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Approximation of non-bounded continuous functions by certain sequences of linear positive operators or polynomials

by

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1. The problem of approximating continuous functions which are defined on a finite closed interval by certain sequences of linear positive operators has been investigated systematically by P. P. Korovkin [5], [6]. In this paper we shall show that certain classes of non-bounded continuous functions defined on the whole axis $(-\infty, \infty)$ can also be approximated by linear positive operators. As examples we shall mention that certain types of generalized Landau polynomials and Chlodovsky's modification of Bernstein polynomials, etc., all may be regarded as particular cases of our general result. Finally, it will be shown that the generalized Landau polynomial operators can even approximate every non-bounded continuous function $f(x)$ with $f(x) = O(e^{|x|})$ $(x \to \pm\infty)$.

2. Let $L_n\big(f(t); x\big)$ $(n = 1, 2, 3, \ldots)$ be a sequence of linear operators (additive and homogeneous operators) transforming $f(t)$ to functions of x defined on a certain set S of real numbers. It is always assumed that the sequence is *ultimately positive*, i. e. for all large n and every non-negative definite function $f(t)$ we have

$$L_n(f; x) \geqslant 0 \qquad (x \in S).$$

In what follows we shall always denote by S the interval $-1 \leqslant x \leqslant 1$, unless stated to the contrary.

THEOREM 1. *Let a_n be increasing to $+\infty$ with n, and let the following limit relation*

$$\lim_{n \to \infty} L_n\big((a_n t)^k; a_n^{-1} x\big) = x^k \tag{1}$$

hold uniformly for all values of x in every finite interval, where $k = 0, 1, 2, m, m+1, m+2$; and m is a certain non-negative even integer. Then for every continuous function $f(x)$ defined on $(-\infty, \infty)$ and satisfying the condition

$$f(x) = O(|x|^m) \qquad (x \to \pm\infty) \tag{2}$$

we have the limit relation

(3) $$\lim_{n\to\infty} L_n\big(f(a_n t);\ a_n^{-1}x\big) = f(x) \qquad (-\infty < x < \infty).$$

Moreover, (3) *holds uniformly on any finite interval of* x.

The particular case with $m = 0$ gives

COROLLARY 1. *If* (1) *holds for* $k = 0, 1, 2$, *then the limit relation* (3) *is valid for every continuous function* $f(x)$ *which is bounded on* $(-\infty, \infty)$.

Also, we easily get

COROLLARY 2. *Denote by* W *the class of all the continuous functions* $f(x)$ *satisfying the condition of the type* $f(x) = O(|x|^N)$ $(x \to \pm\infty)$, $N = 0, 1, 2, \ldots$ *Then the necessary and sufficient condition for* (3) *to be valid for all functions of* W *is that* (1) *should hold for every non-negative* k.

Proof. It is clear that $L_n\big(f(a_n t);\ a_n^{-1}x\big)$ becomes a linear positive operator for large values of n, whenever x belongs to a finite interval. We shall show that (3) holds uniformly for all x in any given interval $[a, b]$.

Given any $\varepsilon > 0$, there is a $\delta > 0$ such that

(4) $$|f(a_n t) - f(x)| < \varepsilon \qquad (a \leqslant x \leqslant b)$$

whenever $|a_n t - x| < \delta$ (i. e. $|t - a_n^{-1}x| < \delta a_n^{-1}$). By condition (2) we can find a positive constant M such that

(5) $$|f(x)| \leqslant M(1 + |x|^m)$$

is valid for all x $(-\infty < x < \infty)$. Consequently, we may infer from (4) and (5) that whatever t may be,

$$\begin{aligned} |f(a_n t) - f(x)| &< \varepsilon + \delta^{-2}(a_n t - x)^2\big(|f(a_n t)| + |f(x)|\big) \\ &\leqslant \varepsilon + \delta^{-2}(a_n t - x)^2 M\big(|a_n t|^m + \max(|a|^m, |b|^m) + 2\big) \\ &= \varepsilon + \delta^{-2}(a_n t - x)^2 (a_n t)^m M + \delta^{-2}(a_n t - x)^2 A, \end{aligned}$$

where $A = M\big(2 + \max(|a|^m, |b|^m)\big)$ and $a \leqslant x \leqslant b$.

L_n being positive and linear, we have

$$\begin{aligned} &\big|L_n\big(f(a_n t) - f(x);\ a_n^{-1}x\big)\big| \\ &\leqslant L_n\big(\varepsilon + \delta^{-2}(a_n t - x)^2(a_n t)^m M + \delta^{-2}(a_n t - x)^2 A;\ a_n^{-1}x\big) \\ &= \varepsilon L_n(1, a_n^{-1}x) + \delta^{-2} M L_n\big((a_n t)^m (a_n t - x)^2;\ a_n^{-1}x\big) + \delta^{-2} A L_n\big((a_n t - x)^2;\ a_n^{-1}x\big). \end{aligned}$$

We want to show that

$$L_n\big((a_n t)^m (a_n t - x)^2;\ a_n^{-1}x\big) \to 0, \qquad L_n\big((a_n t - x)^2;\ a_n^{-1}x\big) \to 0.$$

It suffices to consider the first of the above expressions. Actually, we have

$$\begin{aligned} L_n\big((a_n t)^m (a_n t - x)^2;\ a_n^{-1}x\big) &= L_n\big((a_n t)^{m+2};\ a_n^{-1}x\big) + \\ &\quad + L_n\big((a_n t)^m x^2;\ a_n^{-1}x\big) + L_n\big(-2(a_n t)^{m+1}x;\ a_n^{-1}x\big), \end{aligned}$$

of which the right-hand side will tend uniformly to $x^{m+2} + x^m \cdot x^2 - 2x^{m+1} \cdot x \equiv 0$ with respect to the interval $[a, b]$, in accordance with (1). Hence we may write, in view of (1) with $k = 0$,

(6) $$\big|L_n\big(f(a_n t);\ a_n^{-1}x\big) - f(x) L_n(1;\ a_n^{-1}x)\big| < \varepsilon + \varepsilon_n,$$

where ε_n is a certain sequence decreasing to zero with $1/n$. Since $|f(x)|$ is bounded on $[a, b]$, we may rewrite (6) in the form

(7) $$\big|L_n\big(f(a_n t);\ a_n^{-1}x\big) - f(x)\big| < \varepsilon + \varepsilon_n + \delta_n,$$

where again $\delta_n \downarrow 0$ with $1/n$. Thus (3) is implied by (7).

Irrespective of some technical modifications involved, the proof given above actually follows the same essential idea as that used in the proofs of Korovkin's results. In an analogous manner we can establish the following

THEOREM 2. *Let* a_n *be increasing to* $+\infty$ *with* n, *and let the relation* (1) *hold for all* x *with* $k = 0, 1, 2$. *Then the limit relation* (3) *is valid for all functions of the Lipschitz class* $\mathrm{Lip}\,\alpha$ $(0 < \alpha \leqslant 1)$.

Actually, since the values of t satisfy either the condition $|a_n t - x| < \delta$ or $|a_n t - x| \geqslant \delta$, we easily see that for any given $\varepsilon > 0$, we can find a number $\delta > 0$ such that the inequality

$$|f(a_n t) - f(x)| < \varepsilon + \delta^{\alpha-2}(a_n t - x)^2 \cdot M$$

holds for all values of t, M being the Lipschitz constant for $f(x)$. The rest of the proof is similar to that of Theorem 1.

Theorems 1 and 2 assert that certain sequences of linear operators $L_n\big(f(t); x\big)$ defined on some finite interval of x, sometimes may be modified to such a form $\Phi_n(f; x) \equiv L_n\big(f(a_n t);\ a_n^{-1}x\big)$ that $\Phi_n(f; x)$ becomes capable of approximation to non-bounded continuous functions $f(x)$ defined on $(-\infty, \infty)$. Of course the existence and choice of the required sequence a_n always depend upon the structures of the given operators L_n.

The following are simple examples.

Example 1. Corresponding to

$$B_n\big(f(t); x\big) = \sum_{0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \qquad (0 \leqslant x \leqslant 1)$$

we may introduce the modified Bernstein polynomials of the form

$$B_n(f(a_n t);\, a_n^{-1}x) = \sum_{k=0}^{n} f\left(\frac{a_n k}{n}\right)\binom{n}{k}\left(\frac{x}{a_n}\right)^k\left(1-\frac{x}{a_n}\right)^{n-k}.$$

These polynomials were first investigated by Chlodovsky [1] and were proved to be capable of approximation to continuous functions defined on $(0, \infty)$ with a suitable choice of a_n (see, also, Lorentz [7; p. 36]).

Example 2. By a suitable change of variables the Bernstein polynomials can be defined in the interval $(-1, 1)$, so that a similar modification can be obtained, namely

$$\bar{B}_n(f(a_n t);\, a_n^{-1}x) = 2^{-n}\cdot\sum_{k=0}^{n} f\left(\frac{(2k-n)a_n}{n}\right)\binom{n}{k}(1+a_n^{-1}x)^k(1-a_n^{-1}x)^{n-k},$$

the polynomials being defined on $(-\infty, \infty)$.

Example 3. The Landau polynomials

$$L_n(f(t);\, x) = (n/\pi)^{1/2}\int_0^1 f(t)\,[1-(t-x)^2]^n\,dt \qquad (0 \leqslant x \leqslant 1)$$

may be generalized to the following form:

$$L_n^{(1)}(f(a_n t);\, a_n^{-1}x) = (n/\pi)^{1/2}\int_{-1}^{1} f(a_n t)\,[1-(t-a_n^{-1}x)^2]^n\,dt.$$

Example 4. As a summation analogous to $L_n^{(1)}$ we may also introduce the following type of polynomials:

$$L_n^{(2)}(f(a_n t);\, a_n^{-1}x) = (n\pi)^{-1/2}\sum_{k=-n}^{n} f\left(\frac{a_n k}{n}\right)\left[1-\left(\frac{k}{n}-\frac{x}{a_n}\right)^2\right]^n.$$

Convergence properties of these generalized Landau polynomials have already been investigated (see [3], [4]). Now applying Theorems 1 and 2 to these polynomials, we may get the following more general result:

THEOREM 3. *Let $0 < \theta < \frac{1}{2}$. Then the limit relations*

(7) $$\lim_{n\to\infty} (n/\pi)^{1/2}\int_{-1}^{1} f(n^\theta t)\,[1-(t-n^{-\theta}x)^2]^n\,dt = f(x),$$

(8) $$\lim_{n\to\infty} (n\pi)^{-1/2}\sum_{k=-n}^{n} f\left(\frac{n^\theta k}{n}\right)\left[1-\left(\frac{k}{n}-\frac{x}{n^\theta}\right)^2\right]^n = f(x)$$

are valid for all continuous functions $f(x)$ $(-\infty < x < \infty)$ with $f(x) = O(|x|^N)$ $(x \to \pm\infty)$, N being any large integer. Also (7) *and* (8) *are valid for all* *$f(x)$ of the class* Lip α. *Moreover, in both cases the relations* (7) *and* (8) *hold uniformly on any finite interval of x.*

Condition (1) is satisfied for all non-negative k with $a_n = n^\theta$ $(0 < \theta < \frac{1}{2})$ and can be verified along the same lines as in the proof given in the author's previous note [3] (cf. also the proof of Theorem 3 in [2]). Therefore we may omit details here.

3. Actually, a refinement of the proof given in [3] may lead to a much more general result concerning the convergence of the generalized Landau polynomials. That is, we have the following

THEOREM 4. *Let $0 < \theta < \frac{1}{3}$. Then the limit relation* (7) *is valid for every continuous function $f(x)$ defined on $(-\infty, \infty)$ with $f(x) = O(e^{|x|})$, where $|x| \to \infty$. Moreover, the limit relation holds uniformly on any finite interval.*

Proof. Notice that $f(x) = O(e^{|x|})$ $(|x| \to \infty)$ implies the existence of a positive constant A such that $|f(x)| < A\cdot e^{|x|}$ is true for all x. For each fixed x and every large n let the interval $[-1, 1]$ be divided into three subintervals $\Delta_1 \equiv [-1, \Phi_n(x)-\varepsilon(n)]$, $\Delta_2 \equiv [\Phi_n(x)-\varepsilon(n), \Phi_n(x)+\varepsilon(n)]$, $\Delta_3 \equiv [\Phi_n(x)+\varepsilon(n), 1]$; and as in the case of [3], we write

$$\int_{-1}^{1} f(n^\theta t)\,[1-(t-\Phi_n(x))^2]^n\,dt = \int_{\Delta_1} + \int_{\Delta_2} + \int_{\Delta_3} = J_1(n)+J_2(n)+J_3(n),$$

where $0 < \theta < \frac{1}{3}$, $\Phi_n(x) = n^{-\theta}x$, $\varepsilon(n) = n^{-\theta-\delta}$, δ being chosen to satisfy the condition

(9) $$\max(0, \tfrac{1}{4}-\theta) < \delta < \tfrac{1}{2}-\tfrac{3}{2}\theta.$$

Given any interval $[a, b]$ of x, we want to show that the limit relations

(10) $$\lim_{n\to\infty}(n/\pi)^{1/2}J_1(n) = \lim_{n\to\infty}(n/\pi)^{1/2}J_3(n) = 0,$$

(11) $$\lim_{n\to\infty}(n/\pi)^{1/2}J_2(n) = f(x)$$

hold uniformly on $[a, b]$. It can be shown that for large n and for $-1 \leqslant t \leqslant \Phi_n(x)-\varepsilon(n)$ we have (see [3])

$$|1-(t-\Phi_n(x))^2|^n \leqslant e^{-n^{1-2(\theta+\delta)}}.$$

This inequality is also true for $\Phi_n(x)+\varepsilon(n) \leqslant t \leqslant 1$. Hence we may infer that

$$\begin{aligned} |n^{1/2}J_1(n)| &\leqslant An^{1/2}\int_{\Delta_1} e^{|n^\theta t|}\cdot|1-(t-\Phi_n(x))^2|^n\,dt \\ &\leqslant An^{1/2}e^{n^\theta}\cdot e^{-n^{1-2(\theta+\delta)}}[\Phi_n(x)-\varepsilon(n)+1] \\ &= O(n^{1/2}\cdot e^{n^\theta[1-n^{1-3\theta-2\delta}]}) = o(1) \qquad (n\to\infty), \end{aligned}$$

where $1-3\theta-2\delta>0$ in accordance with (9), the constant factor implied in the term $o(1)$ being independent of x. Similarly we have

$$|n^{1/2}J_3(n)| \leqslant An^{1/2}\int_{\Delta_3} e^{|n^{\theta}t|}\cdot\left[1-\left(t-\Phi_n(x)\right)^2\right]^n dt$$
$$\leqslant An^{1/2}e^{n^{\theta}}\cdot e^{-n^{1-2(\theta+\delta)}}\cdot[1-\Phi_n(x)-\varepsilon(n)] = o(1).$$

In proving (11), it suffices to consider

$$(n/\pi)^{1/2}\int_{\Delta_2}|f(n^{\theta}t)-f(x)|\cdot\left[1-\left(t-\Phi_n(x)\right)^2\right]^n dt.$$

This does not exceed the value

$$\max_{t\in\Delta_2}|f(n^{\theta}t)-f(x)|\cdot(n/\pi)^{1/2}\int_{\Delta_2}\left[1-\left(t-\Phi_n(x)\right)^2\right]^n dt$$
$$=\max_{t\in\Delta_2}|f(n^{\theta}t)-f(x)|\cdot(1+\varepsilon_n),$$

where the term ε_n tends to zero uniformly on the interval $[a, b]$ (cf. [3]). For $t\in\Delta_2$ we clearly have

$$|n^{\theta}t-x| = n^{+\theta}|t-\Phi_n(x)| \leqslant n^{+\theta}\varepsilon(n) = n^{+\theta}\cdot n^{-\theta-\delta} = n^{-\delta}\to 0 \quad (n\to\infty).$$

Hence (11) follows from the uniform continuity of $f(x)$ on $[a, b]$. This completes the proof of the theorem.

Finally, it may be worthy of mention that both Theorems 3 and 4 can easily be extended to the case of several variables. Moreover, in order to have explicit approximating polynomials for the functions of the Lebesgue class $L^p(-\infty,\infty)$ with $p\geqslant 1$, the following modification of $L_n^{(2)}\left(f(a_n t); a_n^{-1}x\right)$ may be proposed (with $a_n = n^{\theta}$, $0<\theta<\frac{1}{2}$):

$$L_n^{(3)}\left(f(n^{\theta}t); n^{-\theta}x\right) = (n/\pi)^{1/2}\sum_{k=-n}^{n-1}\left[1-\left(\frac{k}{n}-\frac{x}{n^{\theta}}\right)^2\right]^n\int_{k/n}^{(k+1)/n} f(n^{\theta}t)\,dt.$$

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