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Regularly increasing functions in connection with the theory of $L^{*\varphi}$-spaces

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In order to investigate the structure of various spaces of φ-integrable functions [1], [6], information on some properties of continuous positive functions as regards the orders of growth of such functions is necessary. The so-called conditions (Δ_a), (Λ_a) (see [5] and [6]) or indices s_φ, σ_φ (see [10] and [9]) occurring in the theory of spaces $L^{*\varphi}(a,b)$ make it possible to compare the function φ with functions t^n. It may be expected that regularly increasing and slowly varying functions, well-known in various problems of asymptotic behaviour of functions, are of importance in the theory of spaces $L^{*\varphi}(a,b)$. The purpose of this paper is to investigate a number of problems connected with the above-mentioned notions. The main stress is laid on a systematic and elementary presentation of the subject, treated as an introduction to the theory of spaces $L^{*\varphi}(a,b)$. Sections 1 and 2 are closely connected with the fundamental papers [2] and [3] of Karamata concerning regularly increasing functions. We avoid integral representations of these functions, starting from the fundamental lemma 1.3 as in [4]. In section 3 the notion of a regularly increasing function appears in connection with functions complementary in the sense of Young. Here, some additions to a theorem of Krasnoselskiĭ and Rutickiĭ [5] are made. Taking in consideration the purposes of this paper we include some results already published, however, somewhat alternatively. Some results of [10], [9] and [7] are also included.

1. In this section we denote by $f, g, h, \ldots$ real functions defined for $-\infty < u < \infty$. We shall also write

$$\bar{\varrho}_f(\mu) = \overline{\lim_{u\to\infty}}\big(f(u+\mu)-f(u)\big), \qquad \underline{\varrho}_f(\mu) = \underline{\lim}_{u\to\infty}\big(f(u+\mu)-f(u)\big);$$

if $\bar{\varrho}_f(\mu) = \underline{\varrho}_f(\mu)$ for a certain μ, we denote this common value by $\varrho_f(\mu)$.

1.1. The following relations are immediately obtained from these definitions:

(a) $$\bar{\varrho}_f(-\mu) = -\underline{\varrho}_f(\mu),$$

(b) $$\underline{\varrho}_f(\mu_1)+\underline{\varrho}_f(\mu_2) \leqslant \underline{\varrho}_f(\mu_1+\mu_2) \leqslant \bar{\varrho}_f(\mu_1+\mu_2) \leqslant \bar{\varrho}_f(\mu_1)+\bar{\varrho}_f(\mu_2);$$

the above inequalities are valid for arbitrary μ_1, μ_2 with the exception of the case when one of the terms of the sum on the right-hand side (or on the left-hand side) is ∞ and the other $-\infty$.

Let $C_f^0 = \{\mu: \varrho_f(\mu) = 0\}$, $C_f = \{\mu: |\varrho_f(\mu)| < \infty\}$, $B_f = \{\mu: \overline{\lim_{u\to\infty}} |f(u+\mu) - f(u)| < \infty\}$. The above relations imply the following:

1.2. *The sets C_f^0, C_f and B_f are rationally linear, i. e. an arbitrary linear combination with integer coefficients of elements of one of these sets belongs to the same set.*

1.3. *Let f be measurable. If*

(a) $\varrho_f(\mu) = 0$ *for an arbitrary* μ, *then*

$$(*) \qquad f(u+\mu) - f(u)$$

tends to zero uniformly in every finite interval of values of μ as $u \to \infty$ (cf. [2], [4]);

(b) $|\bar{\varrho}_f(\mu)| < \infty$, $|\varrho_f(\mu)| < \infty$ *for an arbitrary* μ,

then the functions (*) *are bounded uniformly in every finite interval of values of μ for sufficiently large u.*

In order to prove (a) let us write $E_{n\varepsilon} = \{\mu: |f(u+\mu) - f(u)| \leqslant \varepsilon$, $\mu_1 \leqslant \mu \leqslant \mu_2$, $u \geqslant n\}$. The sets $E_{n\varepsilon}$ are measurable, $\langle \mu_1, \mu_2\rangle = \bigcup_\nu E_{\nu\varepsilon}$. Hence at least one of the sets $E_{n\varepsilon}$ must be of positive measure, say $E_{m\varepsilon}$. If $\mu', \mu'' \in E_{m\varepsilon}$, we have

$$|f(u+\mu''-\mu') - f(u-\mu')| < \varepsilon, \quad |f(u-\mu') - f(u)| < \varepsilon \quad \text{for} \quad u \geqslant m+\mu_2,$$

whence $|f(u+\mu''-\mu') - f(u)| < 2\varepsilon$. As is well known, there is a $\mu_0 > 0$ such that all $\mu \in \langle -\mu_0, \mu_0\rangle$ may be expressed in the form $\mu = \mu'' - \mu'$, where $\mu', \mu'' \in E_{m\varepsilon}$. Since

$$|f(u+\lambda+\mu) - f(u)| \leqslant |f(u+\lambda+\mu) - f(u+\lambda)| + |f(u+\lambda) - f(u)|$$

and $|f(u+\lambda+\mu) - f(u+\lambda)| \leqslant 2\varepsilon$ when μ belongs to $\langle -\mu_0, \mu_0\rangle$, $u \geqslant m + \mu_2 - \lambda$, $|f(u+\lambda) - f(u)| < \varepsilon$ for $u \geqslant u_\lambda$, we have

$$|f(u+\mu') - f(u)| < 3\varepsilon$$

for $u \geqslant \sup(m+\mu_2-\lambda, u_\lambda)$ and for μ' belonging to an interval obtained by a translation of $\langle -\mu_0, \mu_0\rangle$ by λ. Since $\langle \mu_1, \mu_2\rangle$ may be covered by a finite number of intervals which are translations of $\langle -\mu_0, \mu_0\rangle$, we obtain

$$|f(u+\mu) - f(u)| < 3\varepsilon$$

for sufficiently large u and $\mu \in \langle \mu_1, \mu_2\rangle$.

The proof of part (b) of the theorem follows by analogous arguments.

Remark. The above theorem remains true if we replace the assumption of measurability of f by the assumption that f satisfies the Baire condition.

1.4. *If f is continuous (measurable), then any of the sets C_f^0, C_f, B_f is either of the first category (measure 0) or identical with $(-\infty, \infty)$.*

To prove this theorem, let us first note that if f is continuous, then the sets C_f^0, C_f are $F_{\sigma\delta}$ and B_f is F_σ, and if f is measurable, then the above sets are measurable. The theorem follows from the well-known fact that a Borel set of the second category or a set of positive measure contains a rational basis, i. e. a set R such that an arbitrary u may be written in the form $u = n_1u_1 + n_2u_2 + \ldots + n_ru_r$, n_i being integers and $u_i \in R$. Evidently, a rationally linear set containing a rational basis is identical with $(-\infty, \infty)$, and it is sufficient to apply 1.2.

1.5. A function f will be said *to satisfy the condition* (k_0), *resp.* (k), if $C_f^0 = (-\infty, \infty)$, resp. $C_f = (-\infty, \infty)$. Every function of the form $f(u) = g(u) + \int_0^u h(t)\,dt$, where g, h are continuous functions (resp. where g is measurable and h is locally integrable), $g(u) \to c$ as $u \to \infty$, $h(u) \to 0$ as $u \to \infty$, satisfies condition (k_0). Applying 1.3 (a) we may prove (cf. [4]) that, conversely, an arbitrary continuous (resp. locally integrable) function satisfying (k_0) may be written in the above form; $h(u)$ may be assumed to be equal to $f(u+1) - f(u)$. It may be deduced from the integral representation that the set of continuous functions satisfying condition (k_0) and vanishing for $u \leqslant 0$ is a Banach space with the usual definitions of linear operations and with the norm, say

$$\|f\| = \sup_{(0,\infty)} |h(u)| + \sup_{(0,\infty)} \Big|f(u) - \int_0^u h(t)\,dt\Big|, \qquad \text{where} \qquad h(u) = f(u+1) - f(u).$$

1.51. *If f satisfies condition* (k), *then $\varrho_f(\mu)$ is an additive function (as follows from 1.1 (b)); if, moreover, f is measurable, then $\varrho_f(\mu)$ is also measurable, whence $\varrho_f(\mu) = a\mu$. As follows from 1.9, measurability may be replaced by local boundedness of the function f.*

(It is easily seen that some assumptions regarding function f are necessary in this theorem, since Hamel's function f, for example, obviously satisfies condition (k) but $\varrho_f(\mu) = f(\mu)$ is not a linear function.)

An immediate consequence of the above theorem is that an arbitrary measurable (or locally bounded) function satisfying condition (k) may be expressed in the form $f(u) = au + g(u)$, where $g(u)$ satisfies condition (k_0).

1.52. It may happen for a continuous function f that C_f consists only of numbers of the form $n\mu_0$, where $n = \pm 1, \pm 2, \ldots$ In order to get such a function we take, for example, $\mu_0 = 1$ and a continuous periodic function h with period 1. Then

$$\bar{\varrho}_h(\mu) = \sup_{0\leqslant u\leqslant 1} \big(h(u+\mu) - h(u)\big), \qquad \varrho_h(\mu) = \inf_{0\leqslant u\leqslant 1} \big(h(u+\mu) - h(u)\big).$$

If $\mu = 1$, we have $\bar{\varrho}_h(1) = \underline{\varrho}_h(1) = 0$. Now, if we take, for example, $h(u) = \sin 2\pi u$, then $\bar{\varrho}_h(\mu) \neq \underline{\varrho}_h(\mu)$ for $0 < \mu < 1$.

1.53. *If f is a non-decreasing function for $u \geqslant 0$ and if a number $\mu_0 > 0$ belongs to C_f^0, then f satisfies condition* (k_0).

Indeed, we then have $0 \leqslant \underline{\varrho}_f(\mu) \leqslant \bar{\varrho}_f(\mu) \leqslant \bar{\varrho}_f(\mu')$ for $0 \leqslant \mu \leqslant \mu'$, $\varrho_f(n\mu_0) = 0$ for $n = 1, 2, \ldots$

1.54. A function f is called *locally bounded for large u* if there exists u_0 such that f is bounded in every interval $\langle u_0, u_1\rangle$.

If f is measurable and satisfies condition (k_0) *or* (k), *then f is locally bounded for large u.*

Let f satisfy condition (k_0). By 1.3, $|f(u+\mu)-f(u)| < 1$ for $u \geqslant u_0$, $0 \leqslant \mu \leqslant 1$, whence $|f(u)| \leqslant 1+|f(u_0)|$ for $u \in \langle u_0, u_0+1\rangle$, and, more generally, $|f(u)| \leqslant 1+|f(u_0+n-1)|$ for $u \in \langle u_0+n-1, u_0+n\rangle$, $n = 1, 2, \ldots$ If f satisfies condition (k), then $\varrho_f(\mu) = a\mu$ and the function $f(u)-au$ satisfies condition (k_0), whence it is locally bounded for large u, and so is f.

Remark. The assumption that (k) is satisfied may be replaced in this theorem by the assumption $B_f = (-\infty, \infty)$.

1.6. *Let us assume that f is locally bounded for large u. Then the following inequalities hold:*

$$(+) \qquad \frac{\underline{\varrho}_f(\mu)}{\mu} \leqslant \varliminf_{u\to\infty} \frac{f(u)}{u} \leqslant \varlimsup_{u\to\infty} \frac{f(u)}{u} \leqslant \frac{\bar{\varrho}_f(\mu)}{\mu} \quad \textit{for} \quad \mu > 0,$$

$$(++) \qquad \frac{\bar{\varrho}_f(\mu)}{\mu} \leqslant \varliminf_{u\to\infty} \frac{f(u)}{u} \leqslant \varlimsup_{u\to\infty} \frac{f(u)}{u} \leqslant \frac{\underline{\varrho}_f(\mu)}{\mu} \quad \textit{for} \quad \mu < 0.$$

We give the proof of this classical theorem for completeness, and also because it is sometimes quoted without the exact formulation of the assumptions. Let $\mu > 0$, $\varepsilon > 0$, $\bar{\varrho}_f(\mu) < \infty$. Since

$$f(u+l\mu)-f(u) = [f(u+l\mu)-f(u+(l-1)\mu)]+\ldots+[f(u+\mu)-f(u)]$$
$$< l\bar{\varrho}_f(\mu)+l\varepsilon \quad \text{for} \quad u \geqslant u_0(\varepsilon) \text{ and } l = 1, 2, \ldots,$$

we have

$$f(v)-f(v-l\mu) < l\bar{\varrho}_f(\mu)+l\varepsilon$$

for $u = v-l\mu \geqslant \sup(u_0(\varepsilon), u_0) = \bar{u}$, where u_0 denotes u_0 mentioned in 1.54. Taking $l(v)$ so that $\bar{u} \leqslant v-l\mu \leqslant \bar{u}+\mu$ and $v \to \infty$ we obtain $v/l \to \mu$, $\sup_{\bar{u}\leqslant u\leqslant \bar{u}+\mu} |f(u)|/v \to 0$, whence

$$\varlimsup_{v\to\infty} \frac{f(v)}{v} \leqslant \frac{\bar{\varrho}_f(\mu)}{\mu} + \frac{\varepsilon}{\mu}.$$

The second of the inequalities $(+)$ is proved analogously. 1.1 (a) and $(+)$ immediately imply $(++)$.

Let us remark that the local boundedness of f for large u is a necessary condition of $(+)$ in the case when $-\infty < \underline{\varrho}_f(\mu) \leqslant \bar{\varrho}_f(\mu) < \infty$. Indeed, if f is not locally bounded for large u, we have $\sup_{u_n\leqslant u\leqslant u_n+1} |f(u)| = \infty$ for a sequence $u_n \nearrow \infty$. However, for v_n suitably chosen, $u_n \leqslant v_n \leqslant u_n+1$, we then have $\lim_{v_n\to\infty} |f(v_n)|/v_n = \infty$.

1.7. Given a positive μ, denote by a_μ, resp. b_μ, numbers satisfying the inequalities

$$f(u+\mu)-f(u) \geqslant a_\mu \quad \text{for} \quad u \geqslant u_1(\mu),$$

resp.

$$f(u+\mu)-f(u) \leqslant b_\mu \quad \text{for} \quad u \geqslant u_2(\mu).$$

Let us assume that f satisfies one of the conditions

(a) $-\infty < \underline{\varrho}_f(\mu) \leqslant \bar{\varrho}_f(\mu) < \infty$ for every μ, f is measurable,

(b) f is monotone for $u \geqslant 0$.

The following formulae hold:

$$(*) \qquad \lim_{\mu\to\infty} \frac{\varrho_f(\mu)}{\mu} = \sup_{\mu>0} \frac{\varrho_f(\mu)}{\mu} = \sup_{\mu>0} \frac{a_\mu}{\mu};$$

$$(**) \qquad \lim_{\mu\to\infty} \frac{\bar{\varrho}_f(\mu)}{\mu} = \inf_{\mu>0} \frac{\bar{\varrho}_f(\mu)}{\mu} = \inf_{\mu>0} \frac{b_\mu}{\mu}.$$

As regards the meaning of the symbols $\sup_{\mu>0} a_\mu/\mu$, $\inf_{\mu>0} b_\mu/\mu$, the following convention is here adopted: if there exists a finite value of a_μ (resp. b_μ), we take the supremum (resp. infimum) with respect to all possible choices of a_μ and μ (resp. b_μ and μ), where $\mu > 0$. In other case we put $\sup_{\mu>0} a_\mu/\mu = -\infty$ ($\inf_{\mu>0} b_\mu/\mu = \infty$).

We shall prove the first formula for instance. Assumption (a) means that $B_f = (-\infty, \infty)$ and, by 1.3 (b), for every $\mu_0 > 0$ there exist k, u_0 such that

$$(+) \qquad |f(u+\mu)-f(u)| \leqslant k \quad \text{for} \quad 0 \leqslant \mu \leqslant \mu_0,\ u \geqslant u_0.$$

If $f(u+\mu_0)-f(u) \geqslant a_{\mu_0}$ for $u \geqslant u_1(\mu_0)$, then

$$f(u+n\mu_0)-f(u) \geqslant na_{\mu_0} \quad \text{for} \quad u \geqslant u_1(\mu_0) \text{ and } n = 1, 2, \ldots$$

Hence, choosing $(n-1)\mu_0 \leqslant \mu < n\mu_0$, we obtain

$$f(u+\mu)-f(u) \geqslant na_{\mu_0}+f(u+\mu)-f(u+n\mu_0).$$

Since

$$n a_{\mu_0} > \frac{\mu}{\mu_0} a_{\mu_0} \ \text{for}\ a_{\mu_0} > 0 \quad \text{and} \quad n a_{\mu_0} \geqslant \left(\frac{\mu}{\mu_0}+1\right) a_{\mu_0} \ \text{for}\ a_{\mu_0} \leqslant 0,$$

we have

$$\frac{f(u+\mu)-f(u)}{\mu} \geqslant \frac{a\mu_0}{\mu_0} + \frac{f(u+\mu)-f(u+n\mu_0)}{\mu} \quad \text{for} \quad u \geqslant u_1(\mu_0)$$

and for $a_{\mu_0} > 0$. By (+), we obtain

$$\frac{\underline{\varrho}_f(\mu)}{\mu} \geqslant \frac{a\mu_0}{\mu_0} - \frac{k}{\mu},$$

whence

$$\underline{\lim}_{\mu\to\infty} \frac{\underline{\varrho}_f(\mu)}{\mu} \geqslant \frac{a\mu_0}{\mu_0}.$$

The proof of this inequality for $a_{\mu_0} \leqslant 0$ is similar. Since μ_0 is an arbitrary positive number, we have

$$\underline{\lim}_{\mu\to\infty} \underline{\varrho}_f(\mu)/\mu \geqslant \sup_{\mu>0} a_\mu/\mu.$$

Take any $s < \sup_{\mu>0} \underline{\varrho}_f(\mu)/\mu$. Then $\underline{\varrho}_f(\mu_0)/\mu_0 > s$ for a certain $\mu_0 > 0$, whence

$$f(u+\mu_0)-f(u) \geqslant s\mu_0 = a_{\mu_0} \quad \text{for} \quad u \geqslant u_1(\mu_0),$$

i. e.

$$\sup_{\mu>0} \frac{a_\mu}{\mu} \geqslant \frac{a\mu_0}{\mu_0} = s.$$

Thus we have proved

$$\sup_{\mu>0} \frac{a_\mu}{\mu} \geqslant \sup_{\mu>0} \frac{\underline{\varrho}_f(\mu)}{\mu} \geqslant \overline{\lim}_{\mu\to\infty} \frac{\underline{\varrho}_f(\mu)}{\mu}.$$

The proof of formula (**) is similar.

Now, let us assume (b) to be satisfied. Then we obtain

$$f(u+\mu)-f(u) \geqslant (n-1)a_{\mu_0} + f(u+\mu) - f\big(u+(n-1)\mu_0\big) \geqslant (n-1)a_{\mu_0}$$

for $u \geqslant u_1(\mu_0)$ and any μ satisfying the inequalities $(n-1)\mu_0 \leqslant \mu < n\mu_0$ if f is non-decreasing for $u \geqslant 0$. If f is non-increasing for $u \geqslant 0$, we have $f(u+\mu)-f(u) \geqslant n a_{\mu_0}$ for $u \geqslant u_1(\mu_0)$. Arguments analogous to the preceding ones lead to the inequalities $\underline{\lim}_{\mu\to\infty} \underline{\varrho}_f(\mu)/\mu \geqslant \sup_{\mu>0} a_\mu/\mu$ and the further arguments do not differ from these in the proof under assumption (a).

If f is non-increasing for $u \geqslant 0$, then for $\mu > 0$ a finite constant a_μ may not exist. This is possible if and only if $\underline{\varrho}_f(\mu) = -\infty$ for $\mu > 0$. In this case we have $\lim_{\mu\to\infty} \underline{\varrho}_f(\mu)/\mu = \sup_{\mu>0} \underline{\varrho}_f(\mu)/\mu = -\infty$.

The following statements are consequences of 1.54 (Remark), 1.6 and 1.7:

1.71. *By the assumption that either f is measurable and $-\infty < \underline{\varrho}_f(\mu) \leqslant \bar{\varrho}_f(\mu) < \infty$ for every μ or f is monotone, the following inequalities are satisfied:*

$$(*) \qquad \underline{\lim}_{\mu\to\infty} \frac{\underline{\varrho}_f(\mu)}{\mu} \leqslant \underline{\lim}_{u\to\infty} \frac{f(u)}{u} \leqslant \overline{\lim}_{u\to\infty} \frac{f(u)}{u} \leqslant \overline{\lim}_{u\to\infty} \frac{\bar{\varrho}_f(\mu)}{\mu}.$$

Remark. This remains true also in the case when f is locally bounded for large u and the limits $\lim_{\mu\to\infty} \bar{\varrho}_f(\mu)/\mu$ and $\lim_{\mu\to\infty} \underline{\varrho}_f(\mu)/\mu$ exist.

1.72. *If f is locally bounded for large u and if*

$$(**) \qquad \underline{\lim}_{\mu\to\infty} \frac{\underline{\varrho}_f(\mu)}{\mu} = \overline{\lim}_{\mu\to\infty} \frac{\bar{\varrho}_f(\mu)}{\mu} = g$$

(g may also be equal to ∞), then the relation

$$\lim_{u\to\infty} \frac{f(u)}{u} = g$$

(the generalized l'Hospital rule in Cauchy's form) holds.

1.8. *Let f possess a positive derivative f' for $u \geqslant 0$ and let f' satisfy the condition*

$$(\text{o}) \qquad \lim_{u\to\infty} \frac{f'(u+\mu)}{f'(u)} = 1 \quad \textit{for every} \quad \mu.$$

Then

$$\overline{\lim}_{u\to\infty} \frac{f(u)}{u} = \overline{\lim}_{u\to\infty} f'(u) = \frac{\bar{\varrho}_f(\mu_0)}{\mu_0} \quad \textit{for every} \quad \mu_0 > 0,$$

$$\underline{\lim}_{u\to\infty} \frac{f(u)}{u} = \underline{\lim}_{u\to\infty} f'(u) = \frac{\underline{\varrho}_f(\mu_0)}{\mu_0} \quad \textit{for every} \quad \mu_0 > 0.$$

Given $u \geqslant 0$ and $\mu_0 > 0$, denote by $v(u)$ a number satisfying the conditions $v(u) \in (0, \mu_0)$ and such that

$$f(u+\mu_0) - f(u) = f'\big(u+v(u)\big)\mu_0$$

holds. Then we have

$$\overline{\lim}_{u\to\infty} f'\big(u+v(u)\big) = \frac{\bar{\varrho}_f(\mu_0)}{\mu_0}.$$

Define the function

$$h(u) = \begin{cases} \lg f'(u) & \text{for} \quad u \geqslant 0, \\ \lg f'(0) & \text{for} \quad u < 0. \end{cases}$$

By (o), h satisfies condition (k_0), whence, by 1.3 (a), $h(u+\mu)-h(u)\to 0$ uniformly in $\langle 0,\mu_0\rangle$ as $u\to\infty$, i. e. (o) holds uniformly with respect to $\mu\in\langle 0,\mu_0\rangle$. Thus

$$\overline{\lim_{u\to\infty}} f'\big(u+v(u)\big)=\overline{\lim_{u\to\infty}} f'(u), \quad \underline{\lim_{u\to\infty}} f'\big(u+v(u)\big)=\underline{\lim_{u\to\infty}} f'(u).$$

1.81. *By the same assumptions regarding f as in* 1.8, *if* 1.72 (**) *holds, then the limit of the derivative as $u\to\infty$ exists, namely,*

$$\lim_{u\to\infty} f'(u)=g.$$

1.9. *By the same assumptions regarding f as in* 1.6, *we have*

$$\varrho_f(\mu)=a\mu \quad \text{for} \quad \mu\in C_f;$$

in particular, if f satisfies (k), *then* $\varrho_f(\mu)=a\mu$ *for* $-\infty<\mu<\infty$.

This follows from the fact that, by 1.6, $\varrho_f(\mu)=a\mu$ for $\mu\in C_f$, $\mu\neq 0$, where $a=\lim_{u\to\infty} f(u)/u$.

2. In this section (with the exception of 2.12 and 2.8) $\varphi,\psi,\chi,\varrho,\dots$ always denote measurable positive functions defined for $u>0$. According to [6], such a function is called a *φ-function* if it is continuous and non-decreasing, defined for $u=0$ by $\varphi(0)=0$, and tends to infinity as $u\to\infty$. We shall apply the symbols

$$\underline{h}_\varphi(\lambda)=\underline{\lim_{u\to\infty}}\frac{\varphi(u)}{\varphi(\lambda u)}, \quad \overline{h}_\varphi(\lambda)=\overline{\lim_{u\to\infty}}\frac{\varphi(u)}{\varphi(\lambda u)} \quad \text{for} \quad \lambda>0.$$

If $\underline{h}_\varphi(\lambda)=\overline{h}_\varphi(\lambda)=h_\varphi(\lambda)$, where $h_\varphi(\lambda)$ is finite for $\lambda>0$, and $h\not\equiv 1$, we call φ *regularly increasing*, according to the terminology of [2] and [3]. If $h_\varphi(\lambda)=1$ for $\lambda>0$, φ is a *slowly varying* function. (In the terminology of [2], also slowly varying functions are regularly increasing.) Substituting

$$(*\varphi) \qquad f(u)=\lg\varphi(e^u), \quad -\infty<u<\infty,$$

we reduce the investigation of functions φ to the functions we have considered in section 1 and a number of theorems may be obtained immediately by applying the results of section 1. It is clear that φ is regularly increasing, resp. slowly varying, if and only if the corresponding function $f(u)$ satisfies condition (k), resp. (k_0). If $e^\mu=\lambda$, then $\lg\overline{h}_\varphi(\lambda)=-\underline{\varrho}_f(\mu)=\overline{\varrho}_f(-\mu)$, and similarly $\lg\underline{h}_\varphi(\lambda)=\underline{\varrho}_f(-\mu)$. If $\lambda\to 0+$, then $-\mu=-\lg\lambda\to\infty$, and applying 1.7 we obtain for an arbitrary φ-function the existence of the following limits:

$$(*) \qquad s_\varphi=\lim_{\lambda\to 0+}\frac{\lg\underline{h}_\varphi(\lambda)}{-\lg\lambda}, \quad \sigma_\varphi=\lim_{\lambda\to 0+}\frac{\lg\overline{h}_\varphi(\lambda)}{-\lg\lambda}.$$

The indices s_φ, σ_φ play a part in the theory of the spaces $L^{*\varphi}(a,b)$ ([10], [9]). Obviously, we have $\sigma_\varphi\geqslant s_\varphi\geqslant 0$ for an arbitrary φ-function. Indices (*) may exist also for φ which are not φ-functions (in the terminology of [9]; such φ are called *quasi φ-functions* or briefly $q\varphi$-functions). By 1.7, every non-increasing or non-decreasing φ is a $q\varphi$-function. If φ is regularly increasing then $r_\varphi=s_\varphi=\sigma_\varphi\neq 0$ (r_φ is called the *index of regularity*); if φ is slowly varying then $r_\varphi=s_\varphi=\sigma_\varphi=0$. (In the following we term φ to be *of index* r_φ if either φ is regularly increasing, i. e. $r_\varphi\neq 0$, or φ is slowly varying, i. e. $r_\varphi=0$.) This is obvious for slowly varying functions, since then $\overline{h}_\varphi(\lambda)=h_\varphi(\lambda)=1$, and follows from 2.1 for regularly increasing functions.

2.1. *φ is regularly increasing with the index of regularity r if and only if*

$$(**) \qquad \varphi(u)=u^r\psi(u),$$

where $r\neq 0$ and ψ is slowly varying (see [2]).

The easy proof of sufficiency will be omitted. To prove the necessity we apply 1.51 and we decompose the function $(*\varphi)$ into a sum of a linear function and a function f_0 satisfying condition (k_0). If f_0 satisfies condition (k_0) then $e^{f_0(\lg u)}=\psi(u)$ is slowly varying.

Let us remark in connection with the assumption of measurability of φ in the above theorem that $\varphi_0(u)=e^{h(\lg u)}$, where $h(u)$ is a non-measurable Hamel function, is not regularly increasing and the indices s_{φ_0}, σ_{φ_0} do not exist, although $h_{\varphi_0}(\lambda)=\overline{h}_{\varphi_0}(\lambda)=e^{-h(\lg\lambda)}$, $\lg\underline{h}_{\varphi_0}(\lambda)/-\lg\lambda=h(-\lg\lambda)/-\lg\lambda$ for every $\lambda>0$. If a finite limit s_{φ_0} existed, h would be bounded in a certain interval, whence continuous. If s_{φ_0} were equal to $\pm\infty$, h would be bounded from below (from above) in a certain interval, but this is impossible. Thus the index s_{φ_0} does not exist; it is similarly proved that σ_{φ_0} also does not exist.

If $s_\varphi=\sigma_\varphi=r_\varphi$, where $r_\varphi\neq 0$, $|r_\varphi|<\infty$ for a $q\varphi$-function φ, we call φ *quasi-regularly increasing*; if $s_\varphi=\sigma_\varphi=r_\varphi=0$, we call the $q\varphi$-function φ *quasi-slowly varying*. Also in this case r_φ is called the index (of quasi-regularity). If $s_\varphi=\sigma_\varphi=\pm\infty$, we say that φ is of *infinite index* of quasi-regularity and write $r_\varphi=\pm\infty$.

2.11. *If φ is regularly increasing or slowly varying, then*

$$(+) \qquad \frac{\varphi(\lambda u)}{\varphi(u)}\to\lambda^{r_\varphi}, \quad \textit{as} \quad u\to\infty,$$

uniformly in every interval $0<\lambda'\leqslant\lambda\leqslant\lambda''$.

We apply the substitution $(*\varphi)$ and 1.51, 1.3. We obtain $\varphi(\lambda u)/\varphi(u)\to\lambda^{r_\varphi}$ if $\lambda>0$, $u\to\infty$. Applying the definition of the indices s_φ, σ_φ we obtain $s_\varphi=\sigma_\varphi=r_\varphi$.

2.12. If we replace the assumption of measurability of φ in the definition of a regularly increasing or slowly varying function by the local boundedness of $\lg\varphi$, relation (+) remains true for every $\lambda > 0$, although the uniform convergence may not hold. Identity 2.1 (**), where φ is of index $r_\varphi = 0$, remains also true.

This follows by applying the substitution $(*\varphi)$ and 1.9.

2.13. *If φ is regularly increasing (quasi-regularly increasing) or slowly varying (quasi-slowly varying), then $\lg\varphi$ is locally bounded for large u.*

This follows by applying the substitution $(*\varphi)$ and 1.54.

A function φ is called *locally bounded* if it is bounded in an arbitrary interval $(0, v)$. It follows from 2.13 that replacing a regularly increasing (slowly varying) function φ by a function φ_1 such that $\varphi_1(u) = \varphi(\bar{u})$ for $0 < u < \bar{u}$, $\varphi_1(u) = \varphi(u)$ for $u \geqslant \bar{u}$, where $\bar{u}$ is sufficiently large, we obtain a regularly increasing (slowly varying) function which is locally bounded.

2.2. *Let $\lg\varphi$ be locally bounded for large u.*

(a) *If the limits s_φ, σ_φ exist, then*

$$s_\varphi \leqslant \varliminf_{u\to\infty} \frac{\lg\varphi(u)}{\lg u} \leqslant \varlimsup_{u\to\infty} \frac{\lg\varphi(u)}{\lg u} \leqslant \sigma_\varphi.$$

(b) *If the limits s_φ, σ_φ exist and $s_\varphi = \sigma_\varphi = r_\varphi$ (in particular, if φ is regularly increasing or slowly varying), then*

$$\lim_{u\to\infty} \frac{\lg\varphi(u)}{\lg u} = r_\varphi.$$

The above theorems are obtained immediately by applying the substitution $(*\varphi)$ and 1.54, 1.72.

2.3. Denote by $\mathscr{R}$, resp. $\mathscr{R}_0$, the class of regularly increasing, resp. slowly varying, functions φ.

(a) *If $\varphi, \psi \in \mathscr{R}$, then $r_{\varphi\psi} = r_\varphi + r_\psi$ and $\varphi\psi \in \mathscr{R}$ for $r_{\varphi\psi} \neq 0$ and $\varphi\psi \in \mathscr{R}_0$ for $r_{\varphi\psi} = 0$.*

(b) *If $\varphi, \psi \in \mathscr{R}_0$, then $\varphi\psi \in \mathscr{R}_0$.*

(c) *If $\varphi \in \mathscr{R}$, then $r_{1/\varphi} = -r_\varphi$ and $1/\varphi \in \mathscr{R}$.*

(d) *If $\varphi \in \mathscr{R}_0$, then $1/\varphi \in \mathscr{R}_0$.*

(e) *If $\varphi \in \mathscr{R}$, then $r_{\varphi^k} = kr_\varphi$, $\varphi^k \in \mathscr{R}$, when $k \neq 0$.*

(f) *If $\varphi \in \mathscr{R}_0$, then $\varphi^k \in \mathscr{R}_0$ for an arbitrary real k.*

(g) *If $\varphi, \psi \in \mathscr{R}$, $\psi(u) \to \infty$ as $u \to \infty$, then $\varphi(\psi) \in \mathscr{R}$, $r_{\varphi(\psi)} = r_\varphi r_\psi$.*

(h) *By the same assumptions on φ, ψ as in (g), if at least one of the functions φ, ψ belongs to $\mathscr{R}_0$ and the second one belongs to $\mathscr{R}$, then $\varphi(\psi) \in \mathscr{R}_0$.*

(i) *If φ is a strictly increasing φ-function and $\varphi \in \mathscr{R}$, then $\varphi^{-1} \in \mathscr{R}$ and $r_{\varphi^{-1}} = 1/r_\varphi$* (cf. [2]).

Remark. The above theorems remain true if we omit the assumption of measurability in the definition of a regularly increasing, resp. slowly varying, function, replacing it in (a), (c), (e) by the assumption of local boundedness of the functions $\lg\varphi$, $\lg\psi$ in (g) by the assumption of measurability of φ and local boundedness of $\lg\psi$.

Theorems (a)—(f) follow from the definition of a regularly increasing, resp. slowly varying, function and from 2.12 immediately.

Ad (g). Let $\psi(\lambda u) = \varepsilon(u)\psi(u)$, $\lambda > 0$, whence $\varepsilon(u) \to \lambda^{r_\psi}$ as $u \to \infty$. By 2.11, $\varphi(\mu u)/\varphi(u) \to \mu^{r_\varphi}$, as $u \to \infty$, uniformly in each interval $0 < \mu' \leqslant \mu \leqslant \mu''$. Hence

$$\frac{\varphi(\psi(\lambda u))}{\varphi(\psi(u))} = \frac{\varphi(\varepsilon(u)\psi(u))}{\varphi(\psi(u))} \to (\lambda^{r_\psi})^{r_\varphi} \quad \text{as} \quad u \to \infty.$$

Thus $\varphi(\psi)$ is regularly increasing and $r_{\varphi(\psi)} = r_\varphi r_\psi$.

(h) is proved similarly.

Ad (i). Let $\varphi(u) = v$, $\varphi^{-1}(v) = u$, $1 < \mu < \infty$, $\varphi(\lambda_v u) = \mu v$, where $\lambda_v > 1$ is defined uniquely. There exists a constant λ_0 such that $\lambda_v \leqslant \lambda_0$ for $v \geqslant v_0$; indeed, otherwise we should have $\lambda_{v_n} \geqslant (2\mu)^{1/r_\varphi}$ for a sequence $v_n \to \infty$. If $\varphi^{-1}(v_n) = u_n$, then

$$\mu = \frac{\varphi(\lambda_{v_n} u_n)}{\varphi(u_n)} \geqslant \frac{\varphi((2\mu)^{1/r_\varphi} u_n)}{\varphi(u_n)} \to ((2\mu)^{1/r_\varphi})^{r_\varphi},$$

which is a contradiction. Let $\lambda_{v_n} \to g$; since, by 2.11, $\varphi(\lambda u)/\varphi(u) \to \lambda^{r_\varphi}$ uniformly in $1 \leqslant \lambda \leqslant \lambda_0$, we obtain

$$\mu = \frac{\varphi(\lambda_{v_n} u_n)}{\varphi(u_n)} \to g^{r_\varphi},$$

i. e. $g = \mu^{1/r_\varphi}$. We have thus proved that $\lambda_v \to \mu^{1/r_\varphi}$ as $v \to \infty$; thus $\varphi^{-1}(\mu v)/\varphi^{-1}(v) = \lambda_v u/u \to \mu^{1/r_\varphi}$ for $\mu > 1$ and hence for an arbitrary $\mu > 0$.

2.31. *If φ is a strictly increasing φ-function and $\varphi \in \mathscr{R}_0$, then the following relation holds for the inverse function:*

$$(+) \qquad \frac{\varphi^{-1}(\mu u)}{\varphi^{-1}(u)} \to \infty \quad \textit{as} \quad u \to \infty,$$

for every $\mu > 1$. Conversely, if (+) holds, then $\varphi \in \mathscr{R}_0$.

Let μ, λ_v have the same meaning as in the proof of 2.3 (i). Suppose $\lambda_{v_n} \to g$ for a sequence $v_n \to \infty$, where g is a finite limit. Since $\varphi(\lambda u)/\varphi(u) \to 1$ uniformly in $1 \leqslant \lambda \leqslant g + \varepsilon$, we have $\mu = \varphi(\lambda_{v_n} u_n)/\varphi(u_n) \to 1$, i. e. $\mu = 1$, which is a contradiction. Thus we have proved that $\lambda_v \to \infty$ as $v \to \infty$ and, consequently, (+).

In order to prove the second part of the theorem let us write $\varphi(\lambda u) = \mu_u v$, $\lambda \geqslant 1$. Since $\lambda = \varphi^{-1}(\mu_u v)/\varphi^{-1}(v)$, we have $\mu_u \to 1$ as $u \to \infty$, by (+). Thus $\varphi(\lambda u)/\varphi(u) \to 1$ for $\lambda \geqslant 1$ and hence also for $0 < \lambda \leqslant 1$.

2.4. We now introduce some notions which are of importance, particularly in the theory of the spaces $L^{*\varphi}(a, b)$, but which are also of interest in studying the order of growth of functions. We shall say that φ is *l-equivalent to ψ (equivalent to ψ for large u)*, in symbols $\varphi \overset{l}{\sim} \psi$, if the inequalities

$$(+) \qquad a\varphi(k_1 u) \leqslant \psi(u) \leqslant b\varphi(k_2 u)$$

hold for $u \geqslant u_0$, where a, b, k_1, k_2 are some positive constants (see [6]). $\varphi \sim \psi$, resp. $\varphi \simeq \psi$, will mean that φ and ψ are asymptotically similar, resp. asymptotically equal, i. e. that $\varphi(u)/\psi(u) \to c$ as $u \to \infty$, where $c \neq 0$, resp. $c = 1$.

Evidently, $\varphi \sim \psi$ implies $\varphi \overset{l}{\sim} \psi$ but not conversely. Similarly to $\sim$, $\overset{l}{\sim}$ is also an equivalence relation and elementary rules of calculus for $\sim$ are valid also for $\overset{l}{\sim}$. For instance if φ, ψ are non-decreasing (non-increasing), $\varphi \overset{l}{\sim} \varphi_1$, $\psi \overset{l}{\sim} \psi_1$, then $c'\varphi + c''\psi \overset{l}{\sim} c'\varphi_1 + c''\psi_1$ $(c', c'' > 0)$, $\varphi\psi \overset{l}{\sim} \varphi_1\psi_1$, etc. If φ is a $q\varphi$-function, then every function l-equivalent to φ is also a $q\varphi$-function. It is also easily seen that for a $q\varphi$-function the indices s_φ, σ_φ are invariants of the relation $\overset{l}{\sim}$. However, the property that φ is regularly increasing (resp. slowly varying) remains valid for an asymptotically similar (resp. equal) function, but in general does not remain valid for a function l-equivalent to the given one. The following remark makes clear the advantage of applying the notion of l-equivalence in place of the less general notion of asymptotic equality, when investigating orders of growth of functions.

2.41. *If φ, ψ are strictly increasing φ-functions, $\varphi \overset{l}{\sim} \psi$, then $\varphi^{-1} \overset{l}{\sim} \psi^{-1}$* (see [9]).

If $\varphi \simeq \psi$, then $\varphi^{-1} \sim \psi^{-1}$ does not need to hold. For instance the functions $\varphi(u) = \lg(1+u)$, $\varphi_1(u) = \varrho(u)\lg(1+u)$, where $\varrho(u)$ is a continuous function strictly increasing from 0 to 1, are asymptotically equal. However, if we choose ϱ suitably, their inverse functions are not asymptotically similar. It is sufficient to choose an arbitrary sequence $a_n \nearrow 1$ and v_n, u_n', u_n so that $u_n' < u_n < u_n' + 1 < u_{n+1}$, $u_n = e^{v_n/a_n} - 1$, $(e^{v_n/a_n} - 1)(e^{v_n} - 1)^{-1} > n$, and to define $\varrho(u) = a_n$ for $u_n' \leqslant u \leqslant u_n$, $\varrho(u) =$ a linear function in (u_n, u_{n+1}').

2.42. *If φ is a $q\varphi$-function, then $\bar{\varphi}_r(u) = \varphi(u^r)$ and $\bar{\bar{\varphi}}_r(u) = (\varphi(u))^r$, $r > 0$, are $q\varphi$-functions, and $s_{\bar{\varphi}_r} = s_{\bar{\bar{\varphi}}_r} = rs_\varphi$, $\sigma_{\bar{\varphi}_r} = \sigma_{\bar{\bar{\varphi}}_r} = r\sigma_\varphi$.*

Since

$$\frac{\bar{\varphi}_r(u)}{\bar{\varphi}_r(\lambda u)} = \frac{\varphi(u^r)}{\varphi(\lambda^r u^r)}, \qquad \frac{\bar{\bar{\varphi}}_r(u)}{\bar{\bar{\varphi}}_r(\lambda u)} = \left(\frac{\varphi(u)}{\varphi(\lambda u)}\right)^r,$$

we have $\underline{h}_{\bar{\varphi}_r}(\lambda) = \underline{h}_\varphi(\lambda^r)$, $\bar{h}_{\bar{\varphi}_r}(\lambda) = \bar{h}_\varphi(\lambda^r)$, $\underline{h}_{\bar{\bar{\varphi}}_r}(\lambda) = (\underline{h}_\varphi(\lambda))^r$, $\bar{h}_{\bar{\bar{\varphi}}_r}(\lambda) = (\bar{h}_\varphi(\lambda))^r$,

$$\frac{1}{r} s_{\bar{\varphi}_r} = \frac{1}{r} \lim_{\lambda \to 0+} \frac{\lg \underline{h}_{\bar{\varphi}_r}(\lambda)}{-\lg \lambda} = \lim_{\lambda \to 0+} \frac{\lg \underline{h}_\varphi(\lambda^r)}{-\lg \lambda^r} = s_\varphi,$$

and similarly in the remaining cases.

Let us note that, in spite of the fact that the indices of $\bar{\varphi}_r$ and $\bar{\bar{\varphi}}_r$ are equal, these functions need not be l-equivalent if $r \neq 1$. For instance let $\varphi(u) = \psi(\lg(1+u))$, where ψ is a regularly increasing or slowly varying φ-function. Then $\varphi(u^r)/\varphi(u) \to r^{s_\psi}$ as $u \to \infty$ for $r > 0$. If $\bar{\varphi}_r \overset{l}{\sim} \bar{\bar{\varphi}}_r$, then $\varphi \overset{l}{\sim} \varphi^r$, i. e. $a\varphi(k_1 u)/\varphi(u) \leqslant (\varphi(u))^r/\varphi(u) \leqslant b\varphi(k_2 u)/\varphi(u)$ for large u. But this is impossible for $r \neq 1$, because, according to 2.3 (g), φ is a slowly varying φ-function.

2.5. A function φ is said *to satisfy condition* (Δ_a) *for large u* if $a > 1$ and if the inequality

$$\varphi(au) \leqslant d_a \varphi(u) \quad \text{for} \quad u \geqslant u_0(a)$$

holds for a constant $d_a > 1$. φ is said *to satisfy condition* (Λ_a) *for large u* if $a > 1$ and if the inequality

$$\varphi(au) c_a \leqslant \varphi(u) \quad \text{for} \quad u \geqslant u_0(a)$$

is satisfied for a constant $c_a > 1$. For non-decreasing φ the property that condition (Δ_a) (condition (Λ_a)) holds with an $a > 1$ is an invariant of l-equivalence (cf. [6]).

2.51. *If φ is a $q\varphi$-function, then the conditions*

(a) $s_\varphi > 0$,

(a′) (Λ_a) *is satisfied for sufficiently large* a,

are equivalent, and the conditions

(b) $\sigma_a < \infty$,

(b′) (Δ_a) *is satisfied for sufficiently large* a,

are also equivalent.

In order to prove (a) $\rightleftarrows$ (a′) let us note that $\underline{h}_\varphi(\lambda) = \underline{h}^*_\varphi(a) = \varliminf_{u \to \infty} \varphi(au)/\varphi(u)$, if $a = 1/\lambda$, $0 < \lambda < 1$. If $s_\varphi > 0$ and $s_\varphi > s > 0$, then $\lg \underline{h}^*_\varphi(a) > s \lg a$ for $a \geqslant a_0$, whence $\varphi(au) \geqslant a^s \varphi(u)$ for $u \geqslant u_0(a)$, i. e. we may take $a^s = c_a > 1$. If φ satisfies condition (Λ_{a_0}), then $\varphi(a_0 u) \geqslant c_{a_0}\varphi(u)$ for $u \geqslant u(a_0)$, $a_0 > 1$, $c_{a_0} > 1$; hence $\varphi(a_0^k u) \geqslant (c_{a_0})^k \varphi(u)$ for $u \geqslant u_0(a)$, $k = 1, 2, \ldots$, $\lg \underline{h}^*_\varphi(a) \geqslant k \lg c_{a_0}$ for $a = a_0^k$, $\lg \underline{h}^*_\varphi(a)/\lg a \geqslant \lg c_{a_0}/\lg a_0$, $s_\varphi \geqslant \lg c_{a_0}/\lg a_0 > 0$.

(b) $\rightleftarrows$ (b′) is proved similarly.

Remark. Let us note that for a non-decreasing φ (in particular for a φ-function), (Λ_{a_0}) for an a_0 implies (a′) and (Δ_{a_0}) for an a_0 implies (Δ_a) for every $a > 1$.

2.52. Let φ be a φ-function.

(a) *If* $s_\varphi > 0$, *then*

(+) $$s_\varphi = \sup(\lg c_a/\lg a),$$

where the supremum is taken over all pairs of numbers a, c_a *which occur in the definition of condition* (Λ_a);

(b) *if* $\sigma_\varphi < \infty$, *then*

(++) $$\sigma_\varphi = \inf(\lg d_a/\lg a),$$

where the infimum is taken over all pairs of numbers a, d_a *which occur in the definition of condition* (Δ_a).

This follows from 1.7 (b) by applying the substitution $(*\varphi)$.

2.53. *If conditions* (Λ_a), (Δ_a) *are satisfied for sufficiently large* a, *then* φ *is a* $q\varphi$-*function*, $s_\varphi > 0$, $\sigma_\varphi < \infty$, *and formulae* 2.52 (+), (++) *are satisfied.*

Applying the substitution $(*\varphi)$ in this case we can easily see that the corresponding function f satisfies the inequalities $-\infty < \underline{\varrho}_f(\mu) \leqslant \overline{\varrho}_f(\mu) < \infty$ for large μ, whence these inequalities are satisfied for every μ, by 1.4. Formulae 1.7 (*), (**) yield the proof of existence of the indices and formulae 2.52 (+), (++), simultaneously.

2.6. Given a function ϱ, write

$$s_v^\varrho(u) = \sup_{v\leqslant t\leqslant u} \varrho(t) \quad \text{for} \quad u \geqslant v,$$

$$s_v^\varrho(u) = s_v^\varrho(v)u/v \quad \text{for} \quad 0 < u < v, \text{ if } v > 0,$$

$$s^\varrho(u) = s_0^\varrho(u),$$

$$t_v^\varrho(u) = \sup_{u\leqslant t<\infty} \varrho(t) \quad \text{for} \quad u \geqslant v,$$

$$t_v^\varrho(u) = t_v^\varrho(v) \quad \text{for} \quad 0 < u < v, \text{ if } v > 0.$$

Obviously, if $s^\varrho(u) < \infty$ for $u > 0$ and $s^\varrho(u) \to \infty$ as $u \to \infty$, then $s_v^\varrho \simeq s^\varrho$.

A function ϱ is called *pseudo-increasing for large* u if

(+) $$\varrho(u_2) \geqslant m\varrho(nu_1) \quad \text{for} \quad u_2 \geqslant u_1 \geqslant u_0$$

for some constants $m, n > 0$; it is called *pseudo-decreasing for large* u if

(++) $$\varrho(u_2) \geqslant m\varrho(nu_1) \quad \text{for} \quad u_2 \geqslant u_1 \geqslant u_0.$$

2.61. *A function* ϱ *is pseudo-increasing (pseudo-decreasing) for large* u *if and only if it is* l-*equivalent with a non-decreasing (non-increasing) function.*

The sufficiency follows from the definition of l-equivalence immediately. In order to prove the necessity let us note that (+) implies

$$\varrho\left(\frac{u}{n}\right) \geqslant m s_v^\varrho(u) \geqslant m\varrho(u) \quad \text{for} \quad u \geqslant v = \sup(u_0, nu_0),$$

and (++) implies

$$\varrho(u) \leqslant t_{u_0}^\varrho(u) \leqslant m\varrho(nu) \quad \text{for} \quad u \geqslant u_0,$$

whence $\varrho \overset{l}{-} s_v^\varrho$ in the first case and $\varrho \overset{l}{-} t_{u_0}^\varrho$ in the second case.

From the above it follows that

2.611. *If a function pseudo-increasing for large* u *is not* l-*equivalent to a constant, then* $s_v^\varrho(u) \to \infty$ *as* $u \to \infty$.

2.62. *Let us assume that the function* $\varrho(u)u^\varepsilon$ *is asymptotically equal to a non-decreasing function for an* $\varepsilon > 0$.

(a) *If* ϱ *is pseudo-increasing for large* u, *then the inequality*

(+) $$\varrho(u_2) \geqslant k\varrho(u_1) \quad \textit{for} \quad u_2 \geqslant u_1 \geqslant u^*$$

holds for a constant $k > 0$;

(b) *If* ϱ *is pseudo-decreasing for large* u, *then the inequality*

(++) $$\varrho(u_2) \leqslant k\varrho(u_1) \quad \textit{for} \quad u_2 \geqslant u_1 \geqslant u^*$$

holds for a constant $k > 0$.

We shall prove (a) for example. We may restrict ourselves to the case $\varepsilon = 1$. Let $0 < n \leqslant 1$, $u_2 = au$, $u_1 = u/n$, $a \geqslant 1/n$. Since $u_2 \geqslant u_1$, we have $\varrho(au) \geqslant m\varrho(u)$ for $u \geqslant nu_0$, by 2.6 (+). Since for every $0 < \eta < 1$, $u_2\varrho(u_2) \geqslant (1-\eta)u_1\varrho(u_1)$ for $u_2 \geqslant u_1 \geqslant u(\eta)$, we have $\varrho(au) \geqslant (1-\eta)n\varrho(u)$ for $u \geqslant u(\eta)$ and $1 \leqslant a \leqslant 1/n$. If $n > 1$ and $a \geqslant 1$, then applying the inequality $u_2\varrho(u_2) \geqslant (1-\eta)u_1\varrho(u_1)$ for $u_2 = nu$, $u_1 = u$, we obtain $\varrho(nu) \geqslant (1-\eta)\frac{1}{n}\varrho(u)$ for $u \geqslant u(\eta)$, i. e., by 2.6 (+), $\varrho(au) \geqslant m\varrho(nu) \geqslant \frac{m}{n}(1-\eta)\varrho(u)$ for $u \geqslant \sup(u_0, u(\eta))$. Thus we have proved (+) with a constant $k = \inf(m, n(1-\eta), (1-\eta)m/n)$, where $0 < \eta < 1$ may be chosen arbitrarily.

The arguments in the case (b) are similar.

2.63. *If* φ *is regularly increasing and* $r_\varphi > 0$, *then* $\varphi \simeq s_v^\varphi$ *for a certain* v. *If, moreover,* φ *is locally bounded, then* $\varphi \simeq s^\varphi$.

Choose an arbitrary $a_0 > 1$, $1-\varepsilon \geqslant a_0^{-r_\varphi}$. By 2.11, $\varphi(\lambda u) \geqslant (1-\varepsilon)\lambda^{r_\varphi}\varphi(u) \geqslant (1-\varepsilon)\varphi(u)$ for $u \geqslant \overline{u}$, $1 \leqslant \lambda \leqslant a_0$, i. e. $\varphi(a_0^k u) \geqslant \varphi(u)$ for $u \geqslant \overline{u}$ and $k = 1, 2, \ldots$, $\varphi(a_0^k \lambda u) \geqslant (1-\varepsilon)\varphi(u)$ for $k = 0, 1, 2, \ldots$ and $u \geqslant \overline{u}$. Consequently, $\varphi(au) \geqslant (1-\varepsilon)\varphi(u)$ for $u \geqslant \overline{u}$, $a \geqslant 1$, whence $\varphi(u) \geqslant (1-\varepsilon)s_{\overline{u}}^\varphi(u)$. But $\varphi(u) \to \infty$ as $u \to \infty$, by 2.2 (b); thus $s_{\overline{u}}^\varphi(u) \to \infty$

and if φ is bounded in a neighbourhood of 0 then, according to 2.6, $s^{\varphi}(u) < \infty$ for every $u > 0$ and $s_u^{\varphi} \simeq s^{\varphi}$. Hence in this case $\varphi(u) \geqslant (1-2\varepsilon)s^{\varphi}(u) \geqslant (1-2\varepsilon)\varphi(u)$ for sufficiently large u, $1-2\varepsilon \leqslant \underline{\lim}_{u\to\infty} \varphi(u)/s^{\varphi}(u) \leqslant \overline{\lim}_{u\to\infty} \varphi(u)/s^{\varphi}(u) \leqslant 1/(1-2\varepsilon)$, i. e. $\varphi \simeq s^{\varphi}$.

The first part of the theorem is obtained by modifying φ in a neighbourhood of 0 in order to get a locally bounded function.

2.64. *A function φ is slowly varying if and only if $\varphi(u)u^{\varepsilon}$ is asymptotically equal to a non-decreasing function and $\varphi(u)u^{-\varepsilon}$ is asymptotically equal to a non-increasing function for every $\varepsilon > 0$* (see [2] and [13]).

Sufficiency. Let $a > 1$, $\eta > 0$ be given. We choose $\varepsilon > 0$ so that $1+\eta \geqslant (1+\varepsilon)a^{\varepsilon}$, $(1-\varepsilon)/a^{\varepsilon} \geqslant 1-\eta$. Then the inequalities $a^{\varepsilon}u^{\varepsilon}\varphi(au) \geqslant (1-\varepsilon)u^{\varepsilon}\varphi(u)$ and $a^{-\varepsilon}u^{-\varepsilon}\varphi(au) \leqslant (1+\varepsilon)u^{-\varepsilon}\varphi(u)$ hold for sufficiently large u, whence

$$1+\eta \geqslant \overline{\lim_{u\to\infty}} \frac{\varphi(au)}{\varphi(u)} \geqslant \underline{\lim_{u\to\infty}} \frac{\varphi(au)}{\varphi(u)} \geqslant 1-\eta.$$

Necessity. Given any $\varepsilon > 0$, the functions $\varphi_1(u) = u^{\varepsilon}\varphi(u)$ and $\varphi_2(u) = u^{\varepsilon}/\varphi(u)$ are regularly increasing with index ε. By 2.63, $\varphi_1 \simeq s_{v_1}^{\varphi_1}$, $1/\varphi_2 \simeq 1/s_{v_2}^{\varphi_2}$.

2.65. *A function φ is quasi-slowly varying if and only if for every $\varepsilon > 0$ the function $\varphi(u)u^{\varepsilon}$ is pseudo-increasing for large u and the function $\varphi(u)u^{-\varepsilon}$ is pseudo-decreasing for large u.*

Sufficiency. Take an $\varepsilon > 0$. Then $\varphi_1(u) = \varphi(u)u^{\varepsilon}$ and $\varphi_2(u) = \varphi(u)u^{-\varepsilon}$ are $q\varphi$-functinos, by 2 and 2.61. Hence φ is also a $q\varphi$-function. Since $s_{\varphi_1} = s_{\varphi}+\varepsilon \geqslant 0$, $-\varepsilon+\sigma_{\varphi} = \sigma_{\varphi_2} \leqslant 0$, we have $-\varepsilon \leqslant s_{\varphi} \leqslant \sigma_{\varphi} \leqslant \varepsilon$ and, consequently, $s_{\varphi} = \sigma_{\varphi} = 0$.

Necessity. If $s_{\varphi} = \sigma_{\varphi} = 0$, then $r_{\varphi_1} = s_{\varphi_1} = \sigma_{\varphi_1} = \varepsilon$. It follows from the definition of the indices that if $a \geqslant a_0 \geqslant 1$, the inequalities

$$a^{\varepsilon'} \leqslant \frac{\varphi_1(au)}{\varphi_1(u)} \leqslant a^{\varepsilon''} \quad \text{for} \quad u \geqslant u_0(a)$$

are satisfied for given $\varepsilon'' > \varepsilon > \varepsilon' > 0$. Applying the substitution $(*\varphi_1)$, 1.4, 1.3 (b) we easily show the inequalities

$$c_1 \leqslant \frac{\varphi_1(au)}{\varphi_1(u)} \leqslant c_2$$

to hold uniformly with respect to a in $1 \leqslant a \leqslant a_0$ for $u \geqslant \bar{u}$. Let $1 \leqslant a$, $a_0^k \leqslant a < a_0^{k+1}$ for a $k = 0, 1, 2, \ldots$ Since $a = a_0^k\lambda$, where $1 \leqslant \lambda < a_0$, it follows that

$$\varphi_1(au) = \varphi_1(a_0^k\lambda u) \geqslant (a_0^k)^{\varepsilon}\varphi_1(\lambda u) \geqslant c_1\varphi(u) \quad \text{for} \quad u \geqslant \sup(u_0(a_0), \bar{u}).$$

Similarly we prove that $\varphi_2(u)$ is pseudo-decreasing for large u.

2.7. (a) *Let φ be such that $\varphi(u)u^{\varepsilon-1}$ is asymptotically equal to a non-decreasing function for an $\varepsilon > 0$. The function φ is l-equivalent to a convex φ-function if and only if the inequality*

$$(+) \qquad \frac{\varphi(u_2)}{u_2} \geqslant k\frac{\varphi(u_1)}{u_1} \quad \text{for} \quad u_2 \geqslant u_1 \geqslant u_0$$

is satisfied for a certain constant $k > 0$.

(b) *If φ is a φ-function and we change in $(+)$ the sign $\geqslant$ in $\leqslant$, we obtain a necessary and sufficient condition of l-equivalence of φ to a concave φ-function* (cf. [6] and [7]).

First, we consider the case of l-equivalence to a convex function. Let $\varphi \overset{l}{\sim} \psi$, where ψ is a convex φ-function. Inequality 2.4 $(+)$ holds for $u \geqslant u_0$, whence, for $u_2 \geqslant u_1 = \bar{u} = \sup(u_0, k_2u_0/k_1)$,

$$\frac{\varphi(u_2)}{u_2} \geqslant a\frac{\psi(k_1u_2)}{u_2} \geqslant a\frac{\psi(k_1u_1)}{u_1} \geqslant \frac{a}{b}\frac{\varphi\left(\frac{k_1}{k_2}u_1\right)}{u_1},$$

because $\psi(au) \leqslant a\psi(u)$ for $0 < a \leqslant 1$. Since $\varrho(u) = \varphi(u)/u$ satisfies 2.6 $(+)$ and $\varrho(u)u^{\varepsilon}$ is asymptotically equal to a non-decreasing function for a certain $\varepsilon > 0$, inequality $(+)$ follows from 2.62. In order to prove the sufficiency let us define the function $s(u) = s_v^{\varrho}(u)$, where $\varrho(u) = \varphi(u)/u$ and v is equal to u_0 from $(+)$. Arguments as in the proof of 2.61 imply $\varphi(u)/k \geqslant us(u) \geqslant \varphi(u)$ for $u \geqslant u_0$. The function

$$\psi(u) = \int_0^u s(t)\,dt$$

is a convex φ-function and since $\frac{1}{2}us(\frac{1}{2}u) \leqslant \psi(u) \leqslant us(u)$ for $u > 0$, we have $\varphi \overset{l}{\sim} \psi$.

Now, we consider the case (b). Adding to φ a continous function χ strictly increasing from 0 to 1 as $u \to \infty$, we obtain a φ-function $\bar{\varphi}$ strictly increasing, asymptotically equal to φ and such that the inequality

$$(++) \qquad \frac{\bar{\varphi}(u_2)}{u_2} \leqslant \bar{k}\frac{\bar{\varphi}(u_1)}{u_1} \quad \text{for} \quad u_2 \geqslant u_1 \geqslant \bar{u}_0,\ \bar{k} > 0,$$

holds for certain $\bar{k}$, $\bar{u}$ if and only if the inequality

$$\frac{\varphi(u_2)}{u_2} \leqslant k\frac{\varphi(u_1)}{u_1} \quad \text{for} \quad u_2 \geqslant u_1 \geqslant u_0,\ k > 0,$$

is satisfied for some k, u_0. Obviously, inequality $(++)$ is equivalent to the inequality

$$\frac{\bar{\varphi}^{-1}(v_2)}{v_2} \geqslant \frac{1}{\bar{k}}\frac{\bar{\varphi}^{-1}(v_1)}{v_1} \quad \text{for} \quad v_2 \geqslant v_1 \geqslant \bar{\varphi}(\bar{u}_0).$$

Since, by (a), $\bar{\varphi}^{-1} \overset{l}{\frown} \bar{\psi}$, where $\bar{\psi}$ is a convex φ-function, by 2.41 we have $\bar{\varphi} \overset{l}{\frown} \psi$, where $\psi = \bar{\psi}^{-1}$ is a concave φ-function, and since $\varphi \simeq \bar{\varphi}$ we have $\varphi \overset{l}{\frown} \psi$.

Remark. The following question arises: is it possible to define a function ψ in a way analogous to that in the case (a)? Let $t(u) = t_v^{\varrho}(u)$, where $\varrho(u) = \varphi(u)/u$ and v is equal to u_0. As before, we have $\varphi(u) \leqslant t(u)u \leqslant k\varphi(u)$, but the concave function $\psi(u) = \int_0^u t(\tau)d\tau$ is not necessarily l-equivalent to φ. However, this holds if we assume that $s_\varphi > 0$ or that condition (Λ_a) is satisfied for an $a > 1$ (both these assumptions are equivalent), for then we have $s_t > -1$ and 2.92 may be applied. The same remark concerns the application of $s(u) = \inf_{u_0 \leqslant t \leqslant u} m\varphi(nt)/t$ in place of the function $t(u)$ given in [7] on p. 127. I might notice here that the method of proof in the above-mentioned fragment of [7] may be applied, for example, if we assume (Λ_a).

2.71. (a) *A convex φ-function is superadditive, i. e.*

$$(+) \qquad \varphi(u_1+u_2) \geqslant \varphi(u_1)+\varphi(u_2) \quad \text{for} \quad u_1 \geqslant u_2 \geqslant 0;$$

a superadditive φ-function is l-equivalent to a convex φ-function.

(b) *A concave φ-function is subadditive, i. e. (+) holds, where the sign $\geqslant$ has to be changed into $\leqslant$; a subadditive φ-function is l-equivalent to a concave φ-function.*

Ad (a). Since $\varphi(u_1)u_1^{-1} \leqslant \varphi(u_2)u_2^{-1}$ for $u_2 \geqslant u_1 > 0$, we have

$$\varphi(u_1+u_2) = u_1\frac{\varphi(u_1+u_2)}{u_1+u_2} + u_2\frac{\varphi(u_1+u_2)}{u_1+u_2} \geqslant u_1\frac{\varphi(u_1)}{u_1} + u_2\frac{\varphi(u_2)}{u_2}.$$

Let us suppose that φ is superadditive. Let $u_2 \geqslant u_1 > 0$ and let n denote a non-negative integer such that $2^n u_1 \leqslant u_2 < 2^{n+1}u_1$. It follows from the superadditivity that

$$\frac{\varphi(u_2)}{u_2} \geqslant \frac{\varphi(2^n u_1)}{u_2} \geqslant \frac{2^n}{2^{n+1}}\frac{\varphi(u_1)}{u_1} = \frac{1}{2}\frac{\varphi(u_1)}{u_1},$$

and it is sufficient to apply 2.7 (a).

2.72. In the following properties φ, χ, ψ denote φ-functions, $r > 0$:

A. $\varphi \overset{l}{\frown} \bar{\chi}$, $\bar{\chi}(u) = (\bar{\psi}u^r)$, *$\bar{\psi}$ convex.*

B. $\varphi \overset{l}{\frown} \bar{\bar{\chi}}$, $\bar{\bar{\chi}}(u) = (\bar{\bar{\psi}}(u))^r$, *$\bar{\bar{\psi}}$ convex.*

C. $\varphi \overset{l}{\frown} \chi$, *$\chi$ is superadditive in a generalized sense:*

$$(+) \qquad \chi(u_1+u_2) \geqslant \left[(\chi(u_1))^{1/r}+(\chi(u_2))^{1/r}\right]^r \quad \text{for} \quad u_2 \geqslant u_1 \geqslant 0.$$

D. $\varphi(u) = u^r\varrho(u)$, *where ϱ is pseudo-increasing for large u.*

Properties A_0, B_0 will be obtained from A, B by replacing the word "convex" by "concave", property C_0 will be obtained from C by replacing the sign $\geqslant$ in inequality (+) by $\leqslant$, i. e. by replacing generalized superadditivity by generalized subadditivity. Finally, property D_0 will be obtained from D by replacing the phrase "ϱ is pseudo-increasing for large u" by "ϱ is pseudo-decreasing for large u".

2.73. *Any two of the properties A-D are equivalent; moreover, any two of the properties A_0-D_0 are also equivalent.*

This theorem is a consequence of 2.7, 2.71 by the fact that property D, resp. D_0, means that $\varphi(u^{1/r})$, $(\varphi(u))^{1/r}$ satisfy 2.7 (+), resp. 2.7 (++), with the sign $\geqslant$ replaced by $\leqslant$.

2.74. *Let φ be a φ-function.*

(a) *If $s_\varphi > 0$, then φ possesses property D for every $0 < r < s_\varphi$; if φ possesses property D for a certain r, then $s_\varphi > 0$ and $r \leqslant s_\varphi$.*

(b) *If $\sigma_\varphi < \infty$, then φ possesses property D_0 for every $r > \sigma_\varphi$; if φ possesses property D_0 for a certain r, then $\sigma_\varphi < \infty$, $\sigma_\varphi \leqslant r$* [10].

Let $\varphi(u) = u^r\varrho(u)$, where $r > 0$. By 2.62, ϱ is pseudo-increasing for large u if and only if, for every $a \geqslant 1$,

$$(+) \qquad \varrho(au) \geqslant k\varrho(u), \quad \text{where} \quad k > 0,\ u \geqslant u^*.$$

Since $\varphi(au)/\varphi(u) = a^r\varrho(au)/\varrho(u)$, (+) implies that φ satisfies condition (Λ_a) for sufficiently large a with the constant $c_a = a^r k$. By 2.51, $s_\varphi > 0$, and, by 2.52 (a), $\lg c_a/\lg a = r+\lg k/\lg a \leqslant s_\varphi$, i. e. $r \leqslant s_\varphi$. Let us now assume $s_\varphi > 0$, $0 < s < s_\varphi$; according to 2.52 (a) there exists an $a_0 > 1$ such that $r = \lg c_{a_0}/\lg a_0 > s$, $\varphi(a_0 u) \geqslant c_{a_0}\varphi(u)$ for $u \geqslant u_0(a_0)$. Let $a \geqslant 1$, i. e. $a = a_0^k\lambda$, where k is a non-negative integer, $1 \leqslant \lambda < a_0$. If $u \geqslant u_0(a_0)$, the following inequalities are satisfied:

$$\varphi(au) \geqslant (c_{a_0})^k\varphi(\lambda u) = (a_0^k)^r\varphi(\lambda u) \geqslant \frac{a^r}{a_0^r}\varphi(u),$$

$$\varrho(au) = \frac{\varphi(au)}{a^r u^r} \geqslant \frac{1}{a_0^r}\frac{\varphi(u)}{u^r} = \frac{1}{a_0^r}\varrho(u).$$

Property D is satisfied for an arbitrary $r < s_\varphi$, for it is satisfied for some $r < s_\varphi$ arbitrarily near to s_φ.

The proof of (b) is analogous.

2.8. In this section φ may assume also negative values and is always integrable in an arbitrary interval $(0, u)$; however, speaking about regularly increasing or slowly varying functions etc., we shall have in mind functions $\varphi > 0$, just as we did previously. We shall write $\psi(u) = \int_0^u \varphi(t)\,dt$

and we shall assume $\psi(u) > 0$ for $u > 0$. Moreover, we shall write

$$h(u) = \frac{u\varphi(u)}{\psi(u)} \quad \text{for} \quad u > 0.$$

The following inequalities hold:

$$(+) \qquad \varliminf_{u\to\infty} h(u) \leqslant \frac{\lg \underline{h}_\psi(\lambda)}{-\lg\lambda} \leqslant \frac{\lg \overline{h}_\psi(\lambda)}{-\lg\lambda} \leqslant \varlimsup_{u\to\infty} h(u), \qquad 0 < \lambda < 1.$$

We apply the substitution

$$(*\psi), \qquad e^\mu = a, \qquad e^u = v$$

to the function ψ and we write $\lg\psi(e^u) = f(u)$. Since ψ is absolutely continuous, we get $f'(u) = \psi'(e^u)e^u/\psi(e^u) = h(e^u)$ for almost every u, whence

$$\lg\frac{\psi(av)}{\psi(v)} = f(u+\mu) - f(u) = \int_u^{u+\mu} f'(t)\,dt$$

for $a > 1$. However, $\varlimsup_{v\to\infty} \psi(av)/\psi(v) = \overline{h}_\psi(\lambda)$, where $\lambda = 1/a$. Thus

$$\lg\overline{h}_\psi(\lambda) \leqslant \varlimsup_{u\to\infty} \int_u^{u+\mu} f'(t)\,dt \leqslant -\lg\lambda\cdot\varlimsup_{u\to\infty} f'(u).$$

The inequality $-\lg\lambda\cdot\varliminf_{u\to\infty} f'(u) \leqslant \lg\underline{h}_\psi(\lambda)$, when $0 < \lambda < 1$, is proved similarly.

As an immediate consequence of $(+)$ we obtain

2.81. (a) *If $h(u) \to a$, where $a \neq 0$ is finite, then ψ is regularly increasing and of index $r_\psi = a$; if $a = 0$, ψ is slowly varying.*

(b) *If $h(u) \to \infty$ as $u \to \infty$, then*

$$\frac{\psi(\lambda u)}{\psi(u)} \to \begin{cases} 0 & \textit{for} \quad 0 < \lambda < 1, \\ 1 & \textit{for} \quad \lambda = 1, \\ \infty & \textit{for} \quad \lambda > 1. \end{cases}$$

2.811. *Let $\varphi > 0$ for $u > 0$ and $h(u) \to a$ as $u \to \infty$, a finite. If $a = 1$, then φ is slowly varying, and if $a \neq 1$, $a > 0$, then φ is regularly increasing and $r_\varphi = a-1$.*

By the above assumption, $a\psi(u) \simeq u\varphi(u)$ and since, according to 2.81 (a), ψ is of index $r_\psi = a$, we have $r_\psi = 1 + r_\varphi$, i.e. φ is of index $a-1$.

Remark. From the proof of inequality 2.8 $(+)$ it follows that the inequality remains valid if we restrict ourselves to $u \to \infty$, $u \in (0, \infty) - A$ in $\varlimsup h(u)$ and $\varliminf h(u)$, A being a set of measure 0. The same remark applies to 2.81. Taking into consideration the above remark we obtain the following test of φ being slowly varying, resp. regularly increasing:

2.812. *If $\varphi(u) > 0$ for $u > 0$, φ is absolutely continuous, A denotes the set of u for which $\varphi'(u)$ exists and if*

$$(*) \qquad \frac{u\varphi'(u)}{\varphi(u)} \to a \quad \textit{as} \quad u \in A,\ u \to \infty,$$

then φ is slowly varying when $a = 0$ and regularly increasing when $a \neq 0$.

For instance, the above test may be applied to

$$\varphi(u) = \int_0^u \frac{|\sin t|}{t}\,dt, \qquad a = 0.$$

Hence φ is slowly varying; however, φ' does not possess this property.

Condition $(*)$ in 2.812 with $a = 0$ is not necessary in order that an absolutely continuous function φ be slowly varying. Every absolutely continuous non-decreasing function tending to 1 as $u \to \infty$ is slowly varying, but if in an arbitrary neighbourhood of ∞ there are intervals in which φ is constant and intervals in which $\varphi'(u) \geqslant 1$, then the limit $(*)$ does not exist even if we omit any set of measure 0. However, by applying the integral representation of Karamata [2] it may be shown that every slowly varying function is asymptotically equal to a function satisfying 2.812 $(*)$ with $a = 0$.

2.813. *If $\psi(u) \to \infty$ as $u \to \infty$ for a continuous $q\varphi$-function φ, then the following inequalities hold:*

$$1 + s_\varphi \leqslant s_\psi \leqslant \sigma_\psi \leqslant 1 + \sigma_\varphi;$$

if φ is regularly increasing or slowly varying, then ψ has the same property and the index of ψ is $r_\psi = 1 + r_\varphi$ [9].

L'Hospital's rule (in the form with limit superior and limit inferior) yields

$$(+) \qquad \varliminf_{u\to\infty} \frac{\varphi(u)}{\lambda\varphi(\lambda u)} \leqslant \varliminf_{u\to\infty} \frac{\psi(u)}{\psi(\lambda u)} \leqslant \varlimsup_{u\to\infty} \frac{\psi(u)}{\psi(\lambda u)} \leqslant \varlimsup_{u\to\infty} \frac{\varphi(u)}{\lambda\varphi(\lambda u)},$$

i.e. $\underline{h}_\varphi(\lambda)/\lambda \leqslant \underline{h}_\psi(\lambda) \leqslant \overline{h}_\psi(\lambda) \leqslant \overline{h}_\varphi(\lambda)/\lambda$. Indices s_ψ, σ_ψ exist, for ψ is a φ-function. Hence from the last inequalities we get $1 + s_\varphi \leqslant s_\psi \leqslant \sigma_\psi \leqslant 1 + \sigma_\varphi$. If $\underline{h}_\varphi(\lambda) = \overline{h}_\varphi(\lambda)$ for $\lambda > 0$, then $(+)$ implies $\underline{h}_\psi(\lambda) = \overline{h}_\psi(\lambda)$, and since $r_\varphi = s_\varphi = \sigma_\varphi$, we have $r_\psi = 1 + r_\varphi$.

Remark. The assumption of continuity of φ may be removed by a suitable modification of the proof.

2.814. *If $s_\varphi > -1$ for a $q\varphi$-function, $\lg\varphi$ locally bounded for large u, then $\psi(u) \to \infty$ as $u \to \infty$.*

It follows from 2.2 (a) that if $-s_\varphi < s < 1$, then $\varphi(u) \geqslant u^{-s}$ for sufficiently large u, whence $\psi(u) \to \infty$ as $u \to \infty$.

2.82. *If a function $h(u)$ is slowly varying and φ is a continuous $q\varphi$-function such that $\psi(u)\to\infty$ as $u\to\infty$, then the inequalities*

$$(+)\qquad 1+s_\varphi \leqslant s_\psi = \varliminf_{u\to\infty} h(u) \leqslant \varlimsup_{u\to\infty} h(u) = \sigma_\psi \leqslant 1+\sigma_\varphi$$

are satisfied.

To prove this theorem we apply the substitutions $\lg\psi(e^u) = f(u)$, $e^\mu=\lambda$, $e^u=v$, again. We have

$$\frac{h(\lambda v)}{h(v)} = \frac{f'(u+\mu)}{f'(u)} \to 1$$

as $u\to\infty$ for an arbitrary μ, for $u\to\infty$ implies $v\to\infty$ and h is slowly varying. By 1.8, we have

$$(++)\qquad \begin{aligned} &\varlimsup_{u\to\infty} h(u) = \varlimsup_{u\to\infty} f'(u) = \bar{\varrho}_f(\mu)/\mu \quad \text{for} \quad \mu>0,\\ &\varliminf_{u\to\infty} h(u) = \varliminf_{u\to\infty} f'(u) = \underline{\varrho}_f(\mu)/\mu \quad \text{for} \quad \mu>0, \end{aligned}$$

and by the definition of the indices

$$s_\psi = \lim_{\mu\to\infty} \underline{\varrho}_f(\mu)/\mu, \qquad \sigma_\psi = \lim_{\mu\to\infty} \bar{\varrho}_f(\mu)/\mu.$$

Inequality $(+)$ follows from $(++)$ if we take $\mu\to\infty$ and apply theorem 2.813.

2.83. *If φ is regularly increasing or slowly varying, then there exists a continuous function for $u\geqslant 0$ asymptotically equal to φ.*

Let us define a continuous function $\bar\varphi(u)$ as $\bar\varphi(n)=\varphi(n)$ for $n=1,2,\ldots$, $\bar\varphi(u)$ a linear function between the points $(n,\varphi(n))$ and $(n+1,\varphi(n+1))$ and in the interval $\langle 0,1)$. We have, for $n\leqslant u\leqslant n+1$, $u=n\lambda$, $1\leqslant\lambda\leqslant(n+1)/n$, and, by 2.11,

$$\frac{\varphi(u)}{\bar\varphi(u)} = \frac{\varphi(n\lambda)}{\varphi(n)}\cdot\frac{\varphi(n)}{\bar\varphi(n\lambda)} \to 1 \quad \text{as} \quad u\to\infty.$$

2.84. *If $-\infty<s_\varphi\leqslant\sigma_\varphi<\infty$ for a $q\varphi$-function φ, then there exists a continuous function $\bar\varphi$ l-equivalent to φ.*

The indices s_φ, σ_φ being finite, for an arbitrary interval (λ',λ''), where $\lambda'>0$, there are constants $c_2\geqslant c_1>0$ such that $c_2\geqslant\varphi(\lambda u)/\varphi(u)\geqslant c_1$ for $\lambda\in(\lambda',\lambda'')$ and $u\geqslant u_0$ (we make the substitution $(*\varphi)$ and we apply 1.4, 1.3 (b)). We define $\bar\varphi$ as in 2.83, $\lambda'=1$, $\lambda''=2$. Then we have

$$\frac{c_1}{c_2} \leqslant \frac{\varphi(u)}{\bar\varphi(u)} = \frac{\varphi(n\lambda)}{\varphi(n)}\cdot\frac{\varphi(n)}{\bar\varphi(\lambda n)} \leqslant \frac{c_2}{c_1}$$

for sufficiently large u.

2.85. *If φ is a quasi-regularly increasing (quasi-slowly varying) continuous function with $r_\varphi>-1$ and if h is slowly varying, then φ is regularly increasing (slowly varying).*

By 2.814, 2.813 and 2.82,

$$h(u)\to a = 1+r_\varphi > 0,$$

whence, according to 2.811, φ is regularly increasing (slowly varying, if $r_\varphi=0$).

2.86. (a) *In order that φ be regularly increasing of index $r_\varphi>-1$ it is necessary and sufficient that*

$$(+)\qquad h(u)\to a \quad \text{as} \quad u\to\infty,$$

where $a\neq 1$, $a>0$.

(b) *In order that φ be slowly varying it is necessary and sufficient that $(+)$ holds with $a=1$.*

In both cases the index r_φ and the limit a satisfy the equality $r_\varphi = a-1$ (see [2] and [3]).

Sufficiency follows from 2.811; necessity is obtained from 2.82 by assuming φ to be continuous. If φ is not continuous, then, according to 2.83, $\varphi\simeq\bar\varphi$, where $\bar\varphi$ is a continuous function. Since $r_\varphi = r_{\bar\varphi} > -1$, writing

$$\bar\psi(u) = \int_0^u \bar\varphi(t)\,dt$$

we have $\bar\psi(u)\to\infty$ as $u\to\infty$, $\psi\simeq\bar\psi$. If $\bar h(u) = u\bar\varphi(u)/\bar\psi(u)$, then $\bar h\simeq h$, and since $\bar h(u)\to a = 1+r_{\bar\varphi}$ as $u\to\infty$. we have $h(u)\to a$ as $u\to\infty$.

2.9. In connection with theorem 2.86 and condition $(+)$ which means that $\psi(u)\sim u\varphi(u)$, we shall add some remarks concerning the case when $\sim$ in the last relation is replaced by $\overset{l}{\sim}$. As in the previous section, we assume the existence of the integral $\psi(u)$ for $u\geqslant 0$.

2.91. *If φ is non-decreasing for $u\geqslant u_0$, then*

$$(+)\qquad \psi(u)\overset{l}{\sim} u\varphi(u).$$

The relation $(+)$ follows from the inequalities

$$\tfrac{1}{2}u\varphi(\tfrac{1}{2}u) \leqslant \psi(u)-\psi(u_0) \leqslant (u-u_0)\varphi(u) \quad \text{for} \quad u\geqslant 2u_0$$

and from $\psi(u)\to\infty$ as $u\to\infty$.

If φ is non-increasing, then $(+)$ need not be satisfied. E. g., if $\varphi(u) = (1+u)^{-1}\lg(1+u)$, then $\psi(u) = \frac{1}{2}\big(\lg(1+u)\big)^2$ and ψ is not l-equivalent to $u\varphi(u)$. In this example φ is regularly increasing and $r_\varphi=-1$. However, the following sufficient condition may be deduced:

2.92. *If φ is non-increasing for $u\geqslant u_0$, $s_\varphi>-1$, then* 2.91 $(+)$ *holds.*

According to 2.814, $\psi(u)\to\infty$ as $u\to\infty$. We have $s_{\varphi_1}=1+s_\varphi>0$ for $\varphi_1(u)=u\varphi(u)$, whence, by 2.51, φ_1 satisfies condition (Λ_a) for a certain $a>1$, $c_a>1$. Thus $\varphi(au)\geqslant c_a\varphi(u)/a$ for $u\geqslant\bar{u}\geqslant u_0$; hence

$$\psi(au)-\psi(a\bar{u})=a\int_{\bar{u}}^{u}\varphi(at)\,dt\geqslant c_a\int_{\bar{u}}^{u}\varphi(t)\,dt=c_a\big(\psi(u)-\psi(\bar{u})\big),$$

i. e. $\psi(au)\geqslant c_a\psi(u)+k$. On the other hand,

$$\psi(au)=\psi(u)+\int_u^{au}\varphi(t)\,dt\leqslant\psi(u)+(a-1)u\varphi(u)$$

for $u\geqslant\bar{u}$, whence

$$\psi(u)(c_a-1)+k\leqslant(a-1)u\varphi(u)\quad\text{ for }\quad u\geqslant\bar{u},$$

and since

$$\psi(u)-\psi(u_0)\geqslant(u-u_0)\varphi(u),$$

we obtain $\psi(u)\overset{l}{\sim}u\varphi(u)$.

3. In this section we always assume φ to be a convex φ-function; then $\sigma_\varphi\geqslant s_\varphi\geqslant 1$. The following conditions will be of importance in the sequel:

(o_1) $\qquad\varphi(u)u^{-1}\to 0\quad$ as $\quad u\to 0$;

(∞_1) $\qquad\varphi(u)u^{-1}\to\infty\quad$ as $\quad u\to\infty$.

By the assumptions (o_1), (∞_1) it is known (see [1] and [5]) that the function

$$\varphi^*(v)=\sup_{u\geqslant 0}\big(uv-\varphi(u)\big),$$

complementary to the function φ, may be defined. It is easily proved that φ^* is a convex φ-function for $v\geqslant 0$ satisfying conditions (o_1), (∞_1) and $(\varphi^*)^*=\varphi$.

3.1. *If φ^* is regularly increasing and $\varphi\simeq\varphi_1$, then $\varphi^*\simeq\varphi_1^*$.*

We have $(1-\varepsilon)\varphi(u)\leqslant\varphi_1(u)\leqslant(1+\varepsilon)\varphi(u)$ for $u\geqslant u_0$; hence the complementary functions satisfy the following inequalities ([5], p. 23):

$$(1-\varepsilon)\varphi^*\left(\frac{u}{1-\varepsilon}\right)\geqslant\varphi_1^*(u)\geqslant(1+\varepsilon)\varphi^*\left(\frac{u}{1+\varepsilon}\right)\quad\text{ for }\quad u\geqslant u^*,$$

i. e.

$$(1-\varepsilon)\frac{\varphi^*\left(\dfrac{u}{1-\varepsilon}\right)}{\varphi^*(u)}\geqslant\frac{\varphi_1^*(u)}{\varphi^*(u)}\geqslant(1+\varepsilon)\frac{\varphi^*\left(\dfrac{u}{1+\varepsilon}\right)}{\varphi^*(u)},$$

and since $\varphi^*\big(u(1+\varepsilon)^{-1}\big)/\varphi^*(u)\to(1+\varepsilon)^{-r_{\varphi^*}}$, $\varphi^*\big(u(1-\varepsilon)^{-1}\big)/\varphi^*(u)\to(1-\varepsilon)^{-r_{\varphi^*}}$, we have

$$(1-\varepsilon)^{-r_{\varphi^*}}(1-\varepsilon)\geqslant\varlimsup_{u\to\infty}\frac{\varphi_1^*(u)}{\varphi^*(u)}\geqslant\varliminf_{u\to\infty}\frac{\varphi_1^*(u)}{\varphi^*(u)}\geqslant(1+\varepsilon)^{-r_{\varphi^*}}(1+\varepsilon),$$

whence $\varphi_1^*\simeq\varphi^*$.

3.2. (a) *If φ is regularly increasing, $r_\varphi>1$, then φ^* is regularly increasing and the indices satisfy the relation $1/r_\varphi+1/r_{\varphi^*}=1$.*

(b) *If φ is regularly increasing, $r_\varphi=1$, then $(\varphi^*)^{-1}$ is slowly varying, and $r_{\varphi^*}=\infty$.*

(c) *If φ^{-1} is slowly varying, then $r_\varphi=\infty$, φ^* is regularly increasing and $r_{\varphi^*}=1$.*

Let $p(u)=\varphi(u)/u$ for $u>0$, $p(0)=0$. By (o_1) and (∞_1), $p(u)$ is strictly increasing ([5], p. 18) and if $\varphi_1(u)=\int_0^u p(t)\,dt$, then $\varphi_1^*(u)=\int_0^u p^{-1}(t)\,dt$. According to 2.3, $p(u)$ is regularly increasing of index $r_p=r_\varphi-1>0$, and, by 2.86,

$$\frac{\varphi(u)}{\int_0^u p(t)\,dt}=\frac{u\dfrac{\varphi(u)}{u}}{\int_0^u p(t)\,dt}\to 1+r_p=r_\varphi\quad\text{ as }\quad u\to\infty.$$

As is well known, every pair of numbers $u,v>0$ such that $p^{-1}(v)=u$ satisfies the identity $\varphi_1(u)+\varphi_1^*(v)=uv$, i. e.

$$(+)\qquad\frac{\varphi_1(u)}{up(u)}+\frac{\varphi_1^*(v)}{vp^{-1}(v)}=1,$$

and since $u\to\infty$ as $v\to\infty$ and $up(u)=\varphi(u)$, we have

$$(++)\qquad\frac{\varphi_1^*(v)}{vp^{-1}(v)}\to 1-\frac{1}{r_\varphi}=\frac{r_\varphi-1}{r_\varphi}.$$

By 2.86 (a), $p^{-1}(v)$ is regularly increasing and $r_{p^{-1}}=r_\varphi/r_{\varphi-1}-1$. Hence, by 2.912, φ_1^* is regularly increasing and of index $r_{\varphi_1^*}=r_{p^{-1}}+1=r_\varphi/(r_\varphi-1)$. Taking into account the identity $up(u)=\varphi(u)$, we obtain $\varphi_1(u)/\varphi(u)\to 1/r_\varphi$ as $u\to\infty$, i. e. $r_\varphi\varphi_1\simeq\varphi$. According to 3.1 we have $(r_\varphi\varphi_1)^*\simeq\varphi^*$ and since $\big(r_\varphi\varphi_1(u)\big)^*=r_\varphi\varphi_1^*(u/r_\varphi)\overset{l}{\sim}\varphi_1^*(u)$, the function $(r_\varphi\varphi_1)^*$ is regularly increasing and of index $r_{\varphi_1^*}$. Hence it follows that φ^* is also regularly increasing and of the same index.

In order to prove (c) let us note that according to 2.31 we have $\varphi(\lambda u)/\varphi(u)\to\infty$ as $u\to\infty$, $\lambda>1$, whence also $p(\lambda u)/p(u)\to\infty$ as $u\to\infty$, $\lambda>1$. Thus p^{-1} is slowly varying, and consequently φ_1^* is regularly increasing, $r_{\varphi_1^*}=1$, by 2.813. The inequalities

$$\varphi(u) \leqslant \frac{1}{\lambda-1}\varphi_1(\lambda u), \qquad \varphi_1(u) \leqslant \varphi(u) \quad \text{for} \quad u \geqslant 0$$

hold for an arbitrary $\lambda > 1$. Hence the complementary functions satisfy the inequalities

$$(^{+}_{+}) \qquad \varphi^*(u) \geqslant \frac{1}{\lambda-1}\varphi_1^*\left(\frac{\lambda-1}{\lambda}u\right), \qquad \varphi_1^*(u) \geqslant \varphi^*(u) \quad \text{for} \quad u \geqslant 0.$$

Hence, taking into account the equality $r_{\varphi_1^*} = 1$ we obtain

$$\frac{1}{\lambda-1}\varphi_1^*\left(\frac{\lambda-1}{\lambda}u\right)\Big/\varphi_1^*(u) \to \frac{1}{\lambda} \quad \text{as} \quad u\to\infty,$$

hence

$$1 \geqslant \frac{\varphi^*(u)}{\varphi_1^*(u)} \geqslant \frac{1}{\lambda-1}\,\frac{\varphi_1^*\left(\frac{\lambda-1}{\lambda}u\right)}{\varphi_1^*(u)}.$$

Consequently, we obtain the relation $\varphi^* \simeq \varphi_1^*$. Hence φ^* is regularly increasing, $r_{\varphi_1^*} = r_{\varphi*} = 1$. Since $\underline{h}_\varphi(\lambda) = \overline{h}_\varphi(\lambda) = \infty$ for $0<\lambda<1$, the equation $r_\varphi = \infty$ is obvious.

To prove (b) let us note that, by 2.3 (a), p is slowly varying for $r_p = 0$. Hence $\varphi_1(u)/up(u) \to 1$, and $(+)$ implies $vp^{-1}(v)/\varphi_1^*(v) \to \infty$ as $v\to\infty$. According to 2.81 (b), $\varphi_1^*(\lambda u)/\varphi_1^*(u) \to \infty$ as $u\to\infty$, if $\lambda>1$. Inequality $(^{+}_{+})$ yields

$$\frac{\varphi^*(\mu u)}{\varphi^*(u)} \geqslant \frac{1}{\lambda-1}\varphi_1^*\left(\frac{\lambda-1}{\lambda}\mu u\right)\Big/\varphi_1^*(u), \qquad u>0,\ \mu>0.$$

But given $\mu>1$ we may choose $\lambda>1$ so that $(\lambda-1)\mu/\lambda>1$, whence $\varphi^*(\lambda u)/\varphi^*(u)\to\infty$ as $u\to\infty$ for $\lambda>1$, and, by 2.31, $(\varphi^*)^{-1}$ is slowly varying.

3.3. If $\alpha>1$, we denote by β the conjugate exponent, $1/\alpha+1/\beta = 1$. A regularly increasing function φ of index α may be written in the form

$$\varphi(u) = \frac{u^\alpha}{\alpha}\gamma(u),$$

where γ is slowly varying. Hence, by 3.2,

$$\varphi^*(u) = \frac{u^\beta}{\beta}\gamma^*(u),$$

where γ^* is also a slowly varying function. Under suitable assumptions regarding γ, additional information on the asymptotic behaviour of φ^* for large u can be obtained.

3.4. *Let*

$$\varphi(u) = \frac{u^\alpha}{\alpha}\gamma(u), \qquad \alpha>1,$$

where $\gamma(u) = \omega(\lg(1+u))$ and ω is a regularly increasing or slowly varying function of index $r_\omega = s$. Then

$$\varphi^*(u) \simeq c\cdot\frac{1}{\beta}u^\beta(\gamma(u))^{-\beta/\alpha}, \qquad \textit{where} \qquad c = [(\beta/\alpha)^{-\beta/\alpha}]^s.$$

Let $p(u)$, $\varphi_1(u)$ have the same meaning as in 3.2. According to 2.3 (h), γ is slowly varying. Hence $r_\varphi = \alpha$, $p(u) = (u^{\alpha-1}/\alpha)\gamma(u)$, $r_p = \alpha-1>0$, $r_{p^{-1}} = 1/(\alpha-1) = \beta/\alpha = \beta-1$, i. e. p^{-1} is regularly increasing of the form $p^{-1}(u) = u^{\beta-1}\lambda(u)$, where $\lambda(u)$ is a slowly varying function. Thus we have

$$u = p(u^{\beta-1}\lambda(u)) = \frac{u^{(\alpha-1)(\beta-1)}}{\alpha}(\lambda(u))^{\alpha-1}\gamma(u^{\beta-1}\lambda(u)),$$

and since $(\alpha-1)(\beta-1) = 1$, we have

$$(+) \qquad \alpha(\lambda(u))^{-\alpha/\beta} = \gamma(u^{\beta-1}\lambda(u)).$$

Let $r_\omega = s$; then $\omega(\lambda u)/\omega(u)\to\lambda^s$ as $u\to\infty$, this convergence being uniform in each finite interval $\langle\lambda_1,\lambda_2\rangle$, $\lambda_1>0$, by 2.11. Hence $\gamma(u-1)/\gamma(u) = \omega(\lg u)/\omega(\lg(1+u))\to 1$ as $u\to\infty$, $\gamma(u^{\beta-1}\lambda(u)) \simeq \omega((\beta-1)\lg u+\lg\lambda(u))$. But

$$\frac{(\beta-1)\lg u+\lg\lambda(u)}{\lg u}\to\beta-1 \quad \text{as} \quad u\to\infty,$$

for 2.2 (b) implies $\lg\lambda(u)/\lg u\to 0$ as $u\to\infty$. Thus

$$\frac{\gamma(u^{\beta-1}\lambda(u))}{\gamma(u)} \simeq \frac{\omega((\beta-1)\lg u+\lg\lambda(u))}{\omega(\lg u)} \to (\beta-1)^s.$$

It follows from $(+)$ that $(\lambda(u))^{-\alpha/\beta}/\gamma(u)\to(\beta-1)^s/\alpha$, $\lambda(u)\simeq(\gamma(u))^{-\beta/\alpha}\bar{c}$ as $u\to\infty$, where $\bar{c} = (\beta-1)^{-s\beta/\alpha}\alpha^{\beta/\alpha}\neq 0$. However, $(\alpha\varphi_1)^*\simeq\varphi^*$, $(\alpha\varphi_1)^* = \alpha\varphi_1^*(u/\alpha)$, as follows from the proof of 3.2, and since p^{-1} is regularly increasing, we have $u^\beta\lambda(u)/\varphi_1^*(u)\to 1+r_{p-1} = \beta$ as $u\to\infty$, i. e. $1/\beta\cdot u^\beta\lambda(u)\simeq\varphi_1^*(u)$. Moreover, $\varphi_1^*(u/\alpha)/\varphi_1^*(u)\to(1/\alpha)^{r_{\varphi_1^*}} = \alpha^{-\beta}$ as $u\to\infty$, $\varphi_1^*(u/\alpha)\simeq\alpha^{-\beta}\varphi_1^*(u)$. Finally, we obtain $\varphi^*(u)\simeq 1/\beta\cdot u^\beta((\gamma(u))^{-\beta/\alpha}c$, where $c = \bar{c}\alpha^{-\beta+1} = [(\beta/\alpha)^{-\beta/\alpha}]^s$.

Theorem 3.4 is a strengthened form of a theorem of Krasnosielskij and Rutickij [5], who obtain $\overset{L}{\sim}$ in place of $\simeq$, the constant being unspecified, and who make a little more restrictive assumption regarding γ. If $\omega = 1$, then $\gamma = 1$, $r_\omega = 0$ and $\varphi(u) = u^\alpha/\alpha$, $\varphi^*(u) = u^\beta/\beta$, while 3.4 gives only $\varphi^*(u)\simeq u^\beta/\beta$.

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On the analytic functions in p-normed algebras

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A *p-normed algebra* is a complete metric algebra in which topology is given by the meaning of a p-homogeneous submultiplicative norm $\|x\|$:

$$\|ax\| = |a|^p \|x\|, \tag{1}$$

$$\|xy\| \leqslant \|x\| \|y\|, \tag{2}$$

where a is a scalar, p — fixed real number satisfying $0 < p \leqslant 1$.

It is known that every complete locally bounded algebra is a p-normed algebra. These algebras were considered in papers [4], [5], and [6]. The greater part of Gelfand's theory on commutative complex Banach algebras is also true for p-normed algebras. In this paper we give an extension of Gelfand's theory of analytic functions in Banach algebras onto p-normed algebras [1]. We note that the classical method based upon the concept of abstract Riemann integral cannot be applied here, because the algebras in question are not locally convex (cf. [3]).

Let A be a commutative complex p-normed algebra with a unit designed by e. Let $\mathfrak{M}$ be the compact space of its multiplicative linear functionals (= maximal ideals). The spectrum of an element $x \in A$ is defined as

$$\sigma(x) = \{f(x): f \in \mathfrak{M}\}. \tag{3}$$

It is a compact subset of the complex plane. Here we give the positive answer to the following question stated in [6]:

"Let $\Phi(z)$ be a holomorphic function defined in the neighbourhood U of spectrum $\sigma(x)$ of an element $x \in A$. Does there exist a $y \in A$ such that for every $f \in \mathfrak{M}$

$$f(y) = \Phi\big(f(x)\big)?\text{"} \tag{4}$$

We shall give a step by step construction of such an element y. It is natural to write $y = \Phi(x)$. So we give a natural definition of $\Phi(x)$ in locally bounded algebras.

As a corollary we obtain the generalization of the theorem of Lévy [2] on trigonometrical series.