.
\]

For each functional \( F \) in the dual space \( (\text{lip}_a)^* \) of \( (\text{lip}_a)^* \), we define the function \( \hat{F} \) on \( R \) by

\[
\hat{F}(\sigma) = F(\Phi_\sigma), \quad \sigma \in R.
\]

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(1) In [2] it is shown that \( \text{lip}_a \) is the closed linear subspace of \( \text{Lip}_a \) spanned by trigonometric polynomials.
Note that if \( f \) is in \( \text{lip} \alpha \) and \( F \) is its image under the canonical embedding of \( \text{lip} \alpha \) in \( \text{lip} \alpha^{**} \), the function \( F \) is simply \( f \).

**Theorem 2.1.** The mapping \( F \to F \) is an isomorphism and isometry of \( (\text{lip} \alpha)^{**} \) onto \( \text{lip} \alpha \).

The proof proceeds by a sequence of lemmas. We shall denote by \( \| \cdot \| \) and \( \| \cdot \|^{**} \) the norms induced on \( \text{lip} \alpha^{**} \) and \( \text{lip} \alpha^{**} \) by the norm \( \| \cdot \| \) on \( \text{lip} \alpha \).

**Lemma 2.2.** If \( F \) is a functional in \( (\text{lip} \alpha)^{**} \), then the function \( F \) is in \( \text{lip} \alpha \).

**Proof.** If \( \sigma \in K, \tau \in K, \) and \( f \) in \( \text{lip} \alpha \) satisfies \( \| f \| \leq 1 \), then
\[
\| \Phi_{\sigma}(f) - \Phi_{\tau}(f) \| = | f(\sigma) - f(\tau) | \leq | \sigma - \tau |^{\alpha}.
\]
Thus
\[
\| \Phi_{\sigma} - \Phi_{\tau} \|^{**} \leq | \sigma - \tau |^{\alpha},
\]
and as a consequence,
\[
\| \Phi_{\sigma} - \Phi_{\tau} \| \leq | \sigma - \tau |^{\alpha}.
\]
and so \( F \) is in \( \text{lip} \alpha \).

We next identify the continuous linear functionals of \( \text{lip} \alpha \) by constructing an isometric embedding of \( \text{lip} \alpha \) into a space of continuous functionals supplied with the sup norm.

Let \( W \) be the locally compact topological space \( U \cup \mathcal{V} \), where
\[
U = \{ \varepsilon : 0 \leq \varepsilon \leq 1 \}
\]
and
\[
\mathcal{V} = \{ (\sigma, \tau) : 0 \leq \sigma \leq 1, 0 < \tau - \sigma \leq 1/2 \}.
\]
We denote by \( C_{b}(W) \) the Banach space of complex valued continuous functions on \( W \) that are zero at infinity, supplied with the norm \( \| \cdot \|_{W} \) defined by
\[
\| A \|_{W} = \sup_{x \in W} | A(x) |.
\]
We denote the norm of the dual space \( C_{b}(W)^{*} \) of \( C_{b}(W) \) by \( \| \cdot \|_{W^{*}} \). By the Riesz representation theorem, each element \( \psi \) of \( C_{b}(W)^{*} \) is of the form
\[
\psi(h) = \int_{\mathcal{V}} h \, d\mu, \quad h \in C_{b}(W),
\]
for a unique finite measure \( \mu \) on \( W \), and we define \( \| \mu \|_{W^{*}} \) to be \( \| \psi \|_{W^{*}} \).

For each function \( f \) in \( \text{lip} \alpha \), we denote by \( \tilde{f} \) the function on \( W \) defined by
\[
\tilde{f}(\varepsilon) = f(\varepsilon), \quad \varepsilon \in U,
\]
\[
\tilde{f}(\sigma, \tau) = f(\sigma) - f(\tau), \quad (\sigma, \tau) \in \mathcal{V}.
\]

**Lemma 2.3.** The mapping \( f \to \tilde{f} \) is a linear isometry of \( \text{lip} \alpha \), supplied with the norm \( \| \cdot \|_{W} \), into \( C_{b}(W) \), supplied with the norm \( \| \cdot \|_{W} \).

**Proof.** It is clear that \( f \to \tilde{f} \) is a linear mapping of \( \text{lip} \alpha \) into \( C_{b}(W) \). If \( f \) is in \( \text{lip} \alpha \), \( f \) has period 1, so
\[
\sup \{ | f(\varepsilon) | : \varepsilon \in K \} = \sup \{ | f(\varepsilon) | : \varepsilon \in U \}
\]
and
\[
\sup \{ | f(\sigma) - f(\tau) | : \sigma, \tau \in K \} = \sup \{ | f(\sigma) - f(\tau) | : (\sigma, \tau) \in \mathcal{V} \},
\]
and as a consequence, \( \| f \|_{W} = \| \tilde{f} \|_{W} \).

**Lemma 2.4.** Let \( \Phi \) be a functional in \( (\text{lip} \alpha)^{*} \). Then there exists a measure \( \mu \) on \( W \) with \( \| \mu \|_{W} = \| \Phi \|^{**} \) satisfying
\[
\Phi(f) = \int_{U} f(\varepsilon) \, d\mu(\varepsilon) + \int_{\mathcal{V}} f(\sigma) - f(\tau) \, d\mu(\sigma, \tau)
\]
for all \( f \) in \( \text{lip} \alpha \).

**Proof.** By Lemma 2.3, the linear functional \( \psi \) defined on the subspace
\[
\{ f : f \in \text{lip} \alpha \}
\]
of \( C_{b}(W) \) by
\[
\psi(f) = \Phi(f), \quad f \in \text{lip} \alpha,
\]
has its norm equal to \( \| \Phi \|^{**} \). \( \psi \) can be extended, by the Hahn-Banach theorem, to a linear functional of \( C_{b}(W) \) having the same norm, and thus by the Riesz representation theorem there is a measure \( \mu \) on \( W \) satisfying \( \| \mu \|_{W} = \| \Phi \|^{**} \) and
\[
\Phi(f) = \int_{U} f \, d\mu + \int_{\mathcal{V}} f(\sigma) - f(\tau) \, d\mu(\sigma, \tau)
\]
for all \( f \) in \( \text{lip} \alpha \). But (2.3) is simply another way of writing (2.2).

We shall denote by \( (\text{lip} \alpha)^{*} \) the subspace of \( (\text{lip} \alpha)^{*} \) consisting of all functionals \( \Phi \) of the form
\[
\Phi(f) = \int_{U} f \, d\lambda, \quad f \in \text{lip} \alpha,
\]
for \( \lambda \) a measure on \( U \). The subset of \((\text{lip} a)_{\alpha}^*\) consisting of all functionals of the form (2.4) for \( \lambda \) a measure concentrated at a finite number of points will be denoted by \((\text{lip} a)_{\lambda}^*\). Equivalently, \((\text{lip} a)_{\lambda}^*\) is the linear subspace of \((\text{lip} a)^*\) spanned by \( \{\Phi_\lambda; \sigma \in \mathcal{R}\} \).

**Lemma 2.5.** \((\text{lip} a)_{\alpha}^*\) is dense in \((\text{lip} a)^*\) in its norm topology.

**Proof.** Let \( \Phi \) be a functional in \((\text{lip} a)^*\). By Lemma 2.4 there is a measure \( \mu \) on \( W \) that satisfies

\[
\Phi(f) = \int_W f d\mu, \quad f \in \text{lip} a.
\]

Let

\[ W_1 \subset W_2 \subset \ldots \subset W_n \subset \ldots \]

be a sequence of compact subsets of \( W \) whose union is \( W \). For each positive integer \( n \), we define the functional \( \Phi_n \) in \((\text{lip} a)^*\) by

\[
\Phi_n(f) = \int_{W_n} (f - f_{|W_n}) d\mu_n, \quad f \in \text{lip} a.
\]

Because of Lemma 2.3,

\[
\lim_{n \to \infty} \|\Phi_n - \Phi\| = 0,
\]

so it only remains to show that each \( \Phi_n \) is in \((\text{lip} a)_{\alpha}^*\). But since

\[
\Phi_n(f) = \int_{W_n} f d\mu_n + \int_{W_n} (f - f_{|W_n}) d\mu_n,
\]

for all \( f \) in \( \text{lip} a \), and \( |f - f_{|W_n}| \) is bounded away zero on \( V \cap W_n \), this is indeed the case.

**Lemma 2.6.** \((\text{lip} a)_{\alpha}^*\) is dense in \((\text{lip} a)^*\) in its norm topology.

**Proof.** Let \( \lambda \) be a measure on \( U \) and \( \Phi \) the functional in \((\text{lip} a)^*\) defined by (2.4). By Lemma 2.5, it suffices to show that \( \Phi \) is in the closure in \((\text{lip} a)^*\) of \((\text{lip} a)_{\alpha}^*\). Let \( \mathcal{O}(U) \) be the space of complex valued continuous functions on \( U \). Using the Riesz representation theorem, we identify the space of measures on \( U \) with the dual space \( \mathcal{O}(U)^* \) and denote by \( \|\cdot\|_1 \) the norm on this space of measures induced by the sup norm on \( \mathcal{O}(U) \). Choose any \( \varepsilon > 0 \). We shall denote by \( S \) the unit sphere

\[
\{f; f \in \text{lip} a, \|f\|_1 < 1\}
\]

on \( \text{lip} a \). \( S \) is collection of functions having period 1 on \( E \) that is bounded by 1 and equicontinuous. Thus by Ascoli's theorem, \( S \) is conditionally compact in the topology of uniform convergence, so there is a finite subset \( T \) of \( S \) such that each function \( S \) in \( T \) is uniformly within \( \varepsilon(\|\lambda\|_1)^{-1} \) of some function in \( T \). It is well known (see [1, p. 78]) that the subset of the sphere

\[
\{\eta; \eta \in \mathcal{O}(U)^*, \|\eta\|_1 < \|\lambda\|_1\}
\]

consisting of measures concentrated at a finite number of points of \( U \) is dense in this sphere in the weak* topology of \( \mathcal{O}(U)^* \). Thus there is a measure \( \gamma \) concentrated at a finite number of points of \( U \) that satisfies \( \|\eta\|_1 < \|\lambda\|_1 \) and

\[
\int f d\lambda - \int f d\gamma < \varepsilon, \quad f \in T.
\]

Because of the choice of \( T \),

\[
\int f d\lambda - \int f d\gamma < \varepsilon, \quad f \in S,
\]

and as a consequence, the functional \( \psi \) in \((\text{lip} a)^*\) defined by

\[
\psi(f) = \int f d\gamma, \quad f \in \text{lip} a,
\]

satisfies \( |\psi - \Phi| < \varepsilon \). Since \( \varepsilon \) was arbitrary and \( \psi \) is in \((\text{lip} a)_{\alpha}^*\), we have shown that \( \Phi \) is in the closure of \((\text{lip} a)_{\alpha}^*\), and the proof is complete.

**Corollary 2.7.** The mapping \( F \to \Phi \) of \((\text{lip} a)^*\) into \( \text{lip} a \) is one-one.

**Proof.** It is clear that the mapping is linear. If \( F \) in \((\text{lip} a)^*\) is in the kernel of the mapping, \( F \) is the zero function, so

\[
F(\Phi_\lambda) = F(\sigma) = 0, \quad \sigma \in \mathcal{R}.
\]

But by Lemma 2.6, linear combinations of the \( \Phi_\lambda \) are dense in \((\text{lip} a)^*\) in its norm topology. Thus \( F \) must be the zero functional and the mapping is one-one as claimed.

**Lemma 2.8.** The mapping \( F \to \Phi \) of \((\text{lip} a)^*\) into \( \text{lip} a \) is onto and norm preserving.

**Proof.** Let \( h \) be a function in \( \text{lip} a \). We shall first construct a functional \( F \) in \((\text{lip} a)^*\) satisfying \( F = h \) for each positive integer \( n \), the Fejér kernel \( K_n \) is defined by

\[
K_n(\sigma) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n+1)\pi \sigma}{\pi n}, \quad \sigma \in \mathcal{R}.
\]

(For the properties of the Fejér kernel that we shall use, see [4], Chap. 3). The convolution \( K_n * h \) is the \( n \)-th \((C, 1)\) partial sum of the Fourier series of \( h \). These \((C, 1)\) sums converge uniformly to \( h \), so

\[
\lim_{n \to \infty} K_n * h(\sigma) = h(\sigma), \quad \sigma \in \mathcal{R}.
\]

Moreover, it is simple to check, using the fact that each \( K_n \) is positive and satisfies

\[
\int K_n(\sigma) d\sigma = 1,
\]
that
\[ \|K_n \cdot h\| \leq \|h\|_\alpha. \]

\( K_n \cdot h \) is a trigonometric polynomial and thus in \( \text{lip} \alpha \). We shall denote by \( F_\alpha \) the functional in \( \text{lip} \alpha \) corresponding to \( K_n \cdot h \) under the canonical imbedding of \( \text{lip} \alpha \) in \( \text{lip} \alpha^{**} \); i. e.
\[ F_\alpha(\Phi) = \Phi(K_n \cdot h), \quad \Phi \in (\text{lip} \alpha)^*. \]

Because of (2.6) and the fact that the imbedding of \( \text{lip} \alpha \) in \( \text{lip} \alpha^{**} \) is an isometry,
\[ \|F_\alpha\|_{\alpha^{**}} \leq \|h\|_\alpha. \]

We define
\[ F(\Phi) = \lim_{n \to \infty} F_\alpha(\Phi) \]
for all \( \Phi \in (\text{lip} \alpha)^* \) for which the limit exists. By (2.5) and (2.7), \( F(\Phi) \) exists for all \( \Phi \in (\Phi_\alpha, e R) \), and thus by linearity exists for all \( \Phi \in (\text{lip} \alpha)^* \).

But by Lemma 2.6, \( \text{lip} \alpha^{**} \) is dense in \( (\text{lip} \alpha)^* \) in its norm topology. As a consequence, because of (2.8), \( F(\Phi) \) exists for all \( \Phi \in (\text{lip} \alpha)^* \) and \( F \) is a functional in \( (\text{lip} \alpha)^{**} \) satisfying
\[ \|F\|_{\alpha^{**}} \leq \|h\|_\alpha. \]

Furthermore, \( F = h \), since for each \( \sigma \) in \( R \),
\[ F(\sigma) = F(\Phi_\sigma) = \lim_{n \to \infty} F_\alpha(\sigma) = \lim_{n \to \infty} K_n \cdot h(\sigma) = h(\sigma). \]

By (2.9), \( \|F\|_{\alpha^{**}} \leq \|F\|_\alpha \), so to complete the proof of the lemma it remains only to demonstrate the reverse inequality. For each \( g \) in \( R \),
\[ \|F\|_\alpha = \|F(\Phi_\alpha)\| = \|F\|_{\alpha^{**}} \|\Phi_\alpha\|_\alpha < \|F\|_{\alpha^{**}}. \]

Furthermore, for each \( \sigma \) and \( \tau \) in \( R \),
\[ \|F(\sigma - \tau)\| = \|F(\Phi_\sigma - \Phi_\tau)\| \leq \|F\|_{\alpha^{**}} \|\Phi_\sigma - \Phi_\tau\| < \|F\|_{\alpha^{**}} \|\sigma - \tau\| \]
by (2.1), (2.10) and (2.11) together show that \( \|F\|_\alpha \leq \|F\|_{\alpha^{**}} \) and the proof is complete.

Theorem 2.1 is now immediate consequence of Lemma 2.2, Corollary 2.7 and Lemma 2.8.

§ 3. Extreme points in \( (\text{lip} \alpha)^* \). Our aim in this section is the identification of the extreme points (1) of the unit sphere of the dual of \( \text{lip} \alpha \). Because of Lemma 2.3 it suffices to consider the corresponding problem for a linear space of continuous functions under the sup norm.

(1) \( \sigma \) is an extreme point of a convex set if it is not the mid-point of any segment lying in the set.

Let \( X \) be a locally compact topological space and \( C(X) \) the space of complex valued continuous functions on \( X \) that are zero at infinity. Suppose that \( A \) is a closed linear subspace of \( C(X) \). \( A \) is a Banach space under the sup norm and we shall denote its dual by \( A^* \).

The following result is contained in Lemma V.8.6 of [2]:

**Lemma 3.1.** Each extreme point of the unit sphere of \( A^* \) is of the form
\[ \Phi(g) = l(g) \lambda, \quad g \in A, \]

for some \( \lambda \) in \( X \) and some complex number \( \lambda \) with \( |\lambda| = 1 \).

One further definition is necessary before we are able to state a partial converse to Lemma 3.1. Let \( \sigma \) be a point of \( X \). A function \( h \) in \( A \) is said to peak at \( \sigma \) relative to \( A \) if \( h(\sigma) = 1 \) and
\[ |h(y)| < 1, \quad y \in X, y \neq \sigma, \]
with equality holding only for those \( y \) in \( X \) that satisfy either
\[ g(y) = g(\sigma), \quad \forall g \in A, \]

or
\[ g(y) = -g(\sigma), \quad \forall g \in A. \]

**Lemma 3.2.** Let \( \sigma \) be a point of \( X \). Suppose that there is a function in \( A \) that peaks at \( \sigma \) relative to \( A \). Then the functional \( \Phi \) in \( A^* \) defined by
\[ \Phi(g) = l(g), \quad g \in A, \]
is an extreme point of the unit sphere of \( A^* \).

**Proof.** It is clear that \( \Phi \) is in the unit sphere of \( A^* \). Suppose that \( \Phi = \frac{1}{2}(\psi_1 + \psi_2) \), where \( \psi_1 \) and \( \psi_2 \) are also in the unit sphere. We must show that \( \psi_1 = \psi_2 \). By the Hahn-Banach theorem, the functionals \( \psi_1 \) and \( \psi_2 \) can be extended in a norm preserving manner to \( C(X) \) and thus by the Riesz representation theorem there are measures \( \mu_1 \) and \( \mu_2 \) in the unit sphere of \( C(X)^* \) satisfying
\[ \psi_1(g) = \int_X g \, d\mu_1, \quad g \in A, \quad \mu_1, \quad i = 1, 2. \]

Let \( \psi_1 \) be a function in \( A \) that peaks at \( \sigma \) relative to \( A \). Since \( \mu_1 \) and \( \mu_2 \) are in the unit sphere of \( C(X)^* \),
\[ \int_X h \, d\mu_1 \leq \sup_{g \in A} |h(g)| = 1, \quad i = 1, 2. \]

Thus, because
\[ 1 - l(h) = \Phi(h) = \frac{1}{2}(\psi_1(h) + \psi_2(h)) = \frac{1}{2} \left( \int_X h \, d\mu_1 + \int_X h \, d\mu_2 \right), \]
we must have

\[ \int_{A} h \, d\mu_1 = \int_{A} h \, d\mu_2 = 1. \]

We define the subsets \( Y_+, Y_-, \) and \( Y_0 \) of \( X \) by

\[ Y_+ = \{ y : h(y) = 1 \} \]  
\[ Y_- = \{ y : h(y) = -1 \} \]  
\[ Y_0 = \{ y : h(y) = 0 \} \]

Since (3.1) holds and the \( \mu_i \) are in the unit sphere of \( C_0(X) \), we must have

\[ \mu(Y_+) - \mu(Y_-) = 1, \quad \mu(Y_0) = 0, \quad i = 1, 2. \]

Thus for each \( g \) in \( A \),

\[ \psi_i(g) = \int_{A} g \, d\mu_1 = \int_{Y_+} g \, d\mu_1 + \int_{Y_0} g \, d\mu_1 + \int_{Y_-} g \, d\mu_1 \]

\[ = g(x) \mu(Y_+) - g(x) \mu(Y_-) = g(x) = \Phi_i(g), \quad i = 1, 2. \]

As a consequence, \( \psi_1 = \psi_2 = \Phi \) and \( \Phi \) is extreme as claimed.

**Theorem 3.3.** A functional \( \Phi \) in \( \text{lip} \alpha \) is an extreme point of the unit sphere of \( \text{lip} \alpha^* \) if and only if it is of the form

\[ \Phi(f) = I(f), \quad f \text{ clip}, \]

for \( q \) in \( R \) and \( \lambda \) a complex number with \( |\lambda| = 1 \), or of the form

\[ \Phi(f) = \frac{1}{|\lambda|^2} f(x) - f(x), \quad f \text{ clip}, \]

for \( \alpha \) and \( \tau \) in \( R \), \( 0 < \alpha - \tau \leq \frac{1}{2} \) and \( \lambda \) a complex number with \( |\lambda| = 1 \).

**Proof.** We shall use the notation established in § 2. The functionals \( \Phi \) described in the statement of Theorem 3.3 are precisely those of the form

\[ \Phi(f) = I(f)(x), \quad f \text{ clip}, \]

for \( \alpha \) a point of \( W \) and \( \lambda \) a complex number with \( |\lambda| = 1 \). Lemmas 2.3 and 3.1 applied to \( X = W \) and \( A = \{ f : f \text{ clip} \} \) show that each extreme point of the unit sphere of \( \text{lip} \alpha^* \) is indeed a functional of the form (3.4). To establish the converse, because of Lemma 3.2, it suffices to show that for each point \( x \) of \( W \) it is possible to find some function \( f \) in \( \text{lip} \alpha \) with \( f \) peaking at \( x \) relative to \( A \).

**Case I.** \( \alpha = 0 \). By the invariance of \( \text{lip} \alpha \) and \( \| \cdot \| \), under translation, we may assume that \( 0 < q < 1 \). Let \( f \) be any function in \( \text{lip} \alpha \) satisfying \( f(q) = 1 \), \( |f(\alpha)| < 1 \) if \( \alpha - \alpha \) is not an integer, and \( \|f(q) - f(x)\| < \delta(\alpha - \alpha) \) for \( \alpha, x \in R \). Then \( f(1) = 1 \), \( \|f(\alpha)| < 1 \) if \( \alpha \in W \) and \( y \neq x \), so \( f \) peaks at \( x \) relative to \( A \).

**Case II.** \( \alpha = (\alpha, \tau) \), \( 0 < \alpha < 1 \), \( \alpha - \tau \leq \frac{1}{2} \). By the invariance of \( \text{lip} \alpha \) and \( \| \cdot \| \), under translation, we may assume that \( \alpha = 0 \). Let \( f \) be the function in \( \text{lip} \alpha \) that satisfies \( f(1) = 0 \), \( f(\alpha) = -\tau \), \( f(1) = 0 \), and is linear in the intervals \( [0, \tau] \) and \( [\tau, 1] \). Let \( \alpha' \) be the point \( (1, \alpha) \) of \( W \). Then \( f(\alpha') = f(x') = 1 \) if \( y \in W \), \( y \neq x \), \( y \neq x' \), and \( \|f(x) - f(x')\| \leq \delta(\alpha - \alpha) \) for all \( x \in W \), so \( f \) peaks at \( x \) relative to \( A \).

**Case III.** \( \alpha = (\alpha, \tau) \), \( 0 < \alpha < 1 \), \( \alpha - \tau \leq \frac{1}{2} \). By the invariance of \( \text{lip} \alpha \) and \( \| \cdot \| \), under translation, we may assume that \( (\alpha, \tau) = (\alpha', \tau) \). Let \( f \) be the function in \( \text{lip} \alpha \) that satisfies \( f(1) = 0 \), \( f(\alpha') = -\tau \), \( f(\alpha) = 0 \), and is linear in the intervals \( [1, \alpha'] \) and \( [\alpha', 1] \). Let \( x' \) be the point \( (\alpha', \tau) \) of \( W \). Then \( f(x) = f(x') = 1 \) if \( y \in W \), \( y \neq x \), \( y \neq x' \), and \( \|f(y) - f(x')\| \leq \delta(\alpha - \alpha) \) for all \( x \in W \), so \( f \) peaks at \( x \) relative to \( A \).

This completes the proof of Theorem 3.3.

§ 4. The isometries of \( \text{lip} \alpha \). Let \( q \) be a real number and \( \lambda \) a complex number with \( |\lambda| = 1 \). It is clear that the linear mappings \( U \) and \( V \) of \( \text{lip} \alpha \) onto itself defined by

\[ Uf(x) = \lambda f(x + q), \quad \sigma \in R, \]

and

\[ Vf(x) = \lambda f(x - q), \quad \sigma \in R, \]

satisfy

\[ \|Uf\| = \|f\|, \quad f \text{ clip}, \]

and

\[ \|Vf\| = \|f\|, \quad f \text{ clip}. \]

In this section (*) we establish the following results, which shows that \( \text{lip} \alpha \) has no further isometries:

**Theorem 4.1.** Let \( T \) be a linear isometry of \( \text{lip} \alpha \) onto itself. Then there is a real number \( q \) and a complex number \( \lambda \) with \( |\lambda| = 1 \) so that either

\[ Tf(x) = \lambda f(x + q), \quad \sigma \in R, \]

for all \( f \) in \( \text{lip} \alpha \), or

\[ Tf(x) = \lambda f(x - q), \quad \sigma \in R, \]

for all \( f \) in \( \text{lip} \alpha \).

The remainder of the section is devoted to the proof of this theorem.

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We shall denote by $\text{ext } S^*$ the set of extreme points of the unit sphere of $(\text{lip } \alpha)^*$. Since $T$ is a linear isometry of $\text{lip } \alpha$ onto $\text{lip } \alpha$, its adjoint $T^*$ is a linear isometry of $(\text{lip } \alpha)^*$ onto $(\text{lip } \alpha)^*$ and satisfies

\begin{equation}
T^*(\text{ext } S^*) = \text{ext } S^*.
\end{equation}

**Lemma 4.2.** Let $f$ be a function in $\text{lip } \alpha$. Then $f$ is a constant function if and only if

\begin{equation}
\{\Phi(f) : \Phi \in \text{ext } S^*\}
\end{equation}

consists of at most two numbers.

**Proof.** If $f$ is constant, that (4.2) has at most two elements is clear from Theorem 3.3. For the converse, suppose that (4.2) consists of at most two numbers. Since $f \in \text{lip } \alpha$, $0$ is in the closure of

\[ \left\{ \frac{f(\sigma) - f(\tau)}{|\sigma - \tau|^n} : \sigma, \tau \in \mathbb{R}, \sigma \neq \tau \right\}, \]

and thus by Theorem 3.3, 0 must be in (4.3). If there is no other element in (4.2), by Theorem 3.3 $f$ must be the zero function and we are finished. So we may assume that (4.2) is \{0, $g$\} where $g > 0$. Since $f \in \text{lip } \alpha$, there exists an $\varepsilon > 0$ so that

\[ \frac{|f(\sigma) - f(\tau)|}{|\sigma - \tau|^n} < \varepsilon \]

if $|\sigma - \tau| < \varepsilon$. But since (4.2) is \{0, $g$\}, because of Theorem 3.3, each number

\[ \frac{|f(\sigma) - f(\tau)|}{|\sigma - \tau|^n} \]

is equal to either 0 or $g$. Thus $f(\sigma) = f(\tau)$ if $|\sigma - \tau| < \varepsilon$ and $f$ is constant.

Recall that for $\sigma \in \mathbb{R}$, $\Phi_{\sigma}$ is the functional in $(\text{lip } \alpha)^*$ defined by

\[ \Phi_{\sigma}(f) = f(\sigma), \quad f \in \text{lip } \alpha. \]

**Corollary 4.3.** There is a complex number $\lambda$ with $|\lambda| = 1$ so that

\begin{equation}
T^*(\Phi_{\sigma}; \sigma \in \mathbb{R}) = \{\lambda \Phi_{\sigma} : \sigma \in \mathbb{R}\}.
\end{equation}

**Proof.** Let $g$ be the function in $\text{lip } \alpha$ satisfying

\[ g(\sigma) = 1, \quad \sigma \in \mathbb{R}. \]

By (4.1) and Lemma 4.2, $Tg$ is also a constant function. Suppose that

\[ Tg(\sigma) = \lambda, \quad \sigma \in \mathbb{R}. \]

Then, because of Theorem 3.3 and (4.1),

\[ T^*(\Phi_{\sigma}; \sigma \in \mathbb{R}) = T^*(\Phi_{\lambda}; \Phi_{\sigma} \in \text{ext } S^*, \Phi(Tg) = \lambda) = T^*(\Phi_{\lambda}; T^*\Phi_{\sigma} \in \text{ext } S^*, T^*\Phi(g) = \lambda) = \{\psi \in \text{ext } S^* : T^*\Phi_{\sigma} \psi = \lambda \Phi_{\sigma}, \sigma \in \mathbb{R}\}, \]

so (4.3) holds. Finally $|\lambda| = 1$ since $T$ is an isometry.

**Lemma 4.4.** If $\sigma, \tau \in \mathbb{R}$ and $|\sigma - \tau| \leq \frac{1}{2}$, then $|\Phi_{\sigma} - \Phi_{\tau}|^n \leq |\sigma - \tau|^n$, so it suffices to establish the reverse inequality. Assume first that $|\sigma - \tau| < \frac{1}{2}$. By the invariance of $\text{lip } \alpha$ and $\|f\|_n$, under translation, we may assume that $\sigma = 0$ and $0 < \tau < \frac{1}{2}$.

If $f$ is the function constructed in Case II of Theorem 3.3, $\|f\|_n = 1$ and $|\Phi_{\sigma}(f) - \Phi_{\tau}(f)| = |\sigma - \tau|^n$. As a consequence,

\begin{equation}
|\Phi_{\sigma} - \Phi_{\tau}|^n \geq |\sigma - \tau|^n
\end{equation}

when $|\sigma - \tau| < \frac{1}{2}$. A similar argument using the function constructed in Case III of Theorem 3.3 establishes the inequality (4.4) for $|\sigma - \tau| = \frac{1}{2}$.

One further lemma is required before we are able to complete the proof of Theorem 4.1. Let $\lambda$ be the complex number with $|\lambda| = 1$ satisfying (4.3). Then one can find a real number so that $T^*\Phi_{\lambda} = \lambda \Phi_{\lambda}$. Let $\sigma \in \mathbb{R}$ satisfy $|\sigma| < \frac{1}{2}$. By the choice of $\lambda$, there is some $\tau \in \mathbb{R}$ with

\begin{equation}
T^*\Phi_{\lambda} = \Phi_{\tau},
\end{equation}

and thus a unique $\tau \in \mathbb{R}$ satisfying (4.5) and in addition $0 < \frac{1}{2} < \tau < \frac{1}{2}$. This unique $\tau$ will be denoted by $t(\sigma)$. We have thus defined a mapping

\begin{equation}
t : \{\sigma : -\frac{1}{2} < \sigma < +\frac{1}{2}\} \to \mathbb{R}.
\end{equation}

**Lemma 4.5.** The mapping $t$ satisfies either

\begin{equation}
t(\sigma) = \sigma + \sigma, \quad -\frac{1}{2} < \sigma < +\frac{1}{2},
\end{equation}

or

\begin{equation}
t(\sigma) = \sigma - \sigma, \quad -\frac{1}{2} < \sigma < +\frac{1}{2}.
\end{equation}

**Proof.** Let $\sigma$ satisfy $|\sigma| < \frac{1}{2}$. Then $|\sigma| - |\sigma| = |\sigma|$, so by Lemma 4.4,

\[ |t(\sigma) - \sigma| = |\Phi_{\sigma} - \Phi_{\sigma}|^n = |\Phi_{\sigma} - \Phi_{\tau}|^n = |\Phi_{\sigma} - \Phi_{\tau}|^n = |\sigma|^n. \]

Thus

\begin{equation}
|\sigma - \sigma| = |\sigma|, \quad -\frac{1}{2} < \sigma < +\frac{1}{2}.
\end{equation}

Furthermore, the mapping $t$ is continuous. For if $g$ is the function in $\text{lip } \alpha$ defined by

\[ g(\tau) = e^{i\pi \tau}, \quad \tau \in \mathbb{R}, \]

then, because of Theorem 3.3 and (4.1),

\[ T^*(\Phi_{\sigma}; \sigma \in \mathbb{R}) = T^*(\Phi_{\lambda}; T^*\Phi_{\sigma} \in \text{ext } S^*, \Phi(Tg) = \lambda) = T^*(\Phi_{\lambda}; T^*\Phi_{\sigma} \in \text{ext } S^*, T^*\Phi(g) = \lambda) = \{\psi \in \text{ext } S^* : T^*\Phi_{\sigma} \psi = \lambda \Phi_{\sigma}, \sigma \in \mathbb{R}\}, \]

so (4.3) holds. Finally $|\lambda| = 1$ since $T$ is an isometry.
then $Tg$ is continuous and

$$\delta^{a(t)} = g(t) = \Phi_{t}^{}(g)$$

$$= \lambda^{-1}(T^*\Phi_{t}) g = \lambda^{-1}Tg = -\frac{1}{2} < \sigma < +\frac{1}{2}.$$  

It is now clear that $t$ must satisfy either (4.6) or (4.7) since it is one-one continuous and satisfies (4.8).

We are now able to complete the proof of Theorem 4.1. Suppose that the mapping $t$ satisfies (4.6). Then if $f$ is any function in $\text{lip}_a$,  

$$Tf = \Phi_{t}(Tf) = (T^*\Phi_{t}) (f)$$

$$= 2\Phi_{t} (f) = 2f = f + \sigma, \quad -\frac{1}{2} < \sigma < +\frac{1}{2},$$

and as a consequence, 

$$Tf = f + \sigma, \quad \sigma \in \mathbb{R},$$

for all $f$ in $\text{lip}_a$.

Similarly, if the mapping $t$ satisfies (4.7), then 

$$Tf = f - \sigma, \quad \sigma \in \mathbb{R},$$

for all $f$ in $\text{lip}_a$.

Bibliography


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A remark on an imbedding theorem of Kondrashov type

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1. The present note may be considered as the second part of Paper [1]. An approach developed there in order to obtain an elementary proof of complete continuity of the imbedding of the space $W^s_k(\Omega)$ in $C(\Omega)$ for $s$ large enough (see the definition below) is applied here to study the similar property of the imbedding of $W^s_k(\Omega)$ into the space of functions integrable to the power $p$ over a sufficiently smooth variety contained in $\Omega$, and of a dimension smaller than that of $\Omega$. An elementary proof of the Kondrashov theorem is obtained under conditions imposed on the variety under consideration, which differ from the original ones as presented in [4]. To prove the continuity of the imbedding mentioned, it is natural to impose the geometric conditions I invented by Ehrich; for its complete continuity, the more stringent conditions II seem to be necessary.

Several papers have been published recently in connection with simplifications of imbedding theorems (cf. for references [2]).

In what follows $\Omega$ will denote a fixed bounded domain in $N$-dimensional Euclidean space with points $x, y, \ldots$ and corresponding volume elements $da_x, da_y, \ldots$; $C(\Omega)$ will denote the space of functions continuous on $\Omega$, $C^\infty(\Omega)$ the space of functions with continuous derivatives of all orders on $\Omega$. In $C^\infty(\Omega)$ we introduce the norm

$$||f||_m = \left( \sum_{\Omega} |D^\alpha f|^2 \right)^{1/2}, \quad p > 1,$$

where the summation is extended over all derivatives of $f$ of order not larger than

$$m \left\{ D^\alpha f = \frac{\partial^\alpha f}{\partial x_1^{a_1} \ldots \partial x_N^{a_N}}, \quad |\alpha| = a_1 + \ldots + a_N \right\}.$$ 

By completion of $C^\infty(\Omega)$ in the norm $|| ||_m$ we obtain a Banach space $W^s_k(\Omega)$ of all functions of $L^p(\Omega)$ whose generalised derivatives up to order $m$ all belong to $L^p(\Omega)$. In the occurrence of other norms, we shall indicate