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STUDIA MATHEMATICA, T. XXI. (1962)

On the analytic functions in p-normed algebras

by

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A *p-normed algebra* is a complete metric algebra in which topology is given by the meaning of a p-homogeneous submultiplicative norm $\|x\|$:

$$\|ax\| = |a|^p \|x\|, \tag{1}$$

$$\|xy\| \leqslant \|x\| \|y\|, \tag{2}$$

where a is a scalar, p — fixed real number satisfying $0 < p \leqslant 1$.

It is known that every complete locally bounded algebra is a p-normed algebra. These algebras were considered in papers [4], [5], and [6]. The greater part of Gelfand's theory on commutative complex Banach algebras is also true for p-normed algebras. In this paper we give an extension of Gelfand's theory of analytic functions in Banach algebras onto p-normed algebras [1]. We note that the classical method based upon the concept of abstract Riemann integral cannot be applied here, because the algebras in question are not locally convex (cf. [3]).

Let A be a commutative complex p-normed algebra with a unit designed by e. Let $\mathfrak{M}$ be the compact space of its multiplicative linear functionals (= maximal ideals). The spectrum of an element $x \in A$ is defined as

$$\sigma(x) = \{f(x): f \in \mathfrak{M}\}. \tag{3}$$

It is a compact subset of the complex plane. Here we give the positive answer to the following question stated in [6]:

"Let $\Phi(z)$ be a holomorphic function defined in the neighbourhood U of spectrum $\sigma(x)$ of an element $x \in A$. Does there exist a $y \in A$ such that for every $f \in \mathfrak{M}$

$$f(y) = \Phi(f(x))\,?" \tag{4}$$

We shall give a step by step construction of such an element y. It is natural to write $y = \Phi(x)$. So we give a natural definition of $\Phi(x)$ in locally bounded algebras.

As a corollary we obtain the generalization of the theorem of Lévy [2] on trigonometrical series.

LEMMA 1. *Let $\Phi(\lambda)$ be an analytic function defined on an open subset U of the complex plane, and let x be an element of a p-normed algebra A such that*

$$\sigma(x) \subset K(\lambda_0, r) \subset U, \tag{5}$$

where $K(\lambda_0, r) = \{\lambda : |\lambda - \lambda_0| < r\}$. Then there exists such a $y \in A$ that (4) *holds for every $f \in \mathfrak{M}$.*

Proof. It may easily be verified that $\sigma(x - \lambda_0 e) \subset K(0, r)$. By [5] we have

$$\sup_\lambda \{|\lambda|^p : \lambda \in \sigma(x - \lambda_0 e)\} = \lim \sqrt[n]{\|(x - \lambda_0 e)^n\|}.$$

Hence

$$\|(x - \lambda_0 e)^n\| < \varrho^{pn} \tag{6}$$

for large n and suitable $\varrho < r$. But in $K(\lambda_0, r)$ we can write Φ in the form

$$\Phi(\lambda) = \sum_{n=0}^{\infty} a_n (\lambda - \lambda_0)^n$$

and we have

$$r \leqslant (\limsup \sqrt[n]{|a_n|})^{-1}. \tag{7}$$

Now we put

$$y = \sum_{n=0}^{\infty} a_n (x - \lambda_0 e)^n \quad \text{or} \quad y = \Phi(x). \tag{8}$$

This series is absolutely convergent in A. In fact, by (6) and (7), we have

$$\sum \|a_n (x - \lambda_0 e)^n\| \leqslant \sum |a_n|^p \|(x - \lambda_0 e)^n\| \leqslant \sum |a_n|^p \varrho^{pn}$$

and

$$\varrho^p < r^p \leqslant (\limsup \sqrt[n]{|a_n|})^{-p} = (\limsup \sqrt[n]{|a_n|^p})^{-1}.$$

The desired conclusion follows from the fact that for every $f \in \mathfrak{M}$ we have

$$f(y) = f\Big(\sum a_n (x - \lambda_0 e)^n\Big) = \sum a_n \big(f(x) - \lambda_0\big)^n = \Phi\big(f(x)\big).$$

LEMMA 2. *If Φ is an analytic function defined on an open simply connected bounded subset U of a complex plane, and $\sigma(x) \subset U$ for an $x \in A$, then there exists a $y \in A$ such that* (4) *holds.*

Proof. Let $\varphi(\lambda)$ be a 1-1 conformal mapping of U onto $K(0, 1)$. Put $r = \max\{|\lambda| : \lambda \in \varphi(\sigma(x))\}$. We have $1 - r = 5\varepsilon > 0$. We put $U_k = \varphi^{-1}\big(K(0, 1 - \varepsilon k)\big)$, $k = 1, 2, \ldots, 5$, and $\Gamma_k = \varphi^{-1}\big(S(0, 1 - \varepsilon k)\big)$, where $S(\lambda_0, r) = \{\lambda : |\lambda - \lambda_0| = r\}$. We have $\varphi(\sigma(x)) \subset \varphi(U_4)$, and by a theorem of Runge we can choose such a polynomial $p(\lambda)$ that $|p(\lambda) - \varphi(\lambda)| < \varepsilon$ for $\lambda \in U_1$. The equation $\varphi(\lambda) = \lambda_0$ has exactly one solution for every $\lambda_0 \in K(0, 1 - 2\varepsilon)$; moreover, for a fixed $\lambda_0 \in K(0, 1 - 2\varepsilon)$ we have $\min_{\lambda \in \Gamma_1} |\varphi(\lambda) - \lambda_0| \geqslant \varepsilon$. Hence, by a theorem of Rouché, equation $p(\lambda) = \lambda_0$ has in U_1 exactly one solution. We thus have

$$p\big(\sigma(x)\big) \subset K(0, 1 - 3\varepsilon) \subset p(U_2),$$

and p is a 1-1 conformal mapping of U_2 onto $p(U_2) = K(0, 1 - 2\varepsilon)$. Nowe we may easily verify that $p\big(\sigma(x)\big) = \sigma\big(p(x)\big)$, and so we have

$$\sigma\big(p(x)\big) \subset K(0, 1 - 3\varepsilon) \subset p(U_2).$$

The function $\Psi(\lambda) = \Phi\big(p^{-1}(\lambda)\big)$ is a holomorphic function defined on $p(U_2)$. Thus, by lemma 1, we can define an element $y = \Psi\big(p(x)\big)$, and for every $f \in \mathfrak{M}$ we have

$$f(y) = \Psi\Big(f\big(p(x)\big)\Big) = \Psi\Big(p\big(f(x)\big)\Big) = \Phi\big(f(x)\big), \quad \text{q. e. d.}$$

LEMMA 3. *If $x \in A$, $\sigma(x) \in U$, where U is a simply connected open subset of a closed complex plane whose complement has an interior point λ_0, then for every Φ holomorphic on U we can choose such a $y \in A$ that* (4) *holds.*

Proof. The function $\Psi(\lambda) = \Phi(1/\lambda + \lambda_0)$ is a holomorphic function defined on the bounded simply connected open set $V = (U - \lambda_0)^{-1}$. We have also $\sigma\big((x - \lambda_0 e)^{-1}\big) \subset V$. Thus by lemma 2, we can define a $y = \Psi\big((x - \lambda_0 e)^{-1}\big)$, and for every $f \in \mathfrak{M}$ we have

$$f(y) = \Psi\big(f(x - \lambda_0 e)^{-1}\big) = \Phi\big(f(x)\big), \quad \text{q. e. d.}$$

LEMMA 4. *The conclusion of lemma 3 is true also in the case where U is connected provided its complement consists of a finite number of components, each of them having interior points.*

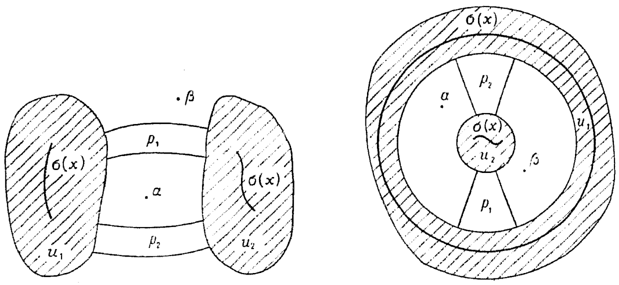
Proof. Using the Cauchy integral formula, we can write every holomorphic function Φ defined on U in the form

$$\Phi(\lambda) = \Phi_1(\lambda) + \Phi_2(\lambda) + \ldots + \Phi_n(\lambda),$$

where Φ_i is a holomorphic function defined on a simply connected set U_i, which is a complement of any component of a complement of U, i. e. $U = \bigcap_{i=1}^{k} U_i$. Thus we can define $y_i = \Phi_i(x)$ in such a way that (4) holds, and the desired element is $y = y_1 + y_2 + \ldots + y_k$.

LEMMA 5. *Let $\sigma(x) \subset U$, and $U = U_1 \cup U_2$, where U_i is a connected open set $(i = 1, 2)$, and $U_1 \cap U_2 = \emptyset$ ($\emptyset$ denotes the void set). If $\sigma(x) \cap U_i \neq \emptyset$ $(i = 1, 2)$, then there exists such an element $z \in A$ that $f(z) = 0$ for every $f \in \mathfrak{M}$ such that $f(x) \in U_1$, and $f(z) = 1$ for every $f \in \mathfrak{M}$ such that $f(x) \in U_2$.*

Proof. Let α and β be two distinct complex numbers which lie in that component of the complement of U which contains the boundary points of U_1 and U_2. Now take two open disjoint sets P_1 and P_2 in such a way that $\alpha, \beta \notin S = U_1 \cup P_1 \cup U_2 \cup P_2$, S is connected, and α and β are not in the same component of the complemention of S (two possible situations are schematically shown in the figure). We now take any branch



$\varphi_1(\lambda)$ of $\log\big(1/(\alpha-\beta)-1/(\lambda-\beta)\big)$ defined on $U_1 \cup P_1 \cup U_2$, put $\varphi_2(\lambda) = \varphi_1(\lambda)$ for $\lambda \in U_1$ and extend it analytically onto P_2 and U_2. We have $\varphi_1-\varphi_2 = 0$ on U_1, and $\varphi_1-\varphi_2 = \varepsilon\cdot 2\pi i$ on U_2, $\varepsilon = 1$ or -1. The functions φ_1 and φ_2 are analytical on the connected open sets $U_1 \cup P_1 \cup U_2$ and $U_1 \cup P_2 \cup U_2$ containing the spectrum of x; thus, by Lemma 4, we can define elements $x_1 = \varphi_1(x)$, $x_2 = \varphi_2(x)$. It is clear that the desired element z of A may be given by the formula $z = (x_1-x_2)/\varepsilon\cdot 2\pi i$, q. e. d.

THEOREM 1. *Let A be a commutative complex p-normed algebra with unit e. Let U be an open subset of a complex plane containing the spectrum $\sigma(x)$ of an element $x \in A$. Then for every holomorphic function $\Phi(\lambda)$ defined on U there exists an element $y = \Phi(x) \in A$ such that* (4) *holds. If A is semisimple, then such an element y is unique.*

Proof. For every $\lambda \in \sigma(x)$ we can choose such real $r_\lambda > 0$ that $K(\lambda, r_\lambda) \subset U$. By the compactness of $\sigma(x)$ we can cover it by a finite number of such neighbourhoods. Thus we can assume that U is a finite sum of sets K, or that U is a finite sum of connected sets $U = U_1 \cup U_2 \cup \ldots \cup U_k$ each complement of U_i having a finite number of components containing interior points. We may also assume that $\sigma(x) \cap U_i \neq \emptyset$ $(i = 1, 2, \ldots, k)$. Now, by lemma 5 and easy induction, we can construct elements $e_1, e_2, \ldots, e_k$ in such a way that

$$e_1+e_2+\ldots+e_k = e$$

and

$$f(e_i) = \begin{cases} 1 & \text{if} \quad f(x) \in U_i, f \in \mathfrak{M}, \\ 0 & \text{if} \quad f(x) \notin U_i, f \in \mathfrak{M}. \end{cases}$$

We now put

$$x_i = (x-\alpha_i e)e_i + \alpha_i e,$$

where $\alpha_i \in U_i \cap \sigma(x)$. We have $\sigma(x_i) \subset U_i$, whence, by lemma 4, we can define an element

$$y_i = \Phi(x_i) - \Phi(\alpha_i)(e-e_i), \qquad i = 1, 2, \ldots, k.$$

We now put $y = y_1+y_2+\ldots+y_k$. Taking any $f \in \mathfrak{M}$ we have $f(x) \in U_n$ for certain n, $1 \leqslant n \leqslant k$. Hence $f(y_n) = \Phi\big(f(x_n)\big) = \Phi\big(f(x)\big)$, and, for $i \neq n$, $f(y_i) = \Phi(\alpha_i) - \Phi(\alpha_i) = 0$. So

$$f(y) = \sum_{i=1}^{k} f(y_i) = \Phi\big(f(x)\big).$$

The uniqueness of y in the case where A is semisimple is obvious, q. e. d.

Applying theorem 1 to the algebras l_p, $0 < p \leqslant 1$, of all sequences $x = (x_n)_{n=-\infty}^{\infty}$ such that $\|x\| = \sum |x_n|^p < \infty$ with convolution multiplication we get by [5], theorem 16, the following generalization of the theorem of Lévy (see also [5], theorem 18):

THEOREM 2. *Let $x(t)$ be a complex function of a real variable $0 \leqslant t \leqslant 2\pi$ equal to its Fourier expansion*

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{int}$$

and let $\sum |x_n|^p < \infty$ for $0 < p \leqslant 1$. Then, if Φ is a holomorphic function defined on an open subset of a complex plane containing the compact set of values of $x(t)$, the function $\Phi\big(x(t)\big)$ may be written in the form

$$\Phi\big(x(t)\big) = \sum y_n e^{int},$$

where $\sum |y_n|^p < \infty$.

In a similar way we get the following theorem (cf. [5], theorem 17):

THEOREM 3. *If $x(\lambda)$ is an analytic function in $K(0,1)$ such that $x(\lambda) = \sum_{n=0}^{\infty} x_n \lambda^n$, where $\sum |x_n|^p < \infty$, $0 < p \leqslant 1$, and if Φ is an analytic function defined in a neighbourhood of the set $x\big(K(0,1)\big)$, then*

$$\Phi\big(x(\lambda)\big) = \sum_{n=0}^{\infty} y_n \lambda^n,$$

where $\sum |y_n|^p < \infty$.

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Basic sequences and reflexivity of Banach spaces

by

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Introduction

R. C. James has given the following characterization of reflexive Banach spaces ([8], theorem 1):

THEOREM (J). *A Banach space* (¹) *E with a basis* $\{x_n\}$ *is reflexive if and only if*

(a) *For every sequence of scalars* $\{a_n\}$ *such that* $\sup_n \|\sum_{i=1}^{n} a_i x_i\| < +\infty$ *the series* $\sum_{i=1}^{\infty} a_i x_i$ *is convergent,*

(b) $\lim_{n\to\infty} \|f\|_n = 0$ *for all functionals* $f \in E^*$, *where* $\|f\|_n$ *denotes the norm of the restriction of f to the closed linear subspace of E spanned by* x_{n+1}, $x_{n+2}, \ldots$

In a recent paper [11], V. Pták has completed the picture of the structure of reflexive Banach spaces given by theorem (J), characterizing reflexivity in terms of bounded biorthogonal systems.

The purpose of the present paper is to continue these investigations of the structure of reflexive and non-reflexive Banach spaces by characterizing the reflexivity of a Banach space with a basis in terms of the behaviour of its basic sequences.

Since we are dealing with Banach spaces having a basis, we shall freely use in our proofs theorem (J).

Let us first recall briefly some definitions and notation, which will be used in the sequel.

If E is a Banach space, we shall denote by E^* its conjugate space.

If $\{z_n\}$ is a sequence of elements in a Banach space E, we shall denote by $[z_n]$, or sometimes by $[z_1, z_2, \ldots]$, the subspace of E spanned by the sequence $\{z_n\}$; by "subspace" we shall always mean "closed linear subspace".

(¹) Throughout this paper, by *Banach space* we shall mean infinite-dimensional Banach space.