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A note on the paper of I. Singer "Basic sequences and reflexivity of Banach spaces"

by

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I. Singer posed in [3] the following question:

Is a Banach space reflexive if its every subspace with a basis is reflexive?

In this note we shall show that the answer to this question is "yes". A similar result was announced without proof by James in [4]. Further we shall give a few characteristics of reflexivity in terms of the behaviour of basic sequences generalizing the analogous results in [3]. Our main result is based on the proposition concerning the choice of basic sequences. This proposition was first proved by C. Bessaga in his thesis. The idea of his proof and our proof is due to Professor Mazur.

Throughout this paper we preserve the terminology and notation of [3]. We recall that a set G of a space Y^* conjugate to a B-space Y is said to be *norming* if $\sup_{g \in S_G} |g(y)| = \|y\|$ for every y in Y, where $S_G = \{g \in G\colon \|g\| = 1\}$.

PROPOSITION. *Let G be a norming set in Y^* and let (y_n) be a sequence in a B-space Y such that*

$$(1) \qquad 0 < \inf_n \|y_n\| \leqslant \sup_n \|y_n\| < +\infty,$$

$$(2) \qquad \lim_n g(y_n) = 0 \quad \textit{for any } g \textit{ in } G.$$

Then there is a subsequence (y_{n_k}) with $y_{n_1} = y_1$ which is a basic sequence in Y.

We shall need the following

LEMMA. *Let (y_n) and G have the same meaning and properties as in the Proposition. Then for every $\varepsilon > 0$, $N > 0$, and for every finite dimensional subspace E of Y there is an index $n > N$ such that*

$$(3) \qquad \|e + ty_n\| \geqslant (1-\varepsilon)\|e\| \quad \textit{for any } e \textit{ in } E \textit{ and every scalar } t.$$

Proof of the lemma. Without loss of generality we may replace condition (1) by

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(1′) $$\|y_n\| = 1 \quad (n = 1, 2, \ldots).$$

Let us put $S_E = \{e \in E : \|e\| = 1\}$. Since the subspace E is of finite dimension, S_E is compact. Hence there are elements $e_1, e_2, \ldots, e_r$ in S_E being an $\varepsilon/3$-net for S_E. Since G is a norming set, there are functionals $g_1, g_2, \ldots, g_r$ such that

(4) $$|g_i(e_i)| > 1 - \varepsilon/3 \quad \text{and} \quad \|g_i\| = 1 \quad (i = 1, 2, \ldots, r).$$

By (2) there is an index $n > N$ such that $|g_i(y_n)| < \varepsilon/6$ for $i = 1, 2, \ldots, r$. We shall show that for such n inequality (3) holds.

By the homogeneity of the norm it is sufficient to restrict our attention to the case where $\|e\| = 1$. We consider two cases

1° $|t| \geqslant 2$. Then $\|ty_n + e\| \geqslant |t|\,\|y_n\| - \|e\| \geqslant 2 - 1 = 1 \geqslant (1-\varepsilon)\|e\|$.

2° $|t| < 2$. Then choosing i such that $\|e - e_i\| < \varepsilon/3$, we have $\|ty_n + e\| \geqslant |g_i(ty_n + e)| \geqslant |g_i(e_i)| - |g_i(ty_n)| - \|g_i\|\,\|e - e_i\| \geqslant 1 - \varepsilon/3 - \frac{2}{6}\varepsilon - \varepsilon/3 = 1 - \varepsilon = (1-\varepsilon)\|e\|$.

Proof of the Proposition. Let (ε_ν) be a sequence of positive numbers such that

$$0 < \varepsilon_\nu < 1 \quad \text{and} \quad \inf_{p > q} \prod_q^p (1 - \varepsilon_\nu) = \delta > 0.$$

We shall define the sequence (y_{n_k}) by induction. Let us put $n_1 = 1$. Suppose that we have chosen $n_1 < n_2 < \ldots < n_k$ $(k > 1)$ in such a way that

(5) $$(1-\varepsilon_r)\|t_1 y_{n_1} + t_2 y_{n_2} + \ldots + t_{r-1} y_{n_{r-1}}\| \leqslant \|t_1 y_{n_1} + t_2 y_{n_2} + \ldots + t_r y_{n_r}\|$$

for any scalars $t_1, t_2, \ldots, t_k$ and for every $1 < r \leqslant k$.

Now we apply the Lemma to the case where $N = n_k$, $\varepsilon = \varepsilon_{k+1}$ and E is the subspace spanned on the elements $y_{n_1}, y_{n_2}, \ldots, y_{n_k}$. We define $n_{k+1} > n_k = N$ as an index satisfying inequality (3) of the Lemma.

It is easily seen that the sequence (y_{n_k}) defined in this way satisfies inequality (5) for $r = 2, 3, \ldots$ Hence

$$\|t_1 y_{n_1} + t_2 y_{n_2} + \ldots + t_p y_{n_p}\| \geqslant \prod_{\nu=q+1}^{p} (1-\varepsilon_\nu) \cdot \|t_1 y_{n_1} + t_2 y_{n_2} + \ldots + t_q y_{n_q}\|$$
$$\geqslant \delta \|t_1 y_{n_1} + t_2 y_{n_2} + \ldots + t_q y_{n_q}\|,$$

for arbitrary scalars $t_1, t_2, \ldots, t_p$ and $q < p$ $(p = 2, 3, \ldots)$. Thus according to [1], p. 111, (y_{n_k}) is a basic sequence.

THEOREM 1. *Let X be a non-reflexive Banach space. Then there is a (non-reflexive) subspace of X with a non-shrinking basis.*

Proof. Since X is non-reflexive, the unit ball of X is not weakly compact (cf. [2], p. 56). We consider two cases

1° There is a weak Cauchy sequence (x_n) in X which has no weak limit in X. Let us put $x_1^{**}(x^*) = \lim_n x^*(x_n)$ for any x^* in X^* and set $x_n^{**} = x_1^{**} - x_{n-1}$ $(n = 2, 3, \ldots)$. Obviously x_1^{**} does not belong to X, because otherwise it would be the weak limit of the sequence (x_n).

Now we apply the Proposition to the case where $Y = X^{**}$, $G = X^*$ and $(y_n) = (x_n^{**})$. Let $(x_{n_k}^{**})$ be a basic sequence and let $x_{n_1}^{**} = x_1^{**}$. Let Z, Z_1, Z_2 denote the subspaces of X^{**} spanned on the sequences $(x_{n_k}^{**})$, (x_{n_k}) and $(x_{n_{k+1}}^{**})$ respectively. Obviously $Z_1 \subset X$ (treated as a subspace of X^{**}). The codimension of Z_2 with respect to Z is obviously equal to one. The codimension of Z_1 in Z is also one, because Z is the smallest linear manifold containing Z_1 and x_1^{**}, and $Z_1 \neq Z$ $(x_1^{**} \notin Z_1, x_1^{**} \in Z)$.

Now we shall prove that the spaces Z_1 and Z_2 are isomorphic. Let $Z_3 = Z_1 \cap Z_2$. Since $Z_1 \neq Z_2$, the codimensions of Z_3 with respect to Z_1 and with respect to Z_2 are equal to one. Thus the space Z_1 as well as the space Z_2 is isomorphic to the Cartesian product of Z_3 by a one-dimensional Banach space; therefore Z_1 is isomorphic to Z_2.

Since the sequence (x_{n_k}) has no weak limit in X, there is an x^{***} in X^{***} such that $\limsup_k |x^{***}(x_{n_k}^{**})| > 0$. Indeed, if $\lim_k x^{***}(x_{n_k}^{**}) = 0$ for any x^{***} in X^{***}, then x_1^{**} would be (in X^{**}) the weak limit of a sequence (x_{n_k}). Hence, by [1], p. 134, x_1^{**} would belong to $Z_1 \subset X$, which contradicts our assumption.

Hence the basic sequences $(x_{n_k}^{**})$ and $(x_{n_{k+1}}^{**})$ are not shrinking. Thus Z_2 is a space with a non-shrinking basis. Since the possessing of a non-shrinking basis is an isomorphic invariant, Z_1 is a subspace of X with a non-shrinking basis.

2° There is a sequence (x_n) in the unit ball of X no subsequence of which is a weak Cauchy sequence. Without loss of generality we may assume that X is separable. Under this assumption, by [1], p. 124, there is in X^* a countable norming set $G = (g_m)$. Using the diagonal procedure, we may choose a subsequence (x_n') of (x_n) such that there exists $\lim_n g_m(x_n')$ for $m = 1, 2, \ldots$ Since the subsequence (x_n') is not a weak Cauchy sequence, there are x^* in X^*, $\delta > 0$ and increasing sequences of indices (p_k) and (q_k) such that $x^*(x_{p_k}' - x_{q_k}') > \delta > 0$ $(k = 1, 2, \ldots)$. Consider the sequence $(y_n) = (x_{p_n}' - x_{q_n}')$. It is easy to establish that this sequence fulfils the assumptions of the Proposition. Thus there is a subsequence (y_{n_k}) which is a basic sequence. Since $x^*(y_{n_k}) > \delta$ $(k = 1, 2, \ldots)$, this basic sequence is non-shrinking.

Remark. It immediately follows from the analysis of the preceding proof that

The unit ball of a Banach space X is conditionally weakly compact if and only if the unit ball of every subspace of X with a basis is conditionally weakly compact.

A Banach space X is weakly complete if and only if every subspace of X with a basis is weakly complete.

THEOREM 2. *Let X be a Banach space. Then the following conditions are equivalent:*

(i) *X is reflexive,*
(ii) *every subspace of X with a basis is reflexive,*
(iii) *every bounded basic sequence in X weakly converges to* 0,
(iv) *every basic sequence in X is shrinking,*
(v) *every basic sequence in X is boundedly complete.*

Proof. (i) $\Rightarrow$ (ii). This is a trivial consequence of the general fact that every subspace of a reflexive space is reflexive (see e. g. [2], p. 56).

(ii) $\Rightarrow$ (iii). Let (x_n) be a bounded basic sequence in X. By (ii) the space $[x_n]$ spanned on the sequence (x_n) is reflexive. Hence, according to a result of James [2], p. 71, the basis (x_n) is shrinking. Thus $\lim_n g(x_n) = 0$ for any g in $[x_n]^*$.

(iii) $\Rightarrow$ (iv). Suppose that (iv) does not hold. Then there is a basic sequence (e_n) which is not shrinking. Thus there are a functional f in $[e_n]^*$ and a sequence (x_n) such that

(6) $$x_n = \sum_{\nu=p_n}^{q_n} t_n^{\nu} e_{\nu}, \quad \text{where} \quad p_1 < q_1 < p_2 < q_2 < \ldots \quad (n = 1, 2, \ldots),$$

(7) $$\|x_n\| = 1 \quad \text{and} \quad \limsup_n |f(x_n)| > 0.$$

It is well known that (x_n) is a bounded basic sequence (as a bounded block sequence of a basic sequence). But by (7) (x_n) does not weakly converge to 0, which contradicts (iii).

(iv) $\Rightarrow$ (i). This implication is an immediate consequence of Theorem 1.

(i) $\Rightarrow$ (v). This implication is an immediate consequence of a result of James [2], p. 71.

(v) $\Rightarrow$ (ii). This implication immediately follows from a result of Singer [3], p. 362.

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Anerkennung der Priorität

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Satz 1 aus unserer Arbeit [4] war teilweise früher bekannt. Er wurde nämlich für den Fall der Bohrschen fastperiodischen Funktionen auf der reellen Achse zuerst von Bochner in [1] bewiesen. Der Beweis läßt sich nicht direkt auf beliebige Gruppen oder auf allgemeinere Klassen fastperiodischer Funktionen übertragen. Weiter muß die Arbeit von Jerison und Rabson [5] zitiert werden, die uns leider entgangen ist und die fast alle Resultate aus [4] enthält, obwohl sie sich ihrem allgemeinen Inhalt nach nicht auf fastperiodische Funktionen bezieht. Satz 4.7 aus [5] umfaßt nämlich unsere Sätze 1 und 2 bis auf Formel (5), die nicht explizit angegeben wird.

Gelegentlich bemerken wir, daß der in [3] angegebene und dem zweiten von uns zugeschriebene Begriff von R-fastperiodischen Funktionen bereits von Doss in [2], wenn auch vermittels anderer Definition eingeführt und mit derselben Bezeichnung verwendet wurde.

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