A note on the paper of I. Singer "Basic sequences and reflexivity of Banach spaces"

by

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I. Singer posed in [3] the following question:
Is a Banach space reflexive if its every subspace with a basis is reflexive?

In this note we shall show that the answer to this question is "yes".
A similar result was announced without proof by James in [4]. Further we shall give a few characteristics of reflexivity in terms of the behaviour of basic sequences generalizing the analogous results in [3]. Our main result is based on the proposition concerning the choice of basic sequences. This proposition was first proved by C. Bessaga in his thesis. The idea of his proof and our proof is due to Professor Mazur.

Throughout this paper we preserve the terminology and notation of [3]. We recall that a set $G$ of a space $Y^*$ conjugate to a $B$-space $Y$ is said to be norming if $\sup_{g \in G} |g(y)| = \|y\|$ for every $y$ in $Y$, where

$$G = \{g \in G : \|g\| = 1\}.$$

**Proposition.** Let $G$ be a norming set in $Y^*$ and let $(y_n)$ be a sequence in a $B$-space $Y$ such that

1. $0 < \inf_{n} \|y_n\| = \sup_{n} |y_n| < +\infty$,
2. $\lim_{n} g(y_n) = 0$ for any $g$ in $G$.

Then there is a subsequence $(y_{n_i})$ with $y_{n_i} = y_1$ which is a basic sequence in $Y$.

We shall need the following

**Lemma.** Let $(y_n)$ and $G$ have the same meaning and properties as in the Proposition. Then for every $\varepsilon > 0$, $N > 0$, and for every finite dimensional subspace $E$ of $Y$ there is an index $n > N$ such that

$$\|y_n\| > (1 - \varepsilon)|e|$$

for any $e$ in $E$ and every scalar $t$.

**Proof** of the lemma. Without loss of generality we may replace condition (1) by
1 There is a weak Cauchy sequence \((a_n)\) in \(X\) which has no weak limit in \(X\). Let us put \(x^* = \lim_{n \to \infty} x^*(a_n)\) for any \(x^* \in X^*\) and set \(a_n^* = a_n - a_{n-1}\). Obviously \(x^*\) does not belong to \(X\), because otherwise it would be the weak limit of the sequence \((a_n)\).

Now we apply the Proposition to the case where \(Y = X^*\), \(G = X^*\), and \((y_n) = (a_n^*)\). Let \((x_n^*)\) be a basic sequence and let \(x_n^* = x_{k_n}^*\). Let \(Z, Z_1, Z_2\) denote the subspaces of \(X^*\) spanned on the sequences \((x_n^*)\), \((a_n)\), and \((x_n^*)\) respectively. Obviously \(Z_1 \subseteq X\) (treated as a subspace of \(X^*\)). The codimension of \(Z_2\) with respect to \(Z\) is obviously equal to one. The codimension of \(Z_1\) in \(Z_2\) is also one, because \(Z_1\) is the smallest linear manifold containing \(Z_2\), and \(x_n^*\) and \(Z_1 \neq Z_2\) span \(Z_2\).

Now we shall prove that the spaces \(Z_1\) and \(Z_2\) are isomorphic. Let \(Z_2 = Z_1 \oplus Z_2\). Since \(Z_1 \neq Z_2\), the codimensions of \(Z_2\) with respect to \(Z_1\) and \(Z_2\) are equal to one. Thus the space \(Z_2\) as well as the space \(Z_1\) is isomorphic to the Cartesian product of \(Z_2\) by a one-dimensional Banach space; therefore \(Z_1\) is isomorphic to \(Z_2\).

Since the sequence \((a_n)\) has no weak limit in \(X\), there is an \(a^*\) in \(X^*\) such that \(\limsup_{n \to \infty} |x^*(a_n^*)| = 0\). Indeed, if \(\lim_{n \to \infty} x^*(a_n) = 0\) for any \(x^* \in X^*\), then \(x^*\) would be (in \(X^*\)) the weak limit of a sequence \((a_n)\). Hence, by [1], p. 334, \(x^*\) would belong to \(X \subseteq X\), which contradicts our assumption.

Hence the basic sequences \((x_n^*)\) and \((x_n^*)\) are not shrinking. Thus \(Z_2\) is a space with a non-shrinking basis. Since the possessing of a non-shrinking basis is an isomorphic invariant, \(Z_2\) is a subspace of \(X\) with a non-shrinking basis.

2 There is a sequence \((a_n)\) in the unit ball of \(X\) no subsequence of which is a weak Cauchy sequence. Without loss of generality we may assume that \(X\) is separable. Under this assumption, by [1], p. 134, there is in \(X\) a countable norming set \(G = (y_n)\). Using the diagonal procedure, we may choose a subsequence \((x_n^*)\) of \((x_n^*)\) such that there exists \(\lim_{n \to \infty} x_n^*(y_n)\) for \(n = 1, 2, \ldots\). Since the subsequence \((x_n^*)\) is not a weak Cauchy sequence, there are \(x^* \in X^*\), \(\delta > 0\) and increasing sequences of indices \((p)\) and \((q)\) such that \(x^*(y_{p_n} - y_{q_n}) > \delta > 0\) for \(n = 1, 2, \ldots\). Consider the sequence \((y_n) = (y_{p_n} - y_{q_n})\). It is easy to establish that this sequence fulfills the assumptions of the Proposition. Thus there is a subsequence \((y_{p_n})\) which is a basic sequence. Since \(x^*(y_{p_n}) > \delta\), this basic sequence is non-shrinking.

Remark. It immediately follows from the analysis of the preceding proof that

The unit ball of a Banach space \(X\) is conditionally weakly compact if and only if the unit ball of every subspace of \(X\) with a basis is conditionally weakly compact.
A Banach space $X$ is weakly complete if and only if every subspace of $X$ with a basis is weakly complete.

**Theorem 2.** Let $X$ be a Banach space. Then the following conditions are equivalent:

(i) $X$ is reflexive,

(ii) every subspace of $X$ with a basis is reflexive,

(iii) every bounded basic sequence in $X$ weakly converges to 0,

(iv) every basic sequence in $X$ is shrinking,

(v) every basic sequence in $X$ is boundedly complete.

**Proof.** (i) $\Rightarrow$ (ii). This is a trivial consequence of the general fact that every subspace of a reflexive space is reflexive (see e.g. [2], p. 56).

(ii) $\Rightarrow$ (iii). Let $(x_n)$ be a bounded basic sequence in $X$. By (ii) the space $\{x_n\}$ spanned on the sequence $(x_n)$ is reflexive. Hence, according to a result of James [2], p. 71, the basis $(x_n)$ is shrinking. Thus $\lim g(x_n) = 0$ for any $g$ in $[x_n]^*$. 

(iii) $\Rightarrow$ (iv). Suppose that (iv) does not hold. Then there is a basic sequence $(x_n)$ which is not shrinking. Thus there are a functional $f$ in $[x_n]^*$ and a sequence $(x_n)$ such that

$$\sum_{n=1}^{\infty} f(x_n) = 0 \quad \text{and} \quad \limsup |f(x_n)| = 0.$$ 

(6) $x_n = \sum_{n=1}^{\infty} e_n$, where $p_1 < q_1 < p_2 < q_2 < \ldots \quad (n = 1, 2, \ldots)$,

(7) $\|x_n\| = 1$ and $\limsup |f(x_n)| > 0$.

It is well known that $(x_n)$ is a bounded basic sequence (as a bounded block sequence of a basic sequence). But by (7) $(x_n)$ does not weakly converge to 0, which contradicts (iii).

(iv) $\Rightarrow$ (i). This implication is an immediate consequence of Theorem 1.

(i) $\Rightarrow$ (v). This implication is an immediate consequence of a result of James [2], p. 71.

(v) $\Rightarrow$ (ii). This implication immediately follows from a result of Singer [3], p. 362.

**References**


**Anerkennung der Priorität.**

von

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**Zitierte Literatur**


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