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Basic sequences and reflexivity of Banach spaces

by

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Introduction

R. C. James has given the following characterization of reflexive Banach spaces ([8], theorem 1):

THEOREM (J). *A Banach space* [1] *E with a basis $\{x_n\}$ is reflexive if and only if*

(a) *For every sequence of scalars $\{a_n\}$ such that* $\sup_n \|\sum_{i=1}^{n} a_i x_i\| < +\infty$ *the series* $\sum_{i=1}^{\infty} a_i x_i$ *is convergent,*

(b) $\lim_{n\to\infty} \|f\|_n = 0$ *for all functionals $f \in E^*$, where $\|f\|_n$ denotes the norm of the restriction of f to the closed linear subspace of E spanned by $x_{n+1}, x_{n+2}, \ldots$*

In a recent paper [11], V. Pták has completed the picture of the structure of reflexive Banach spaces given by theorem (J), characterizing reflexivity in terms of bounded biorthogonal systems.

The purpose of the present paper is to continue these investigations of the structure of reflexive and non-reflexive Banach spaces by characterizing the reflexivity of a Banach space with a basis in terms of the behaviour of its basic sequences.

Since we are dealing with Banach spaces having a basis, we shall freely use in our proofs theorem (J).

Let us first recall briefly some definitions and notation, which will be used in the sequel.

If E is a Banach space, we shall denote by E^* its conjugate space.

If $\{z_n\}$ is a sequence of elements in a Banach space E, we shall denote by $[z_n]$, or sometimes by $[z_1, z_2, \ldots]$, the subspace of E spanned by the sequence $\{z_n\}$; by "subspace" we shall always mean "closed linear subspace".

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(1) Throughout this paper, by *Banach space* we shall mean infinite-dimensional Banach space.

A sequence $\{z_n\} \subset E$ is called a *basic sequence* [2] if $\{z_n\}$ is a basis of the subspace $[z_n]$.

Any sequence of the form

$$y_n = \sum_{i=p_{n-1}+1}^{p_n} a_i x_i, \qquad y_n \neq 0 \;\; (n = 1, 2, \ldots),$$

where $\{p_n\}$ is an increasing sequence of positive integers, $p_0 = 0$, and where $\{a_n\}$ is a sequence of scalars, is called a *block basis* (with respect to $\{x_n\}$) [2]; it is, necessarily [2], a basic sequence ([2]). We shall call *block subspace* of E (with respect to $\{x_n\}$) any subspace spanned by a block basis.

Let E be a Banach space with a basis $\{x_n\}$ (with a basic sequence $\{z_n\}$). We shall say that the basis $\{x_n\}$ (the basic sequence $\{z_n\}$) is *boundedly complete* [4] if it satisfies condition (a) of theorem (J). We shall say that the basis $\{x_n\}$ (the basic sequence $\{z_n\}$) is *shrinking* [4] if it satisfies condition (b) of theorem (J) (and, respectively, condition (b) for $\|f\|_n = \|f\|_{[z_{n+1}, z_{n+2}, \ldots]}$).

Theorem (J) asserts that a Banach space E which has a basis is reflexive if and only if one (and thus every) basis of E is both boundedly complete and shrinking.

In § 1 we shall introduce three types of basic sequences, l_+, P and P^*, and we shall give several characterizations of them, as well as a detailed analysis of the relations between all types of basic sequences considered in this paper (except the type P^0, which occurs only incidentally in a remark in § 2). Though not all of the results of § 1 will be applied in the sequel, we have given them because they might prove interesting for other applications.

In § 2 we shall give various characterizations of reflexivity in terms of basic sequences. E. g. the first of them (the equivalence $1^\circ \Longleftrightarrow 2^\circ$ of theorem 2) involves only boundedly complete basic sequences: since every subspace of a reflexive space is reflexive, (a) of theorem (J) is inherited by all basic sequences in E, and here we prove that this fact characterizes reflexivity. Moreover, we find that it is not necessary to consider all basic sequences in E, but it is sufficient to examine only the bases of block subspaces of E (with respect to a given basis of E). Some of the results of § 2 constitute an improvement, for Banach spaces with bases, of the results of V. Pták [11]; however, our methods differ completely from those of [11].

Finally, § 3 contains some remarks and unsolved problems. The first of these problems (P1 of § 3) has recently been solved by Pełczyński

([2]) Obviously, in general a block basis is not a basis of E; therefore it would be better to call it a *block basic sequence*, or shortly a *block*. However, we shall retain the terminology of [2].

[10]; however, we have left unchanged its formulation and our remarks concerning its solution.

The author wishes to express his gratitude to Dr. A. Pełczyński for reading the manuscript and making a number of valuable remarks.

§ 1. Basic sequences

1. Basic sequences of type l_+. We shall say that a basic sequence $\{z_n\}$ in a Banach space E is *of type* l_+ if $\sup_n \|z_n\| < +\infty$ and if there exists a constant $\eta > 0$ such that we have, for every finite sequence $t_1, \ldots, t_n \geqslant 0$,

$$\Big\| \sum_{i=1}^{n} t_i z_i \Big\| \geqslant \eta \sum_{i=1}^{n} t_i. \tag{1}$$

For any such sequence we must clearly have $\inf_n \|z_n\| \geqslant \eta$.

Proposition 1. *For a basic sequence $\{z_n\}$ with $\sup_n \|z_n\| < +\infty$, the following statements are equivalent:*

1° *$\{z_n\}$ is a basic sequence of type l_+.*

2° *There exists a constant $\eta > 0$ such that we have, for every sequence $t_n \geqslant 0$ $(n = 1, 2, \ldots)$ with $\sum_{i=1}^{\infty} t_i < +\infty$,*

$$\Big\| \sum_{i=1}^{\infty} t_i z_i \Big\| \geqslant \eta \sum_{i=1}^{\infty} t_i. \tag{2}$$

3° *There exists a functional $f \in E^*$ (or, which is equivalent, a functional $f \in [z_n]^*$) such that*

$$f(z_n) \geqslant 1, \quad n = 1, 2, \ldots \tag{3}$$

Proof. The implication $2^\circ \Rightarrow 1^\circ$ is trivial. Conversely, assume that we have 1°, and let $t_i \geqslant 0$ $(n = 1, 2, \ldots)$, $\sum_{i=1}^{\infty} t_i < +\infty$. Then $\sum_{i=1}^{\infty} t_i z_i$ is convergent and from (1) we infer

$$\Big\| \sum_{i=1}^{\infty} t_i z_i \Big\| = \lim_{n\to\infty} \Big\| \sum_{i=1}^{n} t_i z_i \Big\| \geqslant \lim_{n\to\infty} \eta \sum_{i=1}^{n} t_i = \eta \sum_{i=1}^{\infty} t_i,$$

i. e. 2°. Finally, the equivalence $1^\circ \Longleftrightarrow 3^\circ$ is a consequence of a theorem of S. Mazur and W. Orlicz [9].

Remark 1. The equivalence $1^\circ \Longleftrightarrow 2^\circ$ above justifies the term "of type l_+". Moreover, we also have the following:

If $\{z_n\}$ is an *unconditional* basic sequence of type l_+, then $\{z_n\}$ is equivalent to the natural basis of the space l (and thus $[z_n]$ is isomorphic to l).

In fact, if $\sum_{i=1}^{\infty} t_i z_i$ is an arbitrary series such that $\sum_{i=1}^{\infty} |t_i| < +\infty$, then, applying (1) separately for the subseries with all positive t_i's and the subseries with all negative t_i's and then applying the well-known characterization of unconditional bases (see [8], formula (2)), we obtain

$$\sup_n \|z_n\| \sum_{i=1}^{\infty} |t_i| \geqslant \Big\| \sum_{i=1}^{\infty} t_i z_i \Big\| \geqslant C \sum_{i=1}^{\infty} |t_i|,$$

where the constant $C > 0$ depends only on the basic sequence $\{z_n\}$.

2. Basic sequences of types P and P^*. We shall say that a basic sequence $\{y_n\}$ in a Banach space E is *of type P* if $\inf_n \|y_n\| > 0$ and if

$$\sup_n \Big\| \sum_{i=1}^{n} y_i \Big\| < +\infty. \tag{4}$$

For any such sequence we must clearly have $\sup_n \|y_n\| < +\infty$.

We shall say that a basic sequence $\{z_n\}$ in a Banach space E is *of type P^** if $\sup_n \|z_n\| < +\infty$ and if

$$\sup_n \Big\| \sum_{j=1}^{n} h_j \Big\| < +\infty, \tag{5}$$

where $\{h_n\} \subset [z_n]^*$ is the sequence of functionals biorthogonal to $\{z_n\}$.

PROPOSITION 2. *For a basic sequence $\{y_n\}$ with $\inf_n \|y_n\| > 0$, the following statements are equivalent:*

1° *$\{y_n\}$ is a basic sequence of type P.*

2° *The sequence of functionals $\{g_n\} \subset [y_n]^*$ biorthogonal to $\{y_n\}$ is a basic sequence of type P^*.*

3° *$\sup_n \|y_n\| < +\infty$ and the sequence $\{z_n\}$ defined by*

$$z_n = \sum_{i=1}^{n} y_i, \quad n = 1, 2, \ldots, \tag{6}$$

is a basis of $[y_n] = [z_n]$.

4° *There exists a constant $M > 0$ such that we have, for every finite sequence of scalars $t_1, \ldots, t_n$,*

$$\Big| \sum_{i=1}^{n} t_i \Big| \leqslant M \Big\| \sum_{i=1}^{n} t_i g_i \Big\|, \tag{7}$$

where $\{g_n\} \subset [y_n]^*$ denotes again the sequence of functionals biorthogonal to $\{y_n\}$.

5° *There exists a functional $\Phi \in [y_n]^{**}$ (or, which is equivalent, a functional $\Phi \in [g_n]^*$) such that*

$$\Phi(g_n) = 1, \quad n = 1, 2, \ldots \tag{8}$$

6° *The sequence $\{g_1 - g_2, g_2 - g_3, \ldots\}$ is not fundamental* (³) *in $[g_n]$.*

7° *There exists a constant $M > 0$ such that for every monotonic sequence $\{a_i\}$ tending to zero, the sum $\sum_{i=1}^{\infty} a_i y_i$ exists and satisfies*

$$\Big\| \sum_{i=1}^{\infty} a_i y_i \Big\| \leqslant M |a_1|. \tag{9}$$

Proof. Since $\inf_n \|y_n\| > 0$ and since $\{y_n\}$ is a basic sequence, we have $\sup_n \|g_n\| < +\infty$.

Assume now that we have 1°. Then, since the sequence $\{\psi_n\} \subset [g_n]^*$ biorthogonal to $\{g_n\}$ is nothing else but the restriction to $[g_n]$ of the canonical image of $\{y_n\}$ in $[y_n]^{**}$, we have

$$\sup_n \Big\| \sum_{i=1}^{n} \psi_i \Big\| \leqslant \sup_n \Big\| \sum_{i=1}^{n} y_i \Big\| < +\infty,$$

i. e. we have 2°.

Assume now that we have 2°, i. e. that there exists a constant $M > 0$ such that

$$\Big\| \sum_{j=1}^{n} \psi_j \Big\| \leqslant M, \quad n = 1, 2, \ldots,$$

where $\{\psi_n\}$ denotes the sequence of functionals biorthogonal to $\{g_n\}$. Then we have, for every finite sequence $t_1, \ldots, t_n$ of scalars,

$$\Big| \sum_{i=1}^{n} t_i \Big| = \Big| \Big(\sum_{j=1}^{n} \psi_j \Big) \Big(\sum_{i=1}^{n} t_i g_i \Big) \Big| \leqslant M \Big\| \sum_{i=1}^{n} t_i g_i \Big\|,$$

i. e. we have 4°.

The implication 4° ⇒ 5° is a particular case of a classical theorem of E. Helly ([1], p. 55, theorem 4).

On the other hand, if we have 5°, then by [12], lemma 1,

$$\sup_n \Big\| \sum_{i=1}^{n} y_i \Big\| = \sup_n \Big\| \sum_{i=1}^{n} \Phi(g_i) y_i \Big\| < +\infty,$$

i. e. we have 1°. Thus we have proved the equivalence of 1°, 2°, 4° and 5°.

The equivalence 1° ⇔ 3° is a consequence of a theorem of B. R. Gelbaum ([7], § 4, theorem 1) (⁴).

(³) We use the term "fundamental" in the sense of [1], p. 58.

(⁴) In the formulation of [7] it is assumed that $\|y_n\| = 1$ ($n = 1, 2, \ldots$), but it is easy to see that for the proof of the implication 1° ⇒ 3° we need only $\inf_n \|y_n\| > 0$ while for the proof of 3° ⇒ 1° we need only $\sup_n \|y_n\| < +\infty$.

The equivalence $5^\circ \Longleftrightarrow 6^\circ$ is an immediate consequence of a theorem of S. Banach ([1], p. 58, theorem 7).

The implication $7^\circ \Rightarrow 1^\circ$ follows by applying (9) successively to the sequences $\{a_j^{(n)}\}$ $(n = 1, 2, \ldots)$ defined by

$$a_1^{(n)} = \ldots = a_n^{(n)} = 1, \quad a_{n+1}^{(n)} = a_{n+2}^{(n)} = \ldots = 0.$$

Finally, assume that we have 1°, and let $a_1 \geqslant a_2 \geqslant \ldots$ be a sequence tending to zero. Using sequence (6), we have

$$\sum_{i=1}^{n} a_i y_i = a_1 z_1 + \sum_{i=2}^{n} a_i (z_i - z_{i-1}) = \sum_{i=1}^{n-1} (a_i - a_{i+1}) z_i + a_n z_n.$$

Since $\{y_n\}$ is of type P, we have $\sup_n \|z_n\| \leqslant M$, whence, since $a_n \to 0$, we obtain $a_n z_n \to 0$. On the other hand, from $\|z_i\| \leqslant M$, $a_i \geqslant a_{i+1}$ $(i = 1, 2, \ldots)$ and $\sum_{i=1}^{\infty} (a_i - a_{i+1}) = a_1$ it follows that $\sum_{i=1}^{\infty} (a_i - a_{i+1}) z_i$ exists and satisfies $\left\| \sum_{i=1}^{\infty} (a_i - a_{i+1}) z_i \right\| \leqslant M |a_1|$. Consequently, we have 7°, which completes the proof.

PROPOSITION 3. *For a basic sequence $\{z_n\}$ with $\sup_n \|z_n\| < +\infty$, the following statements are equivalent:*

1° *$\{z_n\}$ is a basic sequence of type P^*.*

2° *The sequence of functionals $\{h_n\} \subset [z_n]^*$ biorthogonal to $\{z_n\}$ is a basic sequence of type P.*

3° *The sequence $\{v_n\}$ defined by*

(10) $$v_1 = z_1, \quad v_n = z_{n-1} - z_n, \quad n = 1, 2, \ldots,$$

is a basis of $[z_n] = [v_n]$.

4° *There exists a constant $L > 0$ such that we have, for every finite sequence of scalars $t_1, \ldots, t_n$,*

(11) $$\left| \sum_{i=1}^{n} t_i \right| \leqslant L \left\| \sum_{i=1}^{n} t_i z_i \right\|.$$

5° *There exists a functional $h \in E^*$ (or, which is equivalent, a functional $h \in [z_n]^*$) such that*

(12) $$h(z_n) = 1, \quad n = 1, 2, \ldots$$

6° *The subsequence $\{v_2, v_3, \ldots\}$ of (10) is not fundamental in $[z_n]$.*

7° *There exists a constant $L > 0$ such that, for every monotonic sequence $\{a_i\}$ tending to zero, the sum $\sum_{i=1}^{\infty} a_i h_i$ exists and satisfies*

(13) $$\left\| \sum_{i=1}^{\infty} a_i h_i \right\| \leqslant L |a_1|.$$

8° *There exists a sequence of functionals $\{\chi_n\} \subset E^*$ biorthogonal to $\{z_n\}$ such that*

(14) $$\sup_n \left\| \sum_{i=1}^{n} \chi_i \right\| < +\infty.$$

Proof. Since $\sup_n \|z_n\| = M < +\infty$, we have

$$\|h_n\| \geqslant \frac{|h_n(z_n)|}{\|z_n\|} = \frac{1}{\|z_n\|} \geqslant \frac{1}{M}, \quad n = 1, 2, \ldots,$$

whence $\inf_n \|h_n\| > 0$. This, together with (5), shows that $1^\circ \Rightarrow 2^\circ$.

Assume now that we have 2° and denote by φ_n the restriction of h_n $(n = 1, 2, \ldots)$ to the subspace $[v_2, v_3, \ldots]$ of $[z_n]$. Then $\{\varphi_n\}$ is a basic sequence. By biorthogonality we have $\|h_n\| = \|\varphi_n\|$ for $n = 2, 3, \ldots$, whence, by $\inf_n \|h_n\| > 0$ we infer $\inf_n \|\varphi_n\| > 0$. Furthermore, by (5) we have $\sup_n \left\| \sum_{i=1}^{n} \varphi_i \right\| < +\infty$. Hence, by the implication $1^\circ \Rightarrow 3^\circ$ of proposition 2, the sequence $\{\varphi_1 + \ldots + \varphi_n\}$ is a basis of $[\varphi_n] = [\varphi_1 + \ldots + \varphi_n]$. Since $\{\varphi_1 + \ldots + \varphi_n\}$ is the sequence of functionals biorthogonal to $\{v_2, v_3, \ldots\}$, it follows (see e. g. [1], p. 107, theorem 2) that $\{v_2, v_3, \ldots\}$ is a basic sequence. Since obviously $[v_1, v_2, v_3, \ldots] = [z_n]$, and since $v_1 \notin [v_2, v_3, \ldots]$ (because of $z_1 \notin [z_2, z_3, \ldots]$), it follows that $\{v_1, v_2, v_3, \ldots\}$ is a basis of $[z_n]$. Thus 2° implies 3°.

Assume that we have 3°, and let

$$g_n = \varphi_1 + \ldots + \varphi_n, \quad n = 1, 2, \ldots,$$

where $\{\varphi_n\}$ is as before. Then $\{g_n\}$ is biorthogonal to the basis $\{v_2, v_3, \ldots\}$ of $[v_2, v_3, \ldots]$. Consequently, in order to prove that $\sup_n \|g_n\| < +\infty$, it will be sufficient to prove that $\inf_{2 \leqslant n < +\infty} \|v_n\| > 0$. However, since $\{v_n\}$ is a basic sequence, there exists a constant $K_1 \geqslant 1$ (cf. [2], p. 152) such that

$$\|z_1\| = \|v_1\| \leqslant K_1 \left\| v_1 - \sum_{i=2}^{n} v_i \right\| = K_1 \|z_n\|, \quad n = 2, 3, \ldots,$$

whence $\inf_n \|z_n\| > 0$. On the other hand, since $\{z_n\}$ is a basic sequence, there exists a constant $K_2 \geqslant 1$ such that

$$\|z_{n-1}\| \leqslant K_2 \|z_{n-1} - z_n\| = K_2 \|v_n\|, \quad n = 2, 3, \ldots$$

This, together with $\inf_n \|z_n\| > 0$, proves that $\inf_{2 \leqslant n < +\infty} \|v_n\| > 0$, and hence that $\sup_n \|g_n\| < +\infty$, i. e. that $\sup_n \left\| \sum_{i=1}^{n} \varphi_i \right\| < +\infty$. Since, by biorthogonality, $\|\sum_{i=2}^{n} h_i\| = \|\sum_{i=2}^{n}\varphi_i\|$ for $n=2,3,\ldots$, it follows that $\sup_n \|\sum_{i=1}^{n} h_i\| < +\infty$, i. e. that we have 1°, which completes the proof of the equivalence of 1°, 2° and 3°.

Let us now prove the equivalence 1° $\Longleftrightarrow$ 8°. If we have 1°, then we can take $\chi_1 =$ an extension (to E), with norm $\|h_1\|$, of h_1, $\chi_2 = (\chi_1 + \chi_2) - \chi_1$, where $\chi_1 + \chi_2 =$ an extension with norm $\|h_1 + h_2\|$ of $h_1 + h_2$, and so on.

Conversely, if we have 8°, then we have (5) by virtue of $\sum_{j=1}^{n} h_j = \sum_{j=1}^{n} (\chi_j|_{[z_m]}) = (\sum_{j=1}^{n}\chi_j)|_{[z_m]}$.

The proof of the other equivalences of proposition 3 follows the same lines as that of the corresponding equivalences of proposition 2 (applying [12], lemma 2, instead of [12], lemma 1, in the proof of the implication 5° $\Rightarrow$ 1°).

Remark 2. 1° If $\{z_n\}$ is an *unconditional* basic sequence of type P^*, then $\{z_n\}$ is equivalent to the natural basis of the space l (and thus $[z_n]$ is isomorphic to l).

2° If $\{z_n\}$ is an *unconditional* basic sequence of type P, then $\{z_n\}$ is equivalent to the natural basis of the space c_0 (and thus $[z_n]$ is isomorphic to c_0).

In fact, 1° is a consequence of theorem 1, 4° below and of remark 1, while 2° can be deduced from an attentive examination of the proof of [8], lemma 1.

3. Relations between various types of basic sequences. We have

THEOREM 1. 1° *Every basic sequence of type P is non-boundedly complete. The converse implication is not valid.*

2° *Every non-boundedly complete basic sequence $\{x_n\}$ admits a block basis $\{y_n\}$ of type P.*

3° *Every basic sequence of type P is of type non-l_+. The converse implication is not valid.*

4° *Every basic sequence of type P^* is of type l_+. The converse implication is not valid.*

5° *Every basic sequence of type l_+ (and hence every basic sequence of type P^*) is non-shrinking. The converse implication is not valid.*

6° *Every non-shrinking basic sequence $\{y_n\}$ admits a block basis $\{z_n\}$ of type l_+.*

7° *Every basic sequence $\{z_n\}$ of type l_+ admits a "contraction" $\{w_n\}$ of type P^*.*

8° *A basic sequence cannot be simultaneously of types P and P^*.*

9° *A subspace Z of a Banach space E has a basis $\{y_n\}$ of type P if and only if it has a basis $\{z_n\}$ of type P^*.*

10° *Between the ten types of basic sequences: boundedly complete, shrinking, l_+, P, P^*, non-boundedly complete, non-shrinking, non-l_+, non-P and non-P^*, we have 22 relations of implication (the 10 trivial ones plus the 12 given by 1°, 3°, 4°, 5° and 8° above). All the other 78 relations between them are: non-implication.*

Proof. 1° If a basic sequence $\{y_n\}$ of type P were boundedly complete, then by (4) and the definition of bounded completeness the series $\sum_{i=1}^{\infty} y_i$ would be convergent, which contradicts the property $\inf_n \|y_n\| > 0$ of basic sequences of type P.

On the other hand, it is easy to verify that the basis

$$z_n = \{\underbrace{1, 1, \ldots, 1}_{n}, 0, 0, \ldots\} \quad (n = 1, 2, \ldots) \tag{15}$$

of c_0, the Schauder basis of $C([0,1])$, and the Haar basis of the space $L^1([0,1])$ are neither of type P nor boundedly complete.

2° Let $\{x_n\}$ be a non-boundedly complete basic sequence in E. Let $\{a_n\}$ be a sequence of scalars such that

$$\Big\|\sum_{i=1}^{n} a_i x_i\Big\| \leqslant M, \quad n = 1, 2, \ldots, \tag{16}$$

and that $\sum_{i=1}^{\infty} a_i x_i$ is not convergent. Then there exists an increasing sequence of positive integers $\{q_n\}$ such that

$$\inf_n \Big\|\sum_{i=q_{n-1}+1}^{q_n} a_i x_i\Big\| = K > 0 \quad (q_0 = 0). \tag{17}$$

Let

$$y_n = \sum_{i=q_{n-1}+1}^{q_n} a_i x_i, \quad n = 1, 2, \ldots \tag{18}$$

Then, by (18) and (17), $\{y_n\}$ is a block basic sequence which is, by (17) and (16), of type P.

3° If a basic sequence $\{y_n\}$ of type P were of type l_+, then one would have, by (1) and (4),

$$\sup_n \eta n \leqslant \sup_n \Big\|\sum_{i=1}^{n} y_i\Big\| < +\infty,$$

which is an absurdity.

On the other hand, the Schauder basis of the space $C([0,1])$ and the Haar basis of the space $L^1([0,1])$ are neither of type P nor of type l_+.

4° The first assertion can be derived either from propositions 2 and 3 or directly from the definitions, since for every finite sequence $t_1, \ldots, t_n \geqslant 0$ we have

$$\Big\|\sum_{i=1}^{n} t_i z_i\Big\| \geqslant \frac{1}{\big\|\sum_{j=1}^{n} h_j\big\|}\Big(\sum_{j=1}^{n} h_j\Big)\Big(\sum_{i=1}^{n} t_i z_i\Big) \geqslant \frac{1}{\sup_n \big\|\sum_{j=1}^{n} h_j\big\|}\sum_{i=1}^{n} t_i.$$

On the other hand, we shall exhibit in c_0 a basis of type l_+ which is not of type P^*. Let us recall that B. R. Gelbaum has constructed in [6] the following basis of c_0:

(19) $$u_n = \{(-1)^{n+1}, (-1)^{n+2}, \ldots, (-1)^{2n}, 0, 0, \ldots\}, \quad n = 1, 2, \ldots$$

We now claim that the sequence $\{v_n\}$ defined by

(20) $$v_{4k-3} = \sum_{i=1}^{4k-3} u_i, \quad v_{4k-2} = \sum_{i=1}^{4k-2} u_i,$$
$$v_{4k-1} = 2\sum_{i=1}^{4k-1} u_i, \quad v_{4k} = 2\sum_{i=1}^{4k} u_i \qquad (k = 1, 2, \ldots),$$

where $\{u_n\}$ is the basis (19) of the space c_0, satisfies all our requirements.

In fact, since $\{u_n\}$ is a basis of type P (because of $\|u_n\| = 1$, $\big\|\sum_{i=1}^{n} u_i\big\| = 1$, $n = 1, 2, \ldots$), the sequence $\big\{\sum_{i=1}^{n} u_i\big\}$ is, by the implication $1° \Rightarrow 3°$ of proposition 2, a basis of the space c_0, obviously of type P^*. Hence, by virtue of the implications $1° \Rightarrow 5°$ of proposition 3 and $3° \Rightarrow 1°$ of proposition 1, sequence (20) is a basic sequence of type l_+. However, it is not of type P^*. In fact, if $\{\varphi_n\}$ denotes the sequence of functionals biorthogonal to $\{u_n\}$, then the sequence of functionals biorthogonal to $\{v_n\}$ is nothing else but

$$\psi_{4k-3} = \varphi_{4k-3} - \varphi_{4k-2}, \quad \psi_{4k-2} = \varphi_{4k-2} - \varphi_{4k-1},$$
$$\psi_{4k-1} = \tfrac{1}{2}(\varphi_{4k-1} - \varphi_{4k}), \quad \psi_{4k} = \tfrac{1}{2}(\varphi_{4k} - \varphi_{4k+1}), \quad k = 1, 2, \ldots,$$

whence, since $\varphi_n = \{\underbrace{0, 0, \ldots, 0}_{n-1}, 1, 1, 0, 0, \ldots\}$ $(n = 1, 2, \ldots)$, it is easy to find that

$$\lim_{n\to\infty} \Big\|\sum_{i=1}^{n} \psi_i\Big\| = +\infty.$$

5° Let $\{z_n\}$ be a basic sequence of type l_+. Then there exists, by the implication $1° \Rightarrow 3°$ of proposition 1, a functional $h \in [z_n]^*$ such that

$$h(z_n) \geqslant 1, \quad n = 1, 2, \ldots$$

Since $\sup_n \|z_n\| = M < +\infty$, it follows that

$$\|h\|_n = \|h|_{[z_{n+1}, z_{n+2}, \ldots]}\| \geqslant \frac{1}{M}, \quad n = 1, 2, \ldots,$$

which shows that the basic sequence $\{z_n\}$ is not shrinking.

On the other hand, one can verify that the basis (19) of c_0, the Schauder basis of the space $C([0, 1])$ and the Haar basis of $L^1([0, 1])$ are neither shrinking nor of type l_+.

6° Assume that $\{x_n\}$ is a non-shrinking basic sequence, and let $f \in E^*$ be a functional such that $\lim_{n\to\infty} \|f\|_n \neq 0$. Then there exist a number $\varepsilon > 0$, an increasing sequence of positive integers $\{p_n\}$, a sequence of scalars $\{b_n\}$ and a sequence of elements $\{z_n\} \subset E$ such that

(21) $$z_n = \sum_{i=p_n+1}^{p_{n+1}} b_i x_i, \quad n = 1, 2, \ldots,$$

(22) $$\|z_n\| = 1, \quad n = 1, 2, \ldots,$$

(23) $$f(z_n) > \varepsilon, \quad n = 1, 2, \ldots$$

By (21) and (22) $\{z_n\}$ is a block basis, and from (23) it follows that for every finite sequence $t_1, \ldots, t_n \geqslant 0$ we have

$$\Big\|\sum_{i=1}^{n} t_i z_i\Big\| \geqslant \frac{f}{\|f\|}\Big(\sum_{i=1}^{n} t_i z_i\Big) = \frac{1}{\|f\|}\sum_{i=1}^{n} t_i f(z_i) \geqslant \frac{\varepsilon}{\|f\|}\sum_{i=1}^{n} t_i,$$

i. e. that we have (1) with $\eta = \varepsilon/\|f\|$. This, together with (22) shows that the block basis $\{z_n\}$ is of type l_+.

7° Let $\{z_n\}$ be a basic sequence of type l_+. Then there exists, by the implication $1° \Rightarrow 3°$ of proposition 1, a functional $f \in E^*$ such that

(24) $$f(z_n) \geqslant 1, \quad n = 1, 2, \ldots$$

Let us define a sequence $\{w_n\}$ by

(25) $$w_n = \frac{1}{f(z_n)} z_n, \quad n = 1, 2, \ldots$$

This sequence $\{w_n\}$ is a "contraction" of the basic sequence $\{z_n\}$, since, by (24), $0 < 1/f(z_n) \leqslant 1$ $(n = 1, 2, \ldots)$. Hence $\{w_n\}$ is a basis of $[z_n] = [w_n]$, and

$$\sup_n \|w_n\| = \sup_n \frac{1}{|f(z_n)|}\|z_n\| \leqslant \sup_n \|z_n\| < +\infty.$$

Finally, we have

$$f(w_n) = f\Big(\frac{1}{f(z_n)} z_n\Big) = 1, \quad n = 1, 2, \ldots$$

Consequently, by the implication $5^\circ \Rightarrow 1^\circ$ of proposition 3, $\{w_n\}$ is a basic sequence of type P^*.

8° This is a consequence of 3° and 4°. It can easily be seen also directly from the definitions.

9° This is a consequence of the implications $1^\circ \Rightarrow 3^\circ$ of proposition 2 and $1^\circ \Rightarrow 3^\circ$ of proposition 3, since basis (6) is necessarily of type P^*, while basis (10) is necessarily of type P.

10° The ten trivial implications are: boundedly complete implies boundedly complete, etc. Each of the propositions 1°, 3°, 4° and 8° contains two implications, e. g. 1° contains the following ones: type P implies non-boundedly complete and boundedly complete implies non-P. Finally, 5° contains four implications, and thus we have obtained 22 implications.

On the other hand, ten non-implications are trivial (e. g. $P \not\Rightarrow$ non-P, non-$P \not\Rightarrow P$, etc). Furthermore, the examples of basic sequences given above are sufficient to prove, and we shall not bore the reader with details, that all the other 68 relations between the ten types considered, are: the relation of non-implication. This completes the proof of theorem 1.

§ 2. Characterizations of reflexivity

THEOREM 2. *For a Banach space E with a basis $\{x_n\}$ the following statements are equivalent:*

1° *E is reflexive.*

2° *Every basis of every block subspace* (5) *is boundedly complete.*

3° *No basis of any block subspace is of type P.*

4° *No basis of any block subspace is of type P^*.*

5° *No basis of any block subspace is of type l_+.*

6° *Every basis of every block subspace is shrinking.*

Proof. The implication $1^\circ \Rightarrow 2^\circ$ is an immediate consequence of the fact that every subspace of a reflexive space is reflexive and of the necessity of condition (a) in theorem (J).

The implication $2^\circ \Rightarrow 3^\circ$ is a consequence of theorem 1,1°.

The implication $3^\circ \Rightarrow 4^\circ$ follows from theorem 1,9°.

The implication $4^\circ \Rightarrow 5^\circ$ is a consequence of theorem 1,7°.

In order to prove the implication $5^\circ \Rightarrow 1^\circ$(6), assume that E is non--reflexive. Then, by the sufficiency part of the theorem (J), the basis $\{x_n\}$ is either non-boundedly complete or non-shrinking.

(5) Let us recall that by a "block subspace" we always mean a block subspace with respect to the given basis $\{x_n\}$.

(6) We cannot prove directly the implication $5^\circ \Rightarrow 6^\circ$. In fact, if we assume that E has a block subspace Y with a non-shrinking basis $\{y_n\}$, then, by theorem 1,6° we may conclude that $\{y_n\}$ admits a block basis $\{z_n\}$ of type l_+, but in general $[z_n]$ will not be a block subspace with respect to $\{x_n\}$ (since $\{y_n\}$ is not a block basis).

Assume first that $\{x_n\}$ is non-boundedly complete. Then, by theorem 1,2°, the basis $\{x_n\}$ admits a block basis $\{y_n\}$ of type P. Then, by theorem 1, 9°, the block subspace $Z = [y_n]$ admits a basis $\{z_n\}$ of type P^*. However, by theorem 1,4°, $\{z_n\}$ is of type l_+.

On the other hand, assume now that $\{x_n\}$ is non-shrinking. Then, by theorem 1,6° $\{x_n\}$ admits a block basis $\{z_n\}$ of type l_+.

Thus, if E is non-reflexive, then in either case E contains a block subspace Z with a basis $\{z_n\}$ of type l_+. This completes the proof of the implication $5^\circ \Rightarrow 1^\circ$.

The implication $1^\circ \Rightarrow 6^\circ$ is an immediate consequence of the fact that every subspace of a reflexive space is reflexive and of the necessity of condition (b) in theorem (J).

Finally, the implication $6^\circ \Rightarrow 5^\circ$ follows from theorem 1,5°. This completes the proof of theorem 2.

COROLLARY 1. *Theorem 2 remains valid if we replace in its formulation "every basis of every block subspace" ("no basis of no block subspace") by "every basic sequence" ("no basic sequence").*

In fact, the proof of the implications $1^\circ \Rightarrow 2^\circ \Rightarrow 3^\circ \Rightarrow 4^\circ \Rightarrow 5^\circ$ and $1^\circ \Rightarrow 6^\circ \Rightarrow 5^\circ$ is the same as that of the corresponding ones in theorem 2, while the implication $5^\circ \Rightarrow 1^\circ$ is an immediate consequence of the implication $5^\circ \Rightarrow 1^\circ$ of theorem 2.

Remark 3. In theorem 2 above one cannot replace "every basis of every block subspace" ("no basis of no block subspace") by "every block basis" ("no block basis"). In fact, take $E = c_0$ and $\{x_n\}$ = the natural basis of c_0. Then, according to a remark of [2], every block basis $\{y_n\}$ with $\|y_n\| = 1$ $(n = 1, 2, \ldots)$ is equivalent to the basis $\{x_n\}$. Consequently, every block basis is shrinking and of types non-l_+ and non-P^*. However, c_0 is non-reflexive. Similarly, every block basis with respect to the natural basis $\{x_n\}$ of the non-reflexive space $E = l$ is boundedly complete and of type non-P.

However, it is possible to give characterizations of reflexivity which involve only block bases, e. g. the following

COROLLARY 2. *A Banach space E with a basis $\{x_n\}$ is reflexive if and only if all block bases are neither of type P nor of type l_+.*

In fact, the necessity part is a consequence of the implications $1^\circ \Rightarrow 3^\circ$ and $1^\circ \Rightarrow 5^\circ$ of theorem 2. Conversely, assume that E is non--reflexive. Then, by the sufficiency part of theorem (J), the basis $\{x_n\}$ is either non-boundedly complete or non-shrinking. In the first case there exists, by theorem 1, 2°, a block basis of type P; in the second case there exists, by theorem 1, 6°, a block basis of type l_+.

Remark 4. Even if in theorem 2, or, in corollary 1 above, one starts with an unconditional basis $\{x_n\}$, it is not possible to replace "every (no) basis of every (any) block subspace" by "every (no) unconditional basis of every (any) block subspace", or respectively, "basic sequence" by "unconditional basic sequence". In fact, take $E = c_0$, and $\{x_n\}$ = the natural basis of c_0. Then, since c_0 contains no subspace isomorphic to l, it follows from remark 1, remark 2,1°, and theorem 1,6°, that every unconditional basic sequence is of types non-l_+, non-P^* and shrinking. Similarly, by remark 2, 2° and theorem 1,2° every unconditional basic sequence in the non-reflexive space $E = l$ is of types non-P and boundedly complete.

However, for Banach spaces with unconditional bases we can give

COROLLARY 3. *A Banach space E with an unconditional basis $\{x_n\}$ is reflexive if and only if every basis of every block subspace is either boundedly complete, or shrinking.*

The necessity part is an immediate consequence of the fact that every subspace of a reflexive space is reflexive, and of the necessity part of theorem (J). Conversely, assume that E is non-reflexive. Then by [8], theorem 2 (see also its proof), either E contains a block subspace isomorphic to c_0, or E contains a block subspace isomorphic to l. Thus, in order to complete the proof, it will be sufficient to show that in c_0 and l there exist bases which are neither boundedly complete nor shrinking. However, it is easy to verify that bases (15) and (19) of c_0 and the basis

$$z_n = \{\underbrace{0, 0, \ldots, 0}_{n-2}, 1, -1, 0, 0, \ldots\}, \quad n = 1, 2, \ldots, \tag{26}$$

of l have all these properties. This completes the proof.

Remark 5. Let us mention some connections with the paper [11] of V. Pták.

If we assume only that $\{y_n\}$ is a (not necessarily basic) sequence which admits a sequence of functionals $\{\gamma_n\} \subset E^*$ such that (y_n, γ_n) is a bounded biorthogonal system, then a condition of type P for such sequences $\{y_n\}$ appears in [11], theorem 1, while a condition of type (9) for such sequences appears in [11], theorem 3,1°. Their equivalence, proved in 1° $\Longleftrightarrow$ 7° of our proposition 2 (the same proof remains obviously valid for the case of such sequences), allows some simplifications in [11].

Furthermore, if $\{z_n\}$ is a (not necessarily basic) sequence which admits a sequence $\{\chi_n\} \subset E^*$ such that (z_n, χ_n) is a bounded biorthogonal system, then a condition of type (14) (which is equivalent, by 1° $\Longleftrightarrow$ 8° of proposition 3, to P^*) appears in [11], theorem 2.

Finally, let $\{z_n\}$ be a *basic* sequence such that $\sup_n \|z_n\| < +\infty$ and that if $\{z_n\}$ is considered as a basic sequence in the quotient space $E/B_\perp$ (with the norm $|||x|||$), where B denotes the $\sigma(E^*, E)$-closure of the sub-

space spanned by a sequence $\{\chi_n\} \subset E^*$ biorthogonal to $\{z_n\}$ and where $B_\perp = \{x \in E \mid b(x) = 0 \text{ for all } b \in B\}$ (7), then there exists a constant $\eta > 0$ such that

$$\Big|\Big|\Big|\sum_{i=1}^{\infty} t_i z_i\Big|\Big|\Big| \geqslant \eta \sum_{i=1}^{\infty} t_i \tag{27}$$

for every sequence $t_n \geqslant 0$ $(n = 1, 2, \ldots)$ satisfying $\sum_{i=1}^{\infty} t_i < +\infty$. Then we shall say that $\{z_n\}$ *is of type* P^0. It is easy to verify that *every basic sequence of type* P^* *is also of type* P^0 (but the converse is not true) and that *every basic sequence of type* P^0 *is of type* l_+. Consequently, by virtue of the equivalences 1° $\Longleftrightarrow$ 4° $\Longleftrightarrow$ 5° of theorem 2 and of corollary 1, we have

COROLLARY 4. *For a Banach space E with a basis $\{x_n\}$ the following conditions are equivalent:*

1° *E is reflexive.*

2° *No basis of any block subspace (or, which is equivalent, no basic sequence in E) is of type P^0.*

For (not necessarily basic) sequences $\{z_n\} \subset E$ which admit a sequence $\{\chi_n\} \subset E^*$ such that (z_n, χ_n) is a bounded biorthogonal system, a condition of type P^0 appears in [11], theorem 3,2°.

In conclusion, we see that *the equivalences* 1° $\Longleftrightarrow$ 3° $\Longleftrightarrow$ 4° *of theorem* 2 (*and of corollary* 1) *and* 1° $\Longleftrightarrow$ 2° *of corollary* 4 *constitute an improvement, for Banach spaces with bases, of all theorems of the paper* [11].

We can also interpret this remark as follows: The equivalences 1° $\Longleftrightarrow$ 3° $\Longleftrightarrow$ 4° of theorem 2 (and of corollary 1) and 1° $\Longleftrightarrow$ 2° of corollary 4 remain valid if we replace the basic sequences involved, by (not necessarily basic) sequences $\{z_n\} \subset E$ which admit a sequence $\{\chi_n\} \subset E^*$ such that (z_n, χ_n) be a bounded biorthogonal system. Obviously the implications 2° $\Rightarrow$ 1°, 5° $\Rightarrow$ 1° and 6° $\Rightarrow$ 1° of theorem 2 remain valid in this more general situation. However, the extended implications 1° $\Rightarrow$ 2°, 1° $\Rightarrow$ 5°, 1° $\Rightarrow$ 6° are not more valid. In fact, for the extended non-implication 1° $\not\Rightarrow$ 5° an example is given in [11] (8), and this also proves 1° $\not\Rightarrow$ 6°; on the other hand, 1° $\not\Rightarrow$ 2° follows from [12], theorem 3, according to which every boundedly complete sequence $\{z_n\} \subset E$ which admits a biorthogonal sequence $\{\chi_n\} \subset E^*$ is already a basic sequence.

(7) It is clear that $B_\perp = [\chi_n]_\perp = \{x \in E \mid \chi_n(x) = 0,\ n = 1, 2, \ldots\}$ and that $[\chi_n]_\perp \cap [z_n] = \{0\}$, whence the canonical image of $\{z_n\}$ in $E/B_\perp$ is indeed a basic sequence.

(8) However, let us observe that such an example is not more possible if we require, in addition, that the sequence $\{\chi_n |_{[z_m]}\} \subset [z_n]^*$ be $[z_n]$-total (this happens in particular when $\{z_n\}$ is a basic sequence). In this case the extended implications 1° $\Rightarrow$ 5° and 1° $\Rightarrow$ 6° are also valid.

§ 3. Some remarks and unsolved problems

1. A geometric interpretation of $\|f\|_n$. We shall prove

PROPOSITION 4. *For any $f \epsilon E^*$ and any positive integer n the quantity $\|f\|_n$ occurring in condition* (b) *of theorem* (J) *(and in the definition of shrinking bases) is nothing else but*

$$\|f\|_n = \operatorname{dist}(f, [f_1, \ldots, f_n]), \tag{28}$$

where $\{f_n\} \subset E^*$ is the sequence of functionals biorthogonal to the basis $\{x_n\}$ of E.

Proof. From the isometry $E^*/[f_1, \ldots, f_n] \equiv ([f_1, \ldots, f_n]_\perp)^*$ we have

$$\operatorname{dist}(f, [f_1, \ldots, f_n]) = \sup_{\substack{x \epsilon [f_1, \ldots, f_n]_\perp \\ \|x\| \leqslant 1}} |f(x)| \qquad (f \epsilon E^*).$$

Thus, taking into account the definition of $\|f\|_n$, it is sufficient to prove that

$$[f_1, \ldots, f_n]_\perp = [x_{n+1}, x_{n+2}, \ldots].$$

Now, the inclusion

$$[f_1, \ldots, f_n]_\perp \supseteq [x_{n+1}, x_{n+2}, \ldots]$$

is obvious by biorthogonality. Conversely, assume that $x \epsilon [f_1, \ldots, f_n]_\perp$, i. e. that

$$f_1(x) = \ldots = f_n(x) = 0.$$

Then, since $\{x_n\}$ is a basis of E, we have $x = \sum_{i=1}^{\infty} f_i(x) x_i = \sum_{i=n+1}^{\infty} f_i(x) x_i$, i. e. $x \epsilon [x_{n+1}, x_{n+2}, \ldots]$. Thus we also have the converse inclusion

$$[f_1, \ldots, f_n]_\perp \subseteq [x_{n+1}, x_{n+2}, \ldots],$$

whence the desired equality, and the proof of proposition 4 is complete.

Proposition 4, together with the fact that $\{f_n\}$ is always a basis of $[f_n]$, sheds some new light on theorem 3 (a) of [8], and on lemma 1, p. 70, of [4].

Let us mention that, by extending the above method, formula (28) can also be proved under the weaker assumption that (x_n, f_n) is merely a biorthogonal system with $[x_n] = E$.

2. Duality relations between the types of basic sequences and of their biorthogonal sequences. We have proved (see the equivalences $1^\circ \Longleftrightarrow 2^\circ$ of propositions 2 and 3) that a basic sequence $\{y_n\}$ in a Banach space E is of type P if and only if the sequence of functionals $\{g_n\} \subset [y_n]^*$ biorthogonal to $\{y_n\}$ is of type P^* and that $\{y_n\}$ is of type P^* if and only if $\{g_n\}$ is of type P. We shall now prove a similar result for boundedly complete and shrinking basic sequences.

PROPOSITION 5. *Let $\{y_n\}$ be a basic sequence in a Banach space E, and let $\{g_n\} \subset [y_n]^*$ be the sequence of functionals biorthogonal to $\{y_n\}$. Then*

1° *$\{y_n\}$ is boundedly complete if and only if $\{g_n\}$ is shrinking.*

2° *$\{y_n\}$ is shrinking if and only if $\{g_n\}$ is boundedly complete.*

Proof. Throughout this proof we shall denote by φ the canonical mapping of $[y_n]$ into $[g_n]^*$, i. e. the mapping defined for every $y \epsilon [y_n]$ by

$$[\varphi(y)](g) = g(y) \quad \text{for all} \quad g \epsilon [g_n]. \tag{29}$$

Assume that $\{y_n\}$ is boundedly complete. Then (cf. [4]) $[y_n]$ is canonically isomorphic to $[g_n]^*$, whence $\{\varphi(y_n)\}$ is a basis of $[g_n]^*$. Then, since $\{g_n\}$ is a basis of $[g_n]$ and since $\{\varphi(y_n)\} \subset [g_n]^*$ is the sequence of functionals biorthogonal to $\{g_n\}$, it follows by [8], theorem 3 (a), that $\{g_n\}$ is shrinking.

On the other hand, assume that $\{y_n\}$ is shrinking. Then $\{g_n\}$ is, by [8], theorem 3(a),(b), a boundedly complete basis of $[y_n]^*$.

Conversely, assume now that $\{g_n\}$ is shrinking. Then, by the implication just proved, $\{\varphi(y_n)\}$ is a boundedly complete basis of $[g_n]^*$. Thus, in order to prove that $\{y_n\}$ is boundedly complete, it will be sufficient to prove that φ is an isomorphism of $[y_n]$ onto $[g_n]^*$. However, since $\{g_n\}$ is total with respect to $\{y_n\}$ (i. e. $y \epsilon [y_n]$, $g_n(y) = 0$, $n = 1, 2, \ldots$, imply $y = 0$), φ is one-to-one. On the other hand, since $\{\varphi(y_n)\}$ is a basis of $[g_n]^*$, we have $[g_n]^* = [\varphi(y_n)] \subset \varphi([y_n])$, i. e. φ maps $[y_n]$ onto $[g_n]^*$. Consequently, by [1], p. 41, theorem 5, φ is an isomorphism of $[y_n]$ onto $[g_n]^*$, which completes the proof of 1°.

Finally, by [12], lemma 2, we have

$$\sup_n \Big\| \sum_{i=1}^{n} y^*(y_i) g_i \Big\| < +\infty \quad \text{for all} \quad y^* \epsilon [y_n]^*.$$

Consequently, if $\{g_n\}$ is boundedly complete, then for every $y^* \epsilon [y_n]^*$ the series $\sum_{i=1}^{n} y^*(y_i) g_i$ is convergent. Since its sum is then obviously y^*, it follows that $[y_n]^* \subset [g_n]$, whence, by [8], theorem 3 (a), $\{y_n\}$ is shrinking. This completes the proof.

Let us mention that A. Wilansky has proved the following particular case of one half of the above proposition ([12], theorem 2; see also [8], theorem 3 (b)): A basis $\{x_n\}$ of E is shrinking if and only if its sequence of coefficient functionals $\{f_n\}$ is boundedly complete.

3. We want to raise the following problems concerning possible improvements of the results of this paper.

P1. Does every non-reflexive Banach space contain a non-reflexive subspace with a basis?

An affirmative answer to this problem would imply that corollary 1 is valid for arbitrary Banach spaces (i. e. that we can replace in its formulation "Banach space E which has a basis" by "Banach space")([9]). At the same time, an affirmative answer to this problem would constitute an improvement, respectively an improvement for non-reflexive spaces, of the following known results:

(A) (Eberlein [5]). Every non-reflexive Banach space contains a separable non-reflexive subspace.

(B) (Banach [1], p. 238). Every Banach space contains an infinite dimensional subspace with a basis.

Various proofs of (B) have been given in [2], [7] and [3].

On the other hand, a negative answer to P1 (which seems very improbable) would imply, by virtue of (A), a negative answer to the basis problem.

Remark 6. A. Pełczyński has remarked that a similar problem for unconditional basic sequences has a negative answer, i. e.: *there exists a non-reflexive Banach space E such that every subspace of E having an unconditional basis is reflexive.*

In fact, let E be the non-reflexive space defined by James in [8], p. 253. Since E^* and E^{**} are separable, no subspace of E is isomorphic either to c_0 or to l (by [1], p. 188, theorem 11 and by [2], theorem 4). Hence, according to [8], theorem 2, every subspace of E with an unconditional basis is reflexive.

It is easy to verify that the natural basis of c_0 and the basis (26) of l are of type P (and hence non-boundedly complete), while the basis (15) of c_0 and the natural basis of l are of type P^* (and hence of type l_+ and non-shrinking). This suggests the following problem:

P 2. Is it possible to replace in the results of this paper "basic sequence" (or "basis of block subspace") by "basis of the space E"?

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