Regularly increasing functions in connection with the theory of $L^\infty$-spaces

by

W. MATUSZEWSKA (Poznań)

In order to investigate the structure of various spaces of $\varphi$-integrable functions [1], [6], information on some properties of continuous positive functions as regards the orders of growth of such functions is necessary. The so-called conditions $\langle A_0 \rangle$, $\langle A_1 \rangle$ (see [3]) and $\langle B \rangle$ or indices $\alpha$, $\gamma$ (see [10] and [9]) occurring in the theory of spaces $L^\infty(a, b)$ make it possible to compare the function $\varphi$ with functions $\psi$. It may be expected that regularly increasing and slowly varying functions, well-known in various problems of asymptotic behaviour of functions, are of importance in the theory of spaces $L^\infty(a, b)$. The purpose of this paper is to investigate a number of problems connected with the above-mentioned notions.

The main stress is laid on a systematic and elementary presentation of the subject, treated as an introduction to the theory of spaces $L^\infty(a, b)$. Sections 1 and 2 are closely connected with the fundamental papers [2] and [3] of Karamata concerning regularly increasing functions. We avoid integral representations of these functions, starting from the fundamental lemma 1.3 as in [4]. In section 3 the notion of a regularly increasing function appears in connection with functions complementary in the sense of Young. Here, some additions to a theorem of Krein-Mil'man and Rutickii [5] are made. Taking in consideration the purposes of this paper we include some results already published, however, somewhat alternatively. Some results of [10], [9] and [7] are also included.

1. In this section we denote by $f, g, h, \ldots$ real functions defined for $-\infty < u < \infty$. We shall also write

$$
\varphi_1(\mu) = \lim_{u \to \infty} (f(u + \mu) - f(u)), \quad \varphi_2(\mu) = \lim_{u \to -\infty} (f(u + \mu) - f(u));
$$

if $\varphi_1(\mu) = \varphi_2(\mu)$ for a certain $\mu$, we denote this common value by $\varphi(\mu)$.

1.1. The following relations are immediately obtained from these definitions:

(a) $\varphi_1(-\mu) = -\varphi_1(\mu),$

(b) $\varphi_1(\mu_1) + \varphi_1(\mu_2) \leq \varphi_1(\mu_1 + \mu_2) \leq \varphi_1(\mu_1) + \varphi_1(\mu_2);$. 
the above inequalities are valid for arbitrary \( \mu_1, \mu_2 \) with the exception of the cases when one of the terms of the sum on the right-hand side (or on the left-hand side) is \(+\infty\) and the other \(-\infty\).

Let \( C_{\mu} = \{ \mu : \varrho_0(\mu) = 0 \}, \quad C_{\mu} = \{ \mu : \varrho_0(\mu) < \infty \}, \quad B_{\mu} = \{ \mu : \lim_{\mu \to \infty} f(u + \mu) - f(u) < \infty \}. \) The above relations imply the following:

1.2. The sets \( C_{\mu}, C_{\mu}, \) and \( B_{\mu} \) are rationally linear, i.e., an arbitrary linear combination with integer coefficients of elements of one of these sets belongs to the same set.

1.3. Let \( f \) be measurable. If

\[
\varrho_0(\mu) = 0 \quad \text{for an arbitrary } \mu,
\]

(\( \ast \))

\[
f(u + \mu) - f(u) \quad \text{tends to zero uniformly in every finite interval of values of } \mu \text{ as } u \to \infty
\]

(cf. \([2], [4]\));

(\( b \)) \( \varrho_0(\mu) < \infty, \varrho_0(\mu) < \infty \) for an arbitrary \( \mu \),

then the functions (\( \ast \)) are bounded uniformly in every finite interval of values of \( \mu \) for sufficiently large \( u \).

In order to prove (a) let us write \( E_\mu = \{ \mu : |f(u + \mu) - f(u)| \leq \epsilon, \mu_1 \leq \mu \leq \mu_2, u \geq 0 \} \). The sets \( E_\mu \) are measurable, \( \bigcup_{\mu_1, \mu_2} E_\mu = \bigcup_{\mu_1, \mu_2} E_\mu \). Hence at least one of the sets \( E_\mu \) must be of positive measure, say \( E_\mu \). If \( \mu', \mu'' \in E_\mu \), we have

\[
|f(u + \mu'') - f(u - \mu')| < \epsilon, \quad |f(u - \mu') - f(u)| < \epsilon \quad \text{for } u \geq \mu + 2 \mu_1,
\]

whence \( |f(u + \mu'') - f(u)| < 2\epsilon \). As is well known, there is a \( \mu_0 > 0 \) such that all \( \mu \in -\mu_0, \mu_0 \) may be expressed in the form \( \mu = \mu'' - \mu' \), where \( \mu', \mu'' \in E_\mu \). Since

\[
|f(u + \mu'' + \mu') - f(u)| \leq |f(u + \mu'') - f(u + \mu')| + |f(u + \mu') - f(u)|,
\]

and \( |f(u + \mu'' + \mu') - f(u + \mu)| \leq 2\epsilon \) when \( \mu \) belongs to \( -\mu_0, \mu_0 \), \( u \geq \mu + \mu', \mu'' \), \( u \geq \mu + \mu_1 + \mu'' \), \( f(u + \mu'') - f(u) < \epsilon \) for \( u \geq \mu_1 \), we have

\[
|f(u + \mu') - f(u)| < 2\epsilon
\]

for \( u \geq \sup(\mu + \mu_1 - \mu, u_0) \) and for \( \mu' \) belonging to an interval obtained by a translation of \( -\mu_0, \mu_0 \) by \( \lambda \). Since \( \mu_1, \mu_0 > 0 \) may be covered by a finite number of intervals which are translations of \( -\mu_0, \mu_0 \), we obtain

\[
|f(u + \mu) - f(u)| < 3\epsilon
\]

for sufficiently large \( u \) and \( \mu \in \mu_1, \mu_2 \).

The proof of part (b) of the theorem follows by analogous arguments.

Remark. The above theorem remains true if we replace the assumption of measurability of \( f \) by the assumption that \( f \) satisfies the Baire condition.

1.4. If \( f \) is continuous (measurable), then any of the sets \( C_{\mu}, C_{\mu}, B_{\mu} \) is either of the first category (measure 0) or identical with \( (-\infty, \infty) \).

To prove this theorem, let us first note that if \( f \) is continuous, then the sets \( C_{\mu}, C_{\mu}, B_{\mu} \) are \( F_{\alpha} \) and \( B_{\mu} \) is \( F_{\alpha} \), if \( f \) is measurable, then the above sets are measurable. The theorem follows from the well-known fact that a Borel set of the second category or a set of positive measure contains a rational basis, i.e., a set \( E \) such that an arbitrary \( u \) may be written in the form \( u = n_1 \mu_1 + n_2 \mu_2 + \ldots + n_k \mu_k \), \( n_i \) being integers and \( u_i \in R \). Evidently, a rationally linear set containing a rational basis is identical with \( (-\infty, \infty) \), and it is sufficient to apply 1.2.

1.5. A function \( f \) will be said to satisfy the condition \((k_3)\), resp. \((k)\), if \( C_{\mu} = (-\infty, \infty) \), resp. \( C_{\mu} = (-\infty, \infty) \). Every function of the form

\[
f(u) = g(u) + h(t) \text{dt},
\]

where \( g, h \) are continuous functions (resp. where \( g \) is measurable and \( h \) is locally integrable), \( g(u) \to c \) as \( u \to \infty \), \( h(u) \to 0 \) as \( u \to \infty \), satisfies condition \((k_3)\). Applying 1.3 (a) we may prove (cf. \([4]\)) that, conversely, an arbitrary continuous (resp. locally integrable) function satisfying \((k_3)\) may be written in the above form; \( h(u) \) may be assumed to be equal to \( f(u + 1) - f(u) \). It may be deduced from the integral representation that the set of continuous functions satisfying condition \((k_3)\) and vanishing for \( u < 0 \) is a Banach space with the usual definitions of linear operations and with the norm, say

\[
[f] = \sup_{(v, \infty)} |h(u)| + \sup_{(v, \infty)} |f(u) - h(t) \text{dt}|,
\]

where \( h(u) = f(u + 1) - f(u) \).

1.51. If \( f \) satisfies condition \((k)\), then \( \varrho_0(\mu) \) is an additive function (as follows from 1.1 (b)); \( g \), moreover, is measurable, then \( \varrho_0(\mu) \) is also measurable, whence \( \varrho_0(\mu) = 0 \). As follows from 1.3, \( a_0 \), measurable may be replaced by local boundedness of the function \( f \).

(\( \text{It is easily seen that some assumptions regarding function } f \text{ are necessary in this theorem, since Hamel's function } f, \text{ for example, obviously satisfies condition } (k) \text{ but } \varrho_0(\mu) = f(\mu) \text{ is not a linear function.} \))

An immediate consequence of the above theorem is that an arbitrary measurable (or locally bounded) function satisfying condition \((k)\) may be expressed in the form \( f(u) = \alpha u + g(u) \), where \( g(u) \) satisfies condition \((k_3)\).

1.53. It may happen for a continuous function \( f \) that \( C_{\mu} \) consists only of numbers of the form \( n\mu_0 \), where \( n = \pm 1, \pm 2, \ldots \). In order to get such a function we take, for example, \( \mu_0 = 1 \) and a continuous periodic function \( h \) with period 1. Then

\[
\varrho_0(\mu) = \sup_{0 < \alpha < 1} |h(u + \mu) - h(u)|, \quad \varrho_0(\mu) = \inf_{0 < \alpha < 1} |h(u + \mu) - h(u)|.
\]
The second of the inequalities (+) is proved analogously. 3.1 (a) and
(+) immediately imply (+).

Let us remark that the local boundedness of $f$ for large $u$ is a necessary
condition of (+) in the case when $-\infty < \tilde{g}(\mu) \leq \bar{g}(\mu) < \infty$. Indeed,
if $f$ is not locally bounded for large $u$, we have sup
$[f(u)] = \infty$ for a sequence $u_n \neq \infty$. However, for $u_n$ suitably chosen,
$u_n \leq u_0 < u_n + 1$. Then, we have
$\lim_{u \to \infty} \frac{f(u)}{u_n = \infty}$.

1.7. Given a positive $\mu$, denote by $a_\mu$, resp. $b_\mu$, numbers satisfying the inequalities

\[
f(u + \mu) - f(u) \geq a_\mu \quad \text{for} \quad u \geq u_0(\mu),
\]
resp.
\[
f(u + \mu) - f(u) \leq b_\mu \quad \text{for} \quad u \geq u_0(\mu).
\]

Let us assume that $f$ satisfies one of the conditions
(a) $-\infty < \tilde{g}(\mu) \leq \bar{g}(\mu) < \infty$ for every $\mu$, $f$ is measurable,
(b) $f$ is monotone for $u \geq 0$.

The following formulae hold:

\[
(\ast) \quad \lim_{u \to \infty} \frac{\tilde{g}(\mu)}{\mu} = \sup_{\mu \to \infty} \frac{\tilde{g}(\mu)}{\mu} = \sup_{\mu \to \infty} \frac{a_\mu}{\mu};
\]
\[
(\ast\ast) \quad \lim_{\mu \to \infty} \frac{\bar{g}(\mu)}{\mu} = \inf_{\mu \to \infty} \frac{\bar{g}(\mu)}{\mu} = \inf_{\mu \to \infty} \frac{b_\mu}{\mu}.
\]

As regards the meaning of the symbols $\sup_{\mu \leq \mu} a_\mu$ and $\sup_{\mu \geq \mu} b_\mu$, the fol-
lowing convention is here adopted: if there exists a finite value of $\mu$
(resp. $b_\mu$), we take the supremum (resp. infimum) with respect to all
possible choices of $a_\mu$ and $\mu$ (resp. $b_\mu$) and $\mu$, where $\mu > 0$. In other cases we
put $\sup_{\mu \geq \mu} a_\mu = -\infty$ (inf$_{\mu \geq \mu} b_\mu = \infty$).

We shall prove the first formula for instance. Assumption (a) means
that $B_\mu = (-\infty, \infty)$ and, by 1.3 (b), for every $\mu > 0$ there exists $u_0$
such that

\[
(+) \quad [f(u + \mu) - f(u)] \leq k \quad \text{for} \quad 0 \leq \mu \leq \mu_k, u \geq u_0.
\]

If $f(u + \mu_k) - f(u) \geq u_{0k}$ for $u \geq u_0(\mu_k)$, then
\[
f(u + \mu_k) - f(u) \geq u_0(\mu_k) \quad \text{for} \quad u \geq u_0(\mu_k) \text{ and } u = 3, 2, \ldots
\]

Hence, choosing $(\mu - 1)u_k < \mu < \mu_k$, we obtain

\[
f(u + \mu) - f(u) \geq \mu u_k + f(u + \mu) - f(u + \mu_k).
\]
Since 
\[ n \omega_n > \frac{\mu}{\mu_0} \omega_n \text{ for } \omega_n > 0 \quad \text{and} \quad n \omega_n \geq \left( \frac{\mu}{\mu_0} + 1 \right) \omega_n \text{ for } \omega_n \leq 0, \]
we have 
\[ f(u+\mu) - f(u) \geq \frac{\omega_n}{\mu} \quad \text{for } u \geq \omega_{n_1}(\mu_0), \]
and for \( \omega_n > 0 \). By \((\ast)\), we obtain
\[ \frac{\omega_n}{\mu} \geq \frac{\omega_n}{\mu_0} = \frac{k}{\mu_0}, \]
whence
\[ \lim_{\mu \to \infty} \frac{\omega_n}{\mu} = \frac{\omega_n}{\mu_0}. \]
The proof of this inequality for \( \omega_n \leq 0 \) is similar. Since \( \mu_0 \) is an arbitrary positive number, we have
\[ \lim_{\mu \to \infty} \frac{\omega_n}{\mu} \geq \sup_{\mu > 0} \frac{\omega_n}{\mu_0}. \]
Take any \( s < \sup_{\mu > 0} \frac{\omega_n}{\mu_0} \). Then \( \frac{\omega_n}{\mu_0} > s \) for a certain \( \mu_0 > 0 \), whence
\[ f(u + \mu_0) - f(u) \geq \omega_n = \omega_{n_0} \quad \text{for } u \geq \omega_{n_1}(\mu_0), \]
\[ \sup_{\mu > 0} \frac{\omega_n}{\mu} = \frac{\omega_n}{\mu_0} = s. \]
Thus we have proved
\[ \sup_{\mu > 0} \frac{\omega_n}{\mu} \geq \sup_{\mu > 0} \frac{\omega_n}{\mu_0} = \frac{\omega_n}{\mu_0} = s. \]
The proof of formula \((\ast\ast)\) is similar.

Now, let us assume \((b)\) to be satisfied. Then we obtain
\[ f(u + \mu) - f(u) \geq (n - 1) \omega_n + f(u + \mu) - f(u + (n - 1) \mu_0) \geq \omega_n \quad \text{for } u \geq \omega_{n_1}(\mu_0) \]
and any \( \mu \) satisfying the inequalities \((n - 1) \mu_0 < \mu < n \mu_0 \) if \( f \) is non-decreasing for \( u \geq 0 \). If \( f \) is non-increasing for \( u \geq 0 \), we have
\[ f(u + \mu) - f(u) \geq \omega_n \quad \text{for } u \geq \omega_{n_1}(\mu_0). \]
Arguments analogous to the preceding ones lead to the inequalities \( \lim_{\mu \to \infty} \frac{\omega_n}{\mu} \geq \sup_{\mu > 0} \frac{\omega_n}{\mu} \) and the further arguments do not differ from those in the proof under assumption \((a)\).

If \( f \) is non-increasing for \( u \geq 0 \), then for \( \mu > 0 \) a finite constant \( \omega_n \) may not exist. This is possible if and only if \( \omega_n = -\infty \) for \( \mu > 0 \).

In this case we have \( \lim_{\mu \to \infty} \frac{\omega_n}{\mu} = \sup_{\mu > 0} \frac{\omega_n}{\mu} = -\infty. \)

The following statements are consequences of 1.54 (Remark), 1.6 and 1.7:

1.71. By the assumption that either \( f \) is measurable and \(-\infty < \omega_n < \mu \) for every \( \mu \) or \( f \) is monotone, the following inequalities are satisfied:
\[ \lim_{\mu \to \infty} \frac{\omega_n}{\mu} = \lim_{\mu \to \infty} \frac{\omega_n}{\mu_0} = \frac{\omega_n}{\mu_0}, \]
where \( \omega_n \) is the \( \omega \)-number corresponding to \( \mu \), \( \mu_0 \) the \( \omega \)-number corresponding to \( \mu_0 \), \( \omega_n \) the \( \omega \)-number corresponding to \( \mu \) and \( \omega_n \) the \( \omega \)-number corresponding to \( \mu \).

1.72. If \( f \) is locally bounded for large \( u \) and if
\[ \lim_{\mu \to \infty} \frac{\omega_n}{\mu} = \lim_{\mu \to \infty} \frac{\omega_n}{\mu_0} = g \]
\( (g \text{ may also be equal to } \infty, \) then the relation
\[ \lim_{u \to \infty} \frac{f(u)}{u} = g \]
\( \text{in the generalized lHospital rule in Cauchy's form) holds.} \)

1.8. Let \( f \) possess a positive derivative \( f' \) for \( u > 0 \) and \( f' \) satisfy the condition
\[ \lim_{u \to \infty} \frac{f(u + \mu)}{f(u)} = 1 \quad \text{for every } \mu. \]
Then
\[ \lim_{u \to \infty} \frac{f(u + \mu)}{f(u)} = \lim_{\mu \to \infty} \frac{f(u + \mu)}{f(u)} = \frac{\omega_n}{\mu_0} \quad \text{for every } \mu_0 > 0, \]
\[ \lim_{u \to \infty} f(u) = \lim_{\mu \to \infty} \frac{f(u + \mu_0)}{f(u)} = \frac{\omega_n}{\mu_0} \quad \text{for every } \mu_0 > 0. \]

Given \( u > 0 \) and \( \mu_0 > 0 \), denote by \( v(u) \) a number satisfying the conditions \( v(u) < (3) \) and such that
\[ f(u + \mu_0) - f(u) = f(u + v(u)) \mu_0. \]
holds. Then we have
\[ \lim_{u \to \infty} \frac{f(u + v(u))}{f(u)} = \frac{\omega_n}{\mu_0}. \]

Define the function
\[ h(u) = \begin{cases} \log f(u) & \text{for } u > 0, \\ \log f(0) & \text{for } u < 0. \end{cases} \]
By (o), \( h \) satisfies condition (k)2, whence, by 1.3 (a),\( h(u+\mu)-h(u) \to 0 \) uniformly in \((0, \mu_0)\) as \( u \to +\infty \), i.e. (o) holds uniformly with respect to \( u \in (0, \mu_0) \). Thus
\[
\lim_{u \to +\infty} f'(u+\nu(u)) = \lim_{u \to +\infty} f'(u), \quad \lim_{u \to +\infty} f'(u+\nu(u)) = \lim_{u \to +\infty} f'(u).
\]

1.81. By the same assumptions regarding \( f \) as in 1.8, if 1.72 (**) holds, then the limit of the derivative as \( u \to +\infty \) exists, namely,
\[
\lim_{u \to +\infty} f'(u) = g.
\]

1.9. By the same assumptions regarding \( f \) as in 1.6, we have
\[
\varphi(\mu) = \alpha \mu \quad \text{for} \quad \mu \geq 0\]  
for \( \mu \in (0, \mu_0) \), where \( \alpha = \lim_{u \to +\infty} f(u)/u \).

2. In this section (with the exception of 2.12 and 2.8) \( \varphi, \psi, x, u, \ldots \) always denote measurable positive functions defined for \( u > 0 \). According to [1], such a function is called a \( \varphi \)-function if it is continuous and non-decreasing, defined for \( u = 0 \) by \( \varphi(0) = 0 \), and tends to infinity as \( u \to +\infty \). We shall apply the symbols
\[
h_u(\lambda) = \lim_{u \to +\infty} \frac{\varphi(u)}{\varphi(u+\lambda)}, \quad n_u(\lambda) = \lim_{u \to -\infty} \frac{\varphi(u)}{\varphi(u+\lambda)} \quad \text{for} \quad \lambda > 0.
\]

If \( h_u(\lambda) = n_u(\lambda) = h_u(\lambda) \), then \( h_u(\lambda) \) is finite for \( \lambda > 0 \), and \( \alpha \neq 1 \), we call \( \varphi \) regularly increasing, according to the terminology of [2] and [3].

Let us remark in connection with the assumption of measurability of \( \varphi \) in the above theorem that \( \varphi(u) = e^{\lambda u} \), where \( \lambda \) is a non-measurable function, is not regularly increasing and the indices \( k_u, \alpha_u \) do not exist, although \( h_u(\lambda) = n_u(\lambda) = e^{-\lambda u} \), \( \lambda \) being an unbounded index in a certain interval, whence continuous. If \( \alpha_u = \pm \infty \), \( \lambda \) would be bounded from below (from above) in a certain interval, but this is impossible. Thus the index \( \alpha_u \) does not exist; it is similarly proved that \( \alpha_u \) also does not exist.

Let \( x_u = \alpha_u = r_u \), where \( \alpha_u \neq 0, \alpha_u < \infty \) for a \( \varphi \)-function \( \varphi \), we call \( \varphi \) quasi-regularly increasing; if \( x_u = \alpha_u = r_u = 0 \), we call the \( \varphi \)-function \( \varphi \) quasi-slowly varying. Also in this case \( r_u \) is called the index (of quasi-regularity).

2.11. If \( \varphi \) is regularly increasing or slowly varying, then
\[
\lim_{u \to +\infty} \frac{\varphi(\lambda u)}{\varphi(u)} = \lambda^\alpha, \quad \text{as} \quad u \to +\infty,
\]
ununiformly in every interval \( 0 < \lambda \leq \Lambda \leq \lambda_0 \).

We apply the substitution (**) and 1.51, 1.3. We obtain \( \varphi(\lambda u)/\varphi(u) \to \lambda^\alpha \) if \( \lambda > 0, u \to +\infty \). Applying the definition of the indices \( x_u, \alpha_u \), we obtain \( x_u = \alpha_u = r_u \).
2.12. If we replace the assumption of measurability of $\varphi$ in the definition of a regularly increasing or slowly varying function by the local boundedness of $\log \varphi$, relation (++) remains true for every $\lambda > 0$, although the uniform convergence may not hold. Identity 2.1 (++), where $\psi$ is of index $r_\psi = 0$, remains also true.

This follows by applying the substitution $(\psi)$ and 1.9.

2.13. If $\varphi$ is regularly increasing (quasi-regularly increasing) or slowly varying (quasi-slowly varying), then $\log \varphi$ is locally bounded for large $u$.

This follows by applying the substitution $(\psi)$ and 1.54.

A function $\varphi$ is called locally bounded if it is bounded in an arbitrary interval $(0, \psi)$. It follows from 2.13 that replacing a regularly increasing (slowly varying) function $\varphi$ by a function $\varphi_0$ such that $\varphi_0(u) = \varphi(u)$ for $0 < u < \varepsilon$, $\varphi_0(u) = \varphi(u)$ for $u > \varepsilon$, where $\varepsilon$ is sufficiently large, we obtain a regularly increasing (slowly varying) function which is locally bounded.

2.2. Let $\log \varphi$ be locally bounded for large $u$.

(a) If the limits $\psi_\varphi$, $\sigma_\varphi$ exist, then

$\psi_\varphi \leq \lim_{u \to \infty} \frac{\log \varphi(u)}{\log u} \leq \lim_{u \to \infty} \log \varphi(u) \leq \sigma_\varphi$.

(b) If the limits $\psi_\varphi$, $\sigma_\varphi$ exist and $\psi_\varphi = \sigma_\varphi$ (in particular, if $\varphi$ is regularly increasing or slowly varying), then

$\lim_{u \to \infty} \frac{-\log \varphi(u)}{\log u} = r_\psi$.

The above theorems are obtained immediately by applying the substitution $(\psi)$ and 1.54, 1.72.

2.5. Denote by $\mathcal{R}_\varphi$, resp. $\mathcal{R}_\psi$, the class of regularly increasing, resp. slowly varying, functions $\varphi$.

(a) If $\varphi, \psi \in \mathcal{R}_\varphi$, then $r_\psi = r_\varphi + r_\psi$ and $\varphi \psi \in \mathcal{R}_\varphi$ for $r_\varphi \neq 0$ and $\varphi \psi \in \mathcal{R}_\psi$ for $r_\varphi = 0$.

(b) If $\varphi, \psi \in \mathcal{R}_\varphi$, then $\varphi \psi \in \mathcal{R}_\psi$.

(c) If $\varphi \in \mathcal{R}_\varphi$, then $r_{\varphi^{-1}} = -r_\varphi$ and $1/\varphi \in \mathcal{R}_\varphi$.

(d) If $\varphi \in \mathcal{R}_\varphi$, then $1/\psi \in \mathcal{R}_\varphi$.

(e) If $\varphi \in \mathcal{R}_\varphi$, then $\varphi_k = k \varphi$, $k \psi \in \mathcal{R}_\psi$, when $k \neq 0$.

(f) If $\varphi \in \mathcal{R}_\varphi$, then $q \psi \in \mathcal{R}_\varphi$ for an arbitrary $k$.

(g) If $\varphi, \psi \in \mathcal{R}_\psi$, then $\varphi \psi \in \mathcal{R}_\psi$ when $\psi \to \infty$ as $u \to \infty$.

(h) By the same assumptions on $\varphi, \psi$ as in (g), if at least one of the functions $\varphi, \psi$ belongs to $\mathcal{R}_\varphi$, and the second one belongs to $\mathcal{R}_\psi$, then $\varphi \psi \in \mathcal{R}_\varphi$.

(i) If $\varphi$ is a strictly increasing $\varphi$-function and $\psi \in \mathcal{R}_\varphi$, then $\varphi^{-1} \in \mathcal{R}_\psi$ and $r_{\varphi^{-1}} = 1/r_\varphi$ (cf. [2]).

Remark. The above theorems remain true if we omit the assumption of measurability in the definition of a regularly increasing, resp. slowly varying, function, replacing it in $(\alpha)$, $(\beta)$, $(\gamma)$ by the assumption of local boundedness of the functions $\log \varphi$, $\log \psi$ in $(\alpha)$ by the assumption of measurability of $\varphi$ and local boundedness of $\log \psi$.

Theorems $(\alpha)$—$(\gamma)$ follow from the definition of a regularly increasing, resp. slowly varying, function and from 2.12 immediately.

Ad (g). Let $\psi(2u) = \psi(u)$, $\lambda > 0$, whence $\psi(u) \to \lambda u \vee$ as $u \to \infty$.

By 2.11, $\varphi(\lambda u) / \varphi(u) \to \lambda u$, as $u \to \infty$, uniformly in each interval $0 < u < \lambda u$.

Hence

$\frac{\log \varphi(2u)}{\log \varphi(u)} = \frac{\log \psi(u)}{\log \psi(2u)} \to \lambda u$ as $u \to \infty$.

Thus $\varphi(u)$ is regularly increasing and $r_{\psi(u)} = r_\psi$.

(b) is proved similarly.

Ad (i). Let $\varphi(u) = \psi(u)$, $\varphi^{-1}(u) = u$, $1 < u < \infty$, $\varphi(\lambda u) = \mu u$, where $\lambda > 1$ is defined uniquely. There exists a constant $\lambda_0$ such that $\lambda_0 \leq \lambda_0$ for $\lambda > \lambda_0$; indeed, otherwise we should have $\lambda_0 \geq (\mu u)^{1/r_\varphi}$ for a sequence $u_0 \to \infty$. If $\varphi^{-1}(u_0) = u_0$, then

$\mu = \varphi(\lambda_0 u_0) \geq \varphi((\mu u)^{1/r_\varphi}) = ((\mu u)^{1/r_\varphi})^{r_\psi}$,

which is a contradiction. Let $\lambda_0 \to \lambda$; since, by 2.11, $\varphi(\lambda u) / \varphi(u) \to \lambda u$ uniformly in $1 \leq \lambda \leq \lambda_0$, we obtain

$\mu = \varphi(\lambda_0 u_0) \rightarrow \lambda u_0$,

i.e. $g = \mu u_0$. We have thus proved that $\lambda_0 \rightarrow \mu u_0$ as $u \rightarrow \infty$; thus

$\varphi^{-1}(u) / \varphi(u) = \lambda u / u \rightarrow \mu u_0$ for $\mu u_0 > 1$ and hence for an arbitrary $\mu > 0$.

2.31. If $\varphi$ is a strictly increasing $\varphi$-function and $\psi \in \mathcal{R}_\varphi$, then the following relation holds for the inverse function:

$(\psi)$

$\varphi^{-1}(u) / \psi^{-1}(v) \rightarrow \infty$ as $u \rightarrow \infty$,

for every $u > 1$. Conversely, if $(\psi)$ holds, then $\varphi \in \mathcal{R}_\varphi$.

Let $\lambda_0, \lambda_1$ have the same meaning as in the proof of 2.3 (i). Suppose $\lambda_0 \rightarrow \lambda$ for a sequence $u_0 \rightarrow \infty$, where $g = \mu$ is a finite limit. Since $\varphi(\lambda u) / \varphi(u) \rightarrow 1$ uniformly in $1 \leq \lambda \leq \lambda + 1$, we have $\mu = \varphi(\lambda_0 u_0) / \varphi(u_0) \rightarrow 1$, i.e. $\mu = 1$, which is a contradiction. Thus we have proved that $\lambda_0 \rightarrow \infty$ as $u \rightarrow \infty$ and, consequently, $(\psi)$. 

Regularity increasing functions
In order to prove the second part of the theorem let us write \( \varphi(\lambda u) = \mu \gamma, \lambda \geq 1 \). Since \( \lambda = \varphi^{-1}(\mu \gamma) = \varphi^{-1}(\mu \gamma), \) we have \( \mu \to 1 \) as \( u \to \infty \), by \((\dagger)\). Thus \( \varphi(\lambda u) = \mu \gamma \) for \( \lambda \geq 1 \) and hence also for \( 0 \leq \lambda \leq 1 \).

2.4. We now introduce some notions which are of importance, particularly in the theory of the spaces \( L^p(a, b) \), but which are also of interest in studying the order of growth of functions. We shall say that \( \varphi \) is \( \lambda \)-equivalent to \( \psi \) (equivalent to \( \psi \) for large \( u \)), in symbols \( \varphi \sim \psi \), if the inequalities

\[
(\dagger) \quad a \varphi(ku) \leq \varphi(u) \leq b \psi(ku)
\]

hold for \( u \approx u_k \), where \( a, b, k, k_2 \) are some positive constants (see [6]).

\( \varphi \sim \psi \), resp. \( \varphi \asymp \psi \), will mean that \( \varphi \) and \( \psi \) are asymptotically similar, resp. asymptotically equal, i.e. that \( \varphi(u)/\psi(u) \to e \) as \( u \to \infty \), where \( e \neq 0, \epsilon < 1 \).

Evidently, \( \varphi \sim \psi \) implies \( \varphi \asymp \psi \) but not conversely. Similarly to \( \sim \), \( \asymp \) is also an equivalence relation and elementary rules of calculus for \( \asymp \) are valid also for \( \asymp \). For instance if \( \varphi \asymp \psi \) non-decreasing (non-increasing), \( \varphi \asymp \psi \psi \), then \( \varphi + \psi \asymp \varphi + \psi \psi \), \( \varphi \varphi \psi \asymp \varphi \psi \psi \), etc. If \( \varphi \) is a \( q \)-function, then every function \( \lambda \)-equivalent to \( \varphi \) is also a \( q \)-function. It is also easily seen that for a \( q \)-function the indices \( s_0, s_2 \) are invariants of the relation \( \asymp \). However, the property that \( \varphi \) is regularly increasing (resp. slowly varying) remains valid for an asymptotically similar (resp. equal) function, but in general does not remain valid for a function \( \lambda \)-equivalent to the given one. The following remark makes clear the advantage of applying the notion of \( \lambda \)-equivalence in place of the less general notion of asymptotic equality, when investigating orders of growth of functions.

2.41. If \( \varphi, \psi \) are strictly increasing \( q \)-functions, \( \varphi \asymp \psi \), then \( q^{-1} \asymp q^{-1} \) (see [9]).

If \( \varphi \approx \psi \), then \( q^{-1} \approx q^{-1} \) does not need to hold. For instance the functions \( \varphi(u) = \log(1+u), \psi(u) = \varphi(u) \log(1+u) \), where \( \varphi(u) \) is a continuous function strictly increasing from 0 to 1, are asymptotically equal. However, if we choose \( q \) suitably, their inverse functions are not asymptotically similar. It is sufficient to choose an arbitrary sequence \( \alpha_n \to 1 \) and \( u_n, u', u'' \) so that \( u' < u'' < u'' + 1 < u_n \), \( u_n = \alpha_n^{u''} - 1 \), \( (\alpha_n^{u''} - 1)^{u''+1} > e \), and to define \( \varphi(u) = \alpha_n \) for \( u' < u < u_n \), \( \varphi(u) = \alpha_n^{u''} \) for \( u_n < u < u'' \), \( \varphi(u) = \alpha_n \) for \( u'' \). Now, \( \varphi(u) = \alpha_n \varphi(u) \) is a linear function in \( u_n, u'' \).

2.42. If \( \varphi \) is a \( q \)-function, then \( \varphi(\psi(u)) = \varphi(\psi(u')) \) and \( \varphi(\psi(u)) = \varphi(\psi(u')) \), \( r > 0 \), are \( q \)-functions, and \( \varphi(\psi(u)) = \varphi(\psi(u')) \). Since

\[
\frac{\varphi(u)}{\psi(u)} = \frac{\varphi(u)}{\psi(u')} \quad \text{and} \quad \frac{\varphi(u)}{\psi(u')} = \frac{\varphi(u)}{\psi(u')},
\]

we have \( h_k(\lambda u) = h_k(\lambda u') \), \( h_k(\lambda u) = h_k(\lambda u') \), \( h_k(\lambda u) = h_k(\lambda u') \), \( h_k(\lambda u) = h_k(\lambda u') \), \( h_k(\lambda u) = h_k(\lambda u') \), \( h_k(\lambda u) = h_k(\lambda u') \).

\[
\frac{1}{r} = \lim_{\lambda \to 1} \frac{\log h_k(\lambda)}{\log h_k(\lambda')} = \lim_{\lambda \to 1} \frac{\log h_k(\lambda)}{\log h_k(\lambda')} = s,
\]

and similarly in the remaining cases.

Let us note that, in spite of the fact that the indices of \( \varphi_k \) and \( \varphi_k \) are equal, these functions need not be \( \lambda \)-equivalent if \( r \neq 1 \). For instance let \( \varphi(u) = \varphi(u(1+u)) \), where \( \varphi(u) \) is a regularly increasing or slowly varying \( \varphi \)-function. Then \( \varphi(u') \varphi'(u) \to r' \) as \( u \to \infty \) for \( r > 0 \). If \( \varphi_1 \leq \varphi \), then \( \varphi \) \( \asymp \varphi \), i.e. \( \varphi(\lambda u) \varphi(u) \leq \varphi(\lambda u') \varphi(u') \leq \varphi_1(\lambda u') \varphi(u) \) for large \( u \). But this is impossible for \( r \neq 1 \), because, according to 2.3 (g), \( \varphi \) is a slowly varying \( \varphi \)-function.

2.5. A function \( \varphi \) is said to satisfy condition \( (\Delta) \) for large \( u \) if \( a > 1 \) and if the inequality

\[
\varphi(a u) \leq a \varphi(u) \quad \text{for} \quad u \geq u(a)
\]

holds for a constant \( a > 1 \). \( \varphi \) is said to satisfy condition \( (\Delta) \) for large \( u \) if \( a > 1 \) and if the inequality

\[
\varphi(a u) \leq a \varphi(u) \quad \text{for} \quad u \geq u(a)
\]

is satisfied for a constant \( a > 1 \). For non-decreasing \( \varphi \) the property that condition \( (\Delta) \) (condition \( (\Delta) \)) holds with an \( a > 1 \) is an invariant of \( \lambda \)-equivalence (cf. [6]).

2.51. If \( \varphi \) is a \( q \)-function, then the conditions

\( (a) \) \( \varphi(u) \) is satisfied for sufficiently large \( u \) and the conditions

\( (b) \) \( \varphi(u) \) are equivalent, and the conditions

\( (b') \) \( \varphi(u) \) is satisfied for sufficiently large \( u \) and \( \lambda \)_equivalent.

In order to prove \( (a) \Rightarrow (b) \), let us note that \( h_k(\lambda u) = h_k(\lambda u') \), where \( h_k(\lambda u) = h_k(\lambda u') \), whence \( \varphi(\omega u) \asymp \varphi(\omega u) \) for \( \omega > u(a) \), i.e. we may take \( \omega = m, \omega > u(a) \). If \( \varphi \) satisfies condition \( (\Delta) \), then \( \varphi(u) \asymp \varphi(u) \) for \( u \geq u(a) \), \( \varphi(u) \asymp \varphi(u) \) for \( u \geq u(a) \), \( \varphi(u) \asymp \varphi(u) \) for \( u \geq u(a) \), \( \varphi(u) \asymp \varphi(u) \) for \( u \geq u(a) \), \( \varphi(u) \asymp \varphi(u) \) for \( u \geq u(a) \), \( \varphi(u) \asymp \varphi(u) \) for \( u \geq u(a) \), \( \varphi(u) \asymp \varphi(u) \) for \( u \geq u(a) \), \( \varphi(u) \asymp \varphi(u) \) for \( u \geq u(a) \).

Remark. Let us note that for a non-decreasing \( \varphi \) (in particular for a \( q \)-function), \( (\Delta) \) for an \( a \) implies \( (a') \) and \( (\Delta) \) for an \( a \) implies \( (\Delta) \) for every \( a > 1 \).
25.2. Let \( \varphi \) be a \( p \)-function.

(a) If \( s_\varphi > 0 \), then

\[
\varphi'(u) = \sup_{\alpha, \beta < \infty} (\log \varphi'_{\alpha, \beta}(u))
\]

where the supremum is taken over all pairs of numbers \( \alpha, \beta \) which occur in the definition of condition \( \langle \Delta \rangle \).

(b) If \( s_\varphi < \infty \), then

\[
\varphi'(u) = \inf_{\alpha, \beta < \infty} (\log \varphi'_{\alpha, \beta}(u))
\]

where the infimum is taken over all pairs of numbers \( \alpha, \beta \) which occur in the definition of condition \( \langle \Delta \rangle \).

This follows from 1.7 (b) by applying the substitution \( \varphi' \).

25.3. If conditions \( \langle \Delta \rangle \) are satisfied for sufficiently large \( \alpha \), then \( \varphi \) is a \( q \)-function, \( s_\varphi > 0 \), \( s_\varphi < \infty \), and formulae 2.52 (\( + \)), (\( ++ \)) are satisfied.

Applying the substitution \( \varphi' \) in this case we can easily see that the corresponding function \( f \) satisfies the inequalities \(-\infty < g_f(\mu) < \tilde{g}_f(\mu) < \infty \) for large \( \mu \), whence these inequalities are satisfied for every \( \mu \), by 1.4. Formulae 1.7 (\( + \)), (\( ** \)) yield the proof of existence of the indices and formulae 2.52 (\( + \)), (\( ++ \)), simultaneously.

2.6. Given a function \( \varphi \), write

\[
\varphi'_{\alpha}(u) = \sup_{\alpha < \infty} \varphi'_{\alpha}(u)
\]

\[
\varphi'_{\alpha}(u) = \sup_{\alpha < \infty} \varphi'_{\alpha}(u)
\]

\[
\varphi'_{\alpha}(u) = \inf_{\alpha, \beta < \infty} \varphi'_{\alpha, \beta}(u)
\]

\[
\varphi'_{\alpha, \beta}(u) = \inf_{\alpha, \beta < \infty} \varphi'_{\alpha, \beta}(u)
\]

Obviously, if \( \varphi'_{\alpha}(u) < \infty \) for \( u > 0 \) and \( \varphi'_{\alpha}(u) \to \infty \) as \( u \to \infty \), then

\[
\varphi'_{\alpha}(u) \geq \varphi'_{\alpha}(u)
\]

A function \( \varphi \) is called pseudo-increasing for large \( u \) if

\[
\varphi'(\alpha) > m_\varphi(\alpha) \quad \text{for} \quad \alpha \geq \alpha_0
\]

for some constants \( m_\varphi, \alpha > 0 \); it is called pseudo-decreasing for large \( u \) if

\[
\varphi'(\alpha) > m_\varphi(\alpha) \quad \text{for} \quad \alpha \geq \alpha_0
\]

26.4. A function \( \varphi \) is pseudo-increasing (pseudo-decreasing) for large \( u \) if and only if it is \( l \)-equivalent with a non-decreasing (non-increasing) function.

The sufficiency follows from the definition of \( l \)-equivalence immediately. In order to prove the necessity let us note that (\( + \)) implies

\[
\varphi'(u) > m_\varphi(u) \quad \text{for} \quad u \geq \sup(u_0, \alpha_0)
\]

and (\( ++ \)) implies

\[
\varphi'(u) \leq \varphi'_{\alpha}(u) \leq m_\varphi(u) \quad \text{for} \quad u \geq \alpha_0,
\]

whence \( \varphi \approx \varphi_{\alpha} \) in the first case and \( \varphi \approx \varphi_{\alpha} \) in the second case.

From the above it follows that

26.41. If a function \( \varphi(u) \) is pseudo-increasing for large \( u \) and \( \varphi(u) \geq \varphi_{\alpha}(u) \) for \( u \geq \alpha_0 \), then

(\( + \)) \quad \varphi'(u) \geq k_0(u) \quad \text{for} \quad u \geq u_0

holds for a constant \( k_0 > 0 \);

(\( ++ \)) \quad \varphi'(u) \leq k_0(u) \quad \text{for} \quad u \geq u_0

holds for a constant \( k_0 > 0 \).

We shall prove (a) for example. We may restrict ourselves to the case \( \alpha = 1 \). Let \( 0 < \eta < 1 \), \( \epsilon_0 = \epsilon_0 \), \( \epsilon_0 = \epsilon_0/\epsilon_0 \), \( \epsilon_0 = \epsilon_0/\epsilon_0 \). Since \( u_0 \geq u_0 \), we have \( \varphi'(u) \geq m_\varphi(u) \) for \( u \geq u_0 \), by 2.6 (\( + \)). Since for every \( 0 < \eta < 1 \), \( u_0 \geq u_0 \), \( u_0 \geq u_0 \), we have \( \varphi'(u) \geq m_\varphi(u) \) for \( u \geq u_0 \), and \( 1 \leq \eta < 1/\epsilon_0 \). If \( u \geq 1 \) and \( \alpha_0 \geq 0 \), then applying the inequality \( \varphi'(u) \geq m_\varphi(u) \) for \( u \geq u_0 \), and \( 1 \leq \eta < 1/\epsilon_0 \), we obtain \( \varphi'(u) \geq m_\varphi(u) \) for \( u \geq u_0 \), i.e., by 2.6 (\( + \)), \( \varphi'(u) \geq m_\varphi(u) \geq \sup(u_0, \alpha_0) \). Thus we have proved (\( + \)) with a constant \( k = \inf(u_0, \alpha_0, 1 \eta, 1 \eta, \epsilon_0/\epsilon_0) \), where \( 0 < \eta < 1 \) may be chosen arbitrarily.

The arguments in the case (b) are similar.

2.6.5. If \( \varphi \) is regularly increasing and \( r_0 > 0 \), then \( \varphi \approx \varphi'_{\alpha} \) for a certain \( \alpha \). If moreover, \( \varphi \) is locally bounded, then \( \varphi \approx \varphi'_{\alpha} \).

Choose an arbitrary \( \alpha_0 > 1 \), \(\alpha_0 > 1 \epsilon_0 \). By 2.11, \( \varphi'(u) \geq \varphi'(u) \) for \( u \geq u_0 \), \( 1 \leq \lambda < \alpha_0 \), i.e., \( \varphi'(u) \geq \varphi'(u) \) for \( u \geq u_0 \), and \( k = 1 \). \( 2 \), \( 3 \), \( 4 \), \( 5 \), \( 6 \) and \( \alpha_0 > 1 \). Consequently, \( \varphi'(u) \geq \varphi'(u) \) for \( u \geq u_0 \), \( \alpha_0 > 1 \), whence \( \varphi(u) \geq (1-\epsilon)(u) \). But \( \varphi(u) \to \infty \) as \( u \to \infty \), by 2.6 (b); thus \( \varphi'(u) \to \infty \).
and if \( \phi \) is bounded in a neighbourhood of 0 then, according to 2.6, \( \psi(u) < \infty \) for every \( u > 0 \) and \( c_\varepsilon \leq \psi' \). Hence in this case \( \psi(u) \geq 1 - 2e \psi'(u) \) for sufficiently large \( u \), i.e., \( \phi \simeq \psi \).

The first part of the theorem is obtained by modifying \( \phi \) in a neighbourhood of 0 in order to get a locally bounded function.

2.64. A function \( \phi \) is slowly varying if and only if \( \phi(u)u^\varepsilon \) is asymptotically equal to a non-decreasing function and \( \phi(u)u^{-1} \) is asymptotically equal to a non-decreasing function for every \( \varepsilon > 0 \) (see [2] and [13]).

Sufficiency. Let \( \varepsilon > 0 \) be given. We choose \( k > 0 \) so that \( 1 + \varepsilon \geq (1 + k)^{1/\delta} \). Then the inequalities \( \delta' \psi'(u) \geq (1 + k)^{1/\delta} \psi'(u) \) and \( \delta' \psi(u) \leq (1 + k)^{1/\delta} \psi(u) \) hold for sufficiently large \( u \), whence

\[
1 + \varepsilon \geq \lim_{u \to \infty} \frac{\psi(u)}{\psi(u)} \leq \lim_{u \to \infty} \frac{\psi(u)}{\psi(u)} \geq 1 - \varepsilon.
\]

Necessity. Given any \( \varepsilon > 0 \), the functions \( \varphi_1(u) = u^\varepsilon \psi(u) \) and \( \varphi_2(u) = u^{-1} \psi(u) \) are regularly increasing with index \( \varepsilon \). By 2.53, \( \varphi_1 \simeq \varphi_2 \), \( 1 \varphi_1 \simeq \varphi_2 \).

2.65. A function \( \phi \) is quasi-slowly varying if and only if for every \( \varepsilon > 0 \) the function \( \phi(u)u^\varepsilon \) is pseudo-decreasing for large \( u \) and the function \( \phi(u)u^{-1} \) is pseudo-decreasing for large \( u \).

Sufficiency. Take an \( \varepsilon > 0 \). Then \( \varphi_1(u) = \phi(u)u^\varepsilon \) and \( \varphi_2(u) = \phi(u)u^{-1} \) are regular functions, by 2.6 and 2.61. Hence \( \phi \) is also a \( q \)-function. Since \( \varepsilon_\delta = 0 \), \( -\varepsilon + \varepsilon_\delta = 0 \), \( -\varepsilon + 0 = 0 \), we have \( \varepsilon = \varepsilon_\delta = c_\delta = 0 \).

Necessity. If \( \varepsilon_\delta = \varepsilon_\delta = 0 \), then \( \varepsilon_\delta < \varepsilon_\delta < \varepsilon_\delta = \varepsilon \). It follows from the definition of the indices that if \( \alpha > \alpha_\delta > 1 \), the inequalities

\[
\delta' \leq \frac{\varphi_1(u)}{\varphi_1(u)} \leq \delta \quad \text{for} \quad u \geq \sup(u_\alpha(u))
\]

are satisfied for given \( \varepsilon' > \varepsilon > 0 \). Applying the substitution \( \varphi_1(u) \), 1.4, 1.3 (b) we easily show the inequalities

\[
c_\varepsilon \leq \frac{\varphi_1(u)}{\varphi_1(u)} \leq c_\varepsilon
\]

to hold uniformly with respect to \( \alpha \) in \( 1 \leq \alpha \leq \alpha_\delta \) for \( u \geq \sup(u_\alpha) \). Let \( 1 \leq \alpha < \varepsilon_\delta \), \( \alpha < \varepsilon_\delta \) for \( k = 0, 1, 2, \ldots \). Since \( \alpha = \varepsilon_\delta \), where \( 1 \leq \alpha < \varepsilon_\delta \), it follows that

\[
\varphi_1(u) = \varphi_1(u_\alpha u) \geq (\alpha_\delta / \varphi_1(u_\delta)) \geq c_\varepsilon \psi(u) \quad \text{for} \quad u \geq \sup(u_\alpha, u_\delta).
\]

Similarly we prove that \( \varphi_2(u) \) is pseudo-decreasing for large \( u \).

2.7. (a) Let \( \phi \) be such that \( \phi(u)u^\varepsilon \) is asymptotically equal to a non-decreasing function for \( \varepsilon > 0 \). The function \( \phi \) is \( 1 \)-equivalent to a convex \( \varphi \)-function if and only if the inequality

\[
\frac{\varphi(u_\delta)}{\varphi(u_\delta)} \geq \frac{\varphi(u)}{\varphi(u)} \quad \text{for} \quad u \geq u_\delta \geq u_\delta,
\]

is satisfied for a certain constant \( k > 0 \).

(b) If \( \phi \) is a \( q \)-function and we change in \( (\dagger) \) the sign \( \psi \) to \( \psi \), we obtain one necessary and sufficient condition of \( 1 \)-equivalence of \( \phi \) to a certain \( \varphi \)-function (cf. [8] and [25]).

First, we consider the case of \( 1 \)-equivalence to a convex function. Let \( \psi \leq \psi \), where \( \psi \) is a convex \( \psi \)-function. Inequality 2.4 \( (\dagger) \) holds for \( u \geq u_\delta \), whence, for \( u_\delta \geq u_\delta = \sup(u_\delta, 2 \varphi_\delta(u_\delta)) \),

\[
\frac{\varphi(u_\delta)}{\varphi(u_\delta)} \geq \frac{\psi(u_\delta)}{\varphi(u_\delta)} \geq \frac{\psi(u_\delta)}{\psi(u_\delta)} \geq \frac{\psi(u_\delta)}{\psi(u_\delta)},
\]

because \( \psi(u) \leq \varphi(u) \) for \( 0 < \alpha < 1 \). Since \( \phi(u) - \psi(u)u \) satisfies 2.6 \( (\dagger) \) and \( \phi(u)u \) is asymptotically equal to a non-decreasing function for \( \varepsilon > 0 \), inequality \( (\dagger) \) follows from 2.62. In order to prove the sufficiency let us define the function \( \psi(u) = \psi(u)u \), where \( \phi(u) = \psi(u)u \) and \( \varepsilon \) is equal to \( u_\delta \) from (\dagger). Assume as in the proof of 2.61 imply \( \psi(u)u \) holds for \( u \leq u_\delta \). The function

\[
\psi(u) = \int \phi(t) \, dt
\]

is a convex \( \psi \)-function and since \( \int \phi(t) \, dt \leq \psi(u)u \) for \( u > 0 \), we have \( \psi \leq \psi \).

Now, we consider the case (b). Adding to \( \psi \) a continuous function \( \phi \) strictly increasing from 0 to 1 as \( u \to -\infty \), we obtain a \( \phi \)-function \( \phi \) strictly increasing, asymptotically equal to \( \psi \) and such that the inequality

\[
(\dagger+) \quad \frac{\psi(u_\delta)}{\psi(u_\delta)} \leq \frac{\varphi(u_\delta)}{\psi(u_\delta)} \quad \text{for} \quad u_\delta \geq u_\delta \geq u_\delta, \quad k > 0,
\]

holds for certain \( k_\delta, u_\delta \) if and only if the inequality

\[
\frac{\psi(u_\delta)}{\psi(u_\delta)} \leq \frac{\varphi(u_\delta)}{\psi(u_\delta)} \quad \text{for} \quad u_\delta \geq u_\delta \geq u_\delta, \quad k > 0
\]

is satisfied for some \( k, u_\delta \). Obviously, inequality \( (\dagger+) \) is equivalent to the inequality

\[
\frac{\psi(u_\delta)}{\psi(u_\delta)} \geq \frac{\psi(u_\delta)}{\psi(u_\delta)} \quad \text{for} \quad u_\delta \geq u_\delta \geq u_\delta, \quad k > 0
\]
Since, by (a), \( \varphi^{-1} \overset{\sim}{\rightarrow} \psi \), where \( \psi \) is a convex \( \varphi \)-function, by 2.41 we have \( \varphi \overset{\sim}{\rightarrow} \psi \), where \( \psi = \psi^\varphi \) is a concave \( \varphi \)-function, and since \( \varphi \overset{\sim}{\rightarrow} \varphi \) we have \( \varphi \overset{\sim}{\rightarrow} \varphi \).

**Remark.** The following question arises: is it possible to define a function \( \psi \) in a way analogous to that in the case (a)? Let \( \psi(\xi) = \xi \varphi(\xi) \), where \( \varphi(\xi) = \varphi(\xi)/\xi \) and \( \varphi \) is equal to \( \varphi \). However, this holds if we assume that \( \varphi \) is positive or that condition (\( \Lambda_a \)) is satisfied for an \( a > 0 \) (both these assumptions are equivalent), for then we have \( \varphi \) positive and 2.62 may be applied. The same remark concerns the application of \( \varphi(s) = \inf \varphi(\xi)/\xi \) in place of the function \( \varphi(s) \) given in [7] on p. 127. I might notice here that the method of proof in the above-mentioned fragment of [7] may be applied, for example, if we assume (\( \Lambda_a \)).

2.71. (a) A convex \( \varphi \)-function is superadditive, i.e.

\[
\varphi(u_1 + u_2) \geq \varphi(u_1) + \varphi(u_2) \quad \text{for} \quad u_1 \geq u_2 \geq 0;
\]

a superadditive \( \varphi \)-function is equivalent to a convex \( \varphi \)-function.

(b) A concave \( \varphi \)-function is subadditive, i.e. \((+)\) holds, where the sign \( \geq \) has been changed into \( \leq \); a subadditive \( \varphi \)-function is equivalent to a concave \( \varphi \)-function.

Ad (a). Since \( \varphi(u_1)u_1 \leq \varphi(u_2)u_2 \) for \( u_2 \geq u_1 \), we have

\[
\varphi(u_1 + u_2) = u_1 \varphi(u_1 + u_2) / u_1 + u_2 + \varphi(u_1 + u_2) = u_1 \varphi(u_1) / u_1 + u_2 \geq \varphi(u_2) / u_2.
\]

Let us suppose that \( \varphi \) is superadditive. Let \( u_2 \geq u_1 > 0 \) and let \( n \) denote a non-negative integer such that \( 2^nu_1 \leq u_2 < 2^{n+1}u_1 \). It follows from the superadditivity that

\[
\varphi(u_2) / u_2 \geq \varphi(2^nu_1) / u_2 \geq 2^n \varphi(u_1) / u_1 = 2 \varphi(u_1),
\]

and it is sufficient to apply 2.7 (a).

2.72. In the following properties \( \varphi, \chi, \psi \) denote \( \varphi \)-functions, \( r > 0 \):

A. \( \varphi \overset{\sim}{\rightarrow} \chi \); \( \varphi(\chi) = (\varphi(\psi)) \), \( \varphi \) convex.

B. \( \varphi \overset{\sim}{\rightarrow} \chi \); \( \chi(\psi) = (\chi(\varphi)) \), \( \psi \) convex.

C. \( \varphi \overset{\sim}{\rightarrow} \chi \); \( \chi \) is superadditive in a generalized sense:

\[
\chi(u_1 + u_2) \geq (\chi(u_1))^\varphi + (\chi(u_2))^\varphi \quad \text{for} \quad u_2 \geq u_1 \geq 0.
\]

D. \( \varphi(u) = u^\varphi(u) \), where \( \varphi \) is pseudo-increasing for large \( u \).

Properties \( A, B \) will be obtained from \( A, B \) by replacing the word "convex" by "concave"; property \( C \) will be obtained from \( C \) by replacing the sign \( \geq \) in inequality \((+)\) by \( \leq \), i.e. by replacing generalized superadditivity by generalized subadditivity. Finally, property \( D \) will be obtained from \( D \) by replacing the phrase "\( \varphi \) is pseudo-increasing for large \( u \)" by "\( \varphi \) is pseudo-decreasing for large \( u \)."

2.73. Any two of the properties \( A-D \) are equivalent; moreover, any two of the properties \( A, D \) are also equivalent.

This theorem is a consequence of 2.7, 2.7.1 by the fact that property \( D \), resp. \( D_a \), means that \( \varphi(u) = (\varphi(u))^\varphi \) satisfy 2.7 \((+)\), resp. 2.7 \((\ast)\), with the sign \( \geq \) replaced by \( \leq \).

2.74. Let \( \varphi \) be a \( \varphi \)-function.

(a) If \( \varphi > 0 \), then \( \varphi \) possesses property \( D \) for every \( 0 < r < \varphi \); if \( \varphi \) possesses property \( D \) for a certain \( r \), then \( u_{\varphi} > 0 \) and \( u < \varphi \).

(b) If \( \varphi < \infty \), then \( \varphi \) possesses property \( D \) for every \( r > \varphi \); if \( \varphi \) possesses property \( D_a \) for a certain \( r \), then \( u_{\varphi} < \infty \), \( u_{\varphi} < r \).

Let \( \varphi(u) = u^\varphi(u) \), where \( r > 0 \). By 2.62, \( \varphi \) is pseudo-increasing for large \( u \) if and only if, for every \( a > 1 \),

\[
\varphi(u)^a \geq \varphi(u)^r \quad \text{if} \quad k > 0, \ u > \varphi(u)
\]

Since \( \varphi(u)^a/\varphi(u) = (\varphi(u)^a)/\varphi(u) \), \((+)\) implies that \( \varphi \) satisfies condition (\( \Lambda_a \)) for sufficiently large \( a \) with the constant \( a_0 = a_0^r \). By 2.51, \( a > 0 \), and, by 2.52 (a), \( \varphi(u^a)/\varphi(u) = r^{\varphi(u)/\varphi(u)} \leq \varphi(u)/\varphi(u) \), i.e. \( r \leq \varphi(u) \). Let us now assume \( a > 1 \) such that \( r = \varphi(u)/\varphi(u) > a \), \( \varphi(u) = a \varphi(u) \), \( \varphi(u) = a \varphi(u) \), \( \varphi(u) = a \varphi(u) \), \( \varphi(u) = a \varphi(u) \).

In the above-mentioned fragment of [7] may be applied, for example, if we assume (\( \Lambda_a \)).

2.88. In the following properties \( \varphi, \chi, \psi \) denote \( \varphi \)-functions, \( r > 0 \):

A. \( \varphi \overset{\sim}{\rightarrow} \chi \); \( \varphi^\psi(\chi) = (\varphi^\psi(\psi)^\varphi) \), \( \psi \) convex.

B. \( \varphi \overset{\sim}{\rightarrow} \chi \); \( \varphi^\psi(\chi) = (\chi^\psi(\varphi)^\varphi) \), \( \psi \) convex.

C. \( \varphi \overset{\sim}{\rightarrow} \chi \); \( \chi \) is superadditive in a generalized sense:

\[
\chi(u_1 + u_2) \geq (\chi(u_1))^\varphi + (\chi(u_2))^\varphi \quad \text{for} \quad u_2 \geq u_1 \geq 0.
\]

D. \( \varphi(u) = u^\varphi(u) \), where \( \varphi \) is pseudo-increasing for large \( u \).

Property \( D \) is satisfied for an arbitrary \( r < \varphi \), for it is satisfied for some \( r < \varphi_a \) arbitrarily close to \( \varphi_a \).

The proof of (b) is analogous.

2.89. In this section \( \varphi \) may assume also negative values and is always integrable in an arbitrary interval \((0, u)\); however, speaking about regularly increasing or slowly varying functions etc., we shall have in mind functions \( \varphi > 0 \), just as we did previously. We shall write \( \varphi(u) = \int_0^u \varphi(t) dt \).
and we shall assume $\varphi(u) > 0$ for $u > 0$. Moreover, we shall write

$$h(u) = \frac{u\varphi(u)}{\varphi(u)} \quad \text{for} \quad u > 0.$$ 

The following inequalities hold:

$$\lim_{u \to \infty} h(u) \leq \frac{\log h(u)}{\log \lambda} \leq \frac{\log \varphi(u)}{\log \lambda} \leq \lim_{u \to \infty} h(u), \quad 0 < \lambda < 1. \tag{1}$$

We apply the substitution

$$\varphi(u) = a, \quad \varphi(u) = v$$

to the function $\varphi$ and we write $\log \varphi(e^v) = f(u)$. Since $\varphi$ is absolutely continuous, we get $f'(u) = \varphi'(e^v)e^v/\varphi'(e^v) = h(e^v)$ for almost every $u$, whereas

$$\log \frac{\varphi(u)}{\varphi(u)} = f(u + \mu) - f(u) = \int_u^{u+\mu} f'(t) \, dt$$

for $a > 1$. Hence, $\lim_{u \to \infty} \varphi(\alpha)/\varphi(u) = H_\alpha(\lambda)$, where $\lambda = 1/a$. Thus

$$\log H_{\lambda}(\lambda) \leq \lim_{u \to \infty} \int_0^u f(t) \, dt \leq -\log \lambda \cdot \lim_{u \to \infty} f'(u). \tag{2}$$

The inequality $-\log \lambda \cdot \lim_{u \to \infty} f'(u) \leq \log H_{\lambda}(\lambda)$, when $0 < \lambda < 1$, is proved similarly.

As an immediate consequence of $(1)$ we obtain

2.81. (a) If $h(u) \to a$, where $\varphi \not= 0$ is finite, then $\varphi$ is regularly increasing and of index $\tau_\varphi = a$; if $a = 0$, $\varphi$ is slowly varying.

(b) If $h(u) \to \infty$ as $u \to \infty$, then

$$\frac{\varphi(\lambda u)}{\varphi(u)} \to \begin{cases} 0 & \text{for} \quad 0 < \lambda < 1, \\ 1 & \text{for} \quad \lambda = 1, \\ \infty & \text{for} \quad \lambda > 1. \end{cases} \tag{3}$$

2.811. Let $\varphi > 0$ for $u > 0$ and $h(u) \to a$ as $u \to \infty$, a finite. If $a = 1$, then $\varphi$ is slowly varying, and if $a \not= 1$, $a > 0$, then $\varphi$ is regularly increasing and $\tau_\varphi = a - 1$.

By the above assumption, $\varphi(u) \approx u\varphi(u)$ and since, according to 2.81 (a), $\varphi$ is of index $\tau_\varphi = a$, we have $\tau_\varphi = 1 - \tau_\varphi$, i.e. $\varphi$ is of index $a - 1$.

Remark. From the proof of inequality 2.8 (1) it follows that the inequality remains valid if we restrict ourselves to $u \to \infty$, $u \in (0, \infty) - \lambda$ in $\lim h(u)$ and $\lim h(u)$, $\lambda$ being set of measure 0. The same remark applies to 2.81. Taking into consideration the above remark we obtain the following test of $\varphi$ being slowly varying, resp. regularly increasing:

2.812. If $\varphi(u) > 0$ for $u > 0$, $\varphi$ is absolutely continuous, $A$ denotes the set of $u$ for which $\varphi'(u)$ exists and if

$$\varphi'(u) \to a \quad \text{as} \quad u \to A, \quad u \to \infty, \tag{4}$$

then $\varphi$ is slowly varying when $a = 0$ and regularly increasing when $a = 0$.

For instance, the above test may be applied to

$$\varphi(u) = \int_0^u \sin \frac{t}{u} \, dt, \quad a = 0.$$
2.82. If a function \( h(u) \) is slowly varying and \( \varphi \) is a continuous q-function such that \( \varphi(u) \to \infty \) as \( u \to \infty \), then the inequalities
\[
1 + \varepsilon \leq s_\varphi \leq \lim_{u \to \infty} \frac{h(u)}{\varphi(u)} \leq \lim_{u \to \infty} h(u) = \sigma_\varphi \leq 1 + \sigma_\varphi
\]
are satisfied.

To prove this theorem we apply the substitutions \( \log \psi(u^n) = f(u) \), \( u^n = \lambda \), \( v^n = v \), again. We have
\[
\frac{h(\lambda)}{h(v)} = \frac{f(u + \mu)}{f(u)} \to 1
\]
as \( u \to \infty \) for an arbitrary \( \mu \), for \( u \to \infty \) implies \( v \to \infty \) and \( h \) is slowly varying. By 1.8, we have
\[
\lim_{u \to \infty} h(u) = \lim_{u \to \infty} f(u) = \frac{g(\mu)}{\mu} \quad \text{for} \quad \mu > 0,
\]
\[
(\rightarrow) \quad \lim_{u \to \infty} h(u) = \lim_{u \to \infty} f(u) = \frac{g(\mu)}{\mu} \quad \text{for} \quad \mu > 0,
\]
and by the definition of the indices
\[
s_\varphi = \lim_{u \to \infty} \frac{g(u)}{\mu}, \quad \sigma_\varphi = \lim_{u \to \infty} \frac{g(u)}{\mu}.
\]

Inequality \((\rightarrow)\) follows from \((\rightarrow)\) if we take \( \mu \to \infty \) and apply theorem 2.813.

2.83. If \( \varphi \) is regularly increasing or slowly varying, then there exists a continuous function \( \overline{\varphi}(u) \) asymptotically equal to \( \varphi \).

Let us define a continuous function \( \overline{\varphi}(u) \) as \( \overline{\varphi}(u) = \varphi(nu) \) for \( n = 1, 2, \ldots \). \( \overline{\varphi}(u) \) is a linear function between the points \( (n, \varphi(n)) \) and \( (n+1, \varphi(n+1)) \) and in the interval \( (0, 1) \). We have, for \( n \leq u \leq n+1 \),
\[
\overline{\varphi}(u) = \frac{\varphi(nu)}{\varphi(n)(\varphi(n+1)-\varphi(n))} \to 1 \quad \text{as} \quad u \to \infty.
\]

\[
\overline{\varphi}(u) = \frac{\varphi(nu)}{\varphi(n) - \varphi(n+1)} \to 1 \quad \text{as} \quad u \to \infty.
\]

2.84. If \( -\infty < s_\varphi \leq \sigma_\varphi < \infty \) for a q-function \( \varphi \), then there exists a continuous function \( \overline{\varphi}(u) \) l-equivalent to \( \varphi \).

The indices \( s_\varphi \), \( \sigma_\varphi \) being finite, for an arbitrary interval \( (\lambda', \lambda'') \), where \( \lambda' > 0 \), there are constants \( c_\varphi \geq c_0 > 0 \) such that \( c_\varphi \geq \varphi(\lambda u)/\varphi(u) \geq c_0 \) for \( \lambda \in (\lambda', \lambda'') \) and \( u \geq u_0 \) (we make the substitution \( \varphi(u) \) and we apply 1.4, 1.3 (b)). We define \( \overline{\varphi}(u) \) as in 2.83, \( \lambda' = 1 \), \( \lambda'' = 2 \). Then we have
\[
\frac{c_0}{c_\varphi} \leq \frac{\varphi(u)}{\varphi(nu)} \leq \frac{\varphi(nu)}{\varphi(n) - \varphi(n+1)} \leq c_0
\]
for sufficiently large \( u \).

2.85. If \( \varphi \) is a quasi-regularly increasing (quasi-slowly varying) continuous function with \( r_\varphi > -1 \) and if \( h \) is slowly varying, then \( \varphi \) is regularly increasing (slowly varying).

By 2.814, 2.813 and 2.82,
\[
h(u) \to a = 1 + r_\varphi > 0,
\]
whence, according to 2.811, \( \varphi \) is regularly increasing (slowly varying, if \( r_\varphi = 0 \)).

2.86. (a) In order that \( \varphi \) be regularly increasing of index \( r_\varphi > -1 \) it is necessary and sufficient that
\[
(\rightarrow) \quad \frac{\overline{h}(u)}{\overline{\varphi}(u)} \to a \quad \text{as} \quad u \to \infty,
\]
where \( a \neq 1, a > 0 \).

(b) In order that \( \varphi \) be slowly varying it is necessary and sufficient that \((\rightarrow)\) holds with \( a = 1 \).

In both cases the index \( r_\varphi \) and the limit \( a \) satisfy the equality \( r_\varphi = a - 1 \) (see [2] and [3]).

Sufficiency follows from 2.811; necessity is obtained from 2.82 by assuming \( \varphi \) to be continuous. If \( \varphi \) is not continuous, then, according to 2.83, \( \varphi \equiv \overline{\varphi} \), where \( \overline{\varphi} \) is a continuous function. Since \( r_\varphi = r_\varphi > -1 \), writing
\[
\varphi(u) = \frac{1}{\overline{\varphi}(u)} \int \overline{\varphi}(t) dt
\]
we have \( \overline{\varphi}(u) \to \infty \) as \( u \to \infty \), \( \overline{\varphi} \equiv \varphi \). If \( \overline{h}(u) = \overline{\varphi}(u)/\overline{\varphi}(u) \), then \( \overline{h} \approx \overline{h} \),
and since \( h(u) \to a = 1 + r_\varphi \) as \( u \to \infty \), we have \( h(u) \to a \) as \( u \to \infty \).

2.9. In connection with theorem 2.86 and condition \((\rightarrow)\) which means that \( \varphi(u) \to \overline{\varphi}(u) \), we shall add some remarks concerning the case when \( \sim \) in the last relation is replaced by \( = \). As in the previous section, we assume the existence of the integral \( \varphi(u) \) for \( u > 0 \).

2.91. If \( \varphi \) is non-decreasing for \( u \geq u_0 \), then
\[
(\rightarrow) \quad \varphi(u) \sim \overline{\varphi}(u).
\]

The relation \((\rightarrow)\) follows from the inequalities
\[
\frac{1}{\overline{\varphi}(u)} (\varphi(u) - \overline{\varphi}(u)) \leq (u - u_0) \varphi(u) \quad \text{for} \quad u \geq 2u_0
\]
and from \( \varphi(u) \to \infty \) as \( u \to \infty \).

If \( \varphi \) is non-increasing, then \((\rightarrow)\) need not be satisfied. E.g., if \( \varphi(u) = (1 + u)^{-1} \log(1 + u) \), then \( \varphi(u) = \frac{1}{u} (\log(1 + u)/u) \) and \( \varphi \) is not l-equivalent to \( \overline{\varphi}(u) \). In this example \( \varphi \) is regularly increasing and \( r_\varphi = -1 \). However, the following sufficient condition may be deduced:

2.92. If \( \varphi \) is non-increasing for \( u \geq u_0, r_\varphi > -1 \), then 2.91 \((\rightarrow)\) holds.
According to 2.814, \( \psi(u) \to \infty \) as \( u \to \infty \). We have \( s_p > 1 \) for \( \varphi_k(u) = w_p(u) \), whence, by 2.51, \( \varphi_k \) satisfies condition \((\Lambda_k)\) for a certain \( \alpha > 1 \), \( c_k > 1 \). Thus \( \psi(\alpha u) \geq c_k \psi(u) / u \) for \( u \geq u_k \), hence

\[
\psi(\alpha u) - \psi(u) = a \int_0^u \varphi(t) dt \geq c_k \int_0^u \varphi(t) dt = c_k \psi(u) - \psi(u),
\]

i.e. \( \psi(\alpha u) \geq c_k \psi(u) + k \). On the other hand,

\[
\psi(\alpha u) = \psi(u) + \int_0^u \varphi(t) dt \leq \psi(u) + (\alpha - 1) w_p(u)
\]

for \( u \geq u_k \), whence

\[
\psi(u)(c_k - 1) + k \leq (\alpha - 1) w_p(u) \quad \text{for} \quad u \geq u_k,
\]

and since

\[
\psi(u) - \psi(u_k) \geq (u - u_k) \psi(u),
\]

we obtain \( \psi(u) \leq w_p(u) \).

3. In this section we always assume \( \psi \) to be a convex \( \varphi \)-function; then \( r_p \geq s_p > 1 \). The following conditions will be of importance in the sequel:

(\( \psi \)) \( \psi(u) u^{-1} \to 0 \) as \( u \to 0 \);

(\( \psi \)) \( \psi(u) u^{-1} \to \infty \) as \( u \to \infty \).

By the assumptions (\( \psi \)) and (\( \psi \)) it is known (see [1] and [3]) that the function

\[
\varphi^*(u) = \sup_{\psi \geq 0} \{\psi(u) - \psi(u)\},
\]

complementary to the function \( \psi \), may be defined. It is easily proved that \( \psi^* \) is a convex \( \varphi \)-function for \( \psi > 0 \) satisfying conditions (\( \psi \)) and (\( \psi \)).

5. If \( \psi^* \) is regularly increasing and \( \varphi \leq \psi_1 \), then \( \psi^* \leq \psi_1^* \).

We have \( (1 - \varepsilon) \varphi(u) \leq \psi_1(u) \leq (1 + \varepsilon) \varphi(u) \) for \( u \geq u_k \); hence the complementary functions satisfy the following inequalities ([3], p. 23):

\[
(1 - \varepsilon) \varphi^* \left( \frac{u}{1 + \varepsilon} \right) \leq \psi_1^*(u) \leq (1 + \varepsilon) \varphi^* \left( \frac{u}{1 - \varepsilon} \right) \quad \text{for} \quad u \geq u^*_1,
\]

i.e.

\[
\psi^* \left( \frac{u}{1 - \varepsilon} \right) \leq \psi_1^*(u) \leq (1 + \varepsilon) \varphi^* \left( \frac{u}{1 - \varepsilon} \right) \quad \text{for} \quad u \geq u^*_1,
\]

and since \( \psi^*(u(1 + \varepsilon)^{-1}) \varphi^*(u) \to (1 + \varepsilon)^{-1} \varphi^*(u) \to (1 - \varepsilon)^{-1} \varphi^*(u) \), \( \psi^*(u(1 - \varepsilon)^{-1}) \varphi^*(u) \to (1 - \varepsilon)^{-1} \varphi^*(u) \), we have

\[
(1 - \varepsilon)^{-1} \varphi^*(1 - \varepsilon) \geq \lim_{u \to \infty} \frac{\psi_1^*(u)}{\psi^*(u)} \geq \lim_{u \to \infty} \frac{\psi^*(u)}{\psi_1^*(u)} \geq (1 + \varepsilon)^{-1} \varphi^*(1 + \varepsilon),
\]

whence \( \psi_1^* \approx \psi^* \).

5.2. (a) If \( \psi \) is regularly increasing, \( r_p > 1 \), then \( \psi^* \) is regularly increasing and the indices satisfy the relation \( 1 / r_p + 1 / r_p = 1 \).

(b) If \( \psi \) is regularly increasing, \( r_p = 1 \), then \( (\psi^*)^{-1} \) is slowly varying, and \( r_{p-1} = \infty \).

(c) If \( \psi^{-1} \) is slowly varying, then \( r_p = \infty \), \( \psi^* \) is regularly increasing and \( r_{p-1} = 1 \).

Let \( p(u) = \psi(u)/u \) for \( u > 0 \), \( p(0) = 0 \). By (\( \psi \)) and (\( \psi \)), \( p(u) \) is strictly increasing ([3], p. 18) and if \( \psi_1(u) = \int_0^u p(t) dt \), then \( \psi_1^*(u) = \frac{\psi_1(u)}{\int_0^u p(t) dt} \).

According to 2.3, \( p(u) \) is regularly increasing of index \( r_p = r_p - 1 > 0 \), and, by 2.86,

\[
\frac{\psi(u)}{p(u)} \to \frac{\psi(u)}{u} \to 1 + r_p = r_p \quad \text{as} \quad u \to \infty.
\]

As is well known, every pair of numbers \( u, v > 0 \) such that \( p^{-1}(v) = u \) satisfies the identity \( \psi(u) = \psi_1 \), i.e.

\[
\frac{\psi_1(u)}{u} + \frac{\psi^*(v)}{v} = 1,
\]

and since \( u \to \infty \) as \( v \to \infty \) and \( w(p) = w(u) \), we have

\[
\frac{\psi_1(v)}{v} \to 1 - \frac{1}{r_p} = r_{p-1} \quad \text{as} \quad v \to \infty.
\]

By 2.86, \( p^{-1}(v) \) is regularly increasing and \( r_{p-1} = r_{p-1} - 1 \).

Hence, by 2.812, \( \psi_1^* \) is regularly increasing and of index \( r_{p-1} = r_{p-1} - 1 \).

Taking into account the identity \( wp(u) = \psi(u) \), we obtain

\[
(q(w)) \leq \psi_1 \psi_1^* \leq \psi_1 \psi_1^* \quad \text{and} \quad \psi_1 \psi_1^* \leq \psi_1 \psi_1^* \leq \psi_1 \psi_1^* \psi_1 \psi_1^*.
\]

The function \( (r_p \psi_1^*) \) is regularly increasing and of index \( r_p \psi_1^* \). Hence it follows that \( \psi^* \) is also regularly increasing and of the same index.

In order to prove (e) let us note that according to 2.51 we have \( \psi(\lambda u) \) for \( u \to \infty \), \( \lambda > 1 \), whence also \( p(\lambda u) \to \infty \) as \( u \to \infty \), \( \lambda > 1 \). Thus \( p^* \) is slowly varying, and consequently \( \psi^*_1 \) is regularly increasing, \( r_1 = 1 \), by 2.813. The inequalities
\[ \varphi(u) \leq \frac{1}{\lambda - 1} \varphi_1(\lambda u), \quad \varphi_1(u) \leq \varphi(u) \quad \text{for} \quad u \geq 0 \]

hold for an arbitrary \( \lambda > 1 \). Hence the complementary functions satisfy the inequalities

\[
\varphi^*(w) \geq \frac{1}{\lambda - 1} \varphi_1^*(\lambda \frac{1}{\lambda} - 1) \quad \text{and} \quad \varphi_1^*(w) \leq \varphi^*(w) \quad \text{for} \quad w \geq 0.
\]

Hence, taking into account the equality \( r_{\varphi^*} = 1 \) we obtain

\[
\frac{1}{\lambda - 1} \varphi_1^*\left(\lambda \frac{1}{\lambda} - 1\right) \geq \varphi^*(u) \quad \text{as} \quad u \to \infty,
\]

hence

\[
1 \geq \frac{\varphi^*(u)}{\varphi_1^*(u)} \geq \frac{1}{\lambda - 1} \varphi_1^*\left(\lambda \frac{1}{\lambda} - 1\right) \quad \text{as} \quad u \to \infty.
\]

Consequently, we obtain the relation \( \varphi^* \sim \varphi_1^* \). Hence \( \varphi^* \) is regularly increasing, \( r_{\varphi^*} = r_{\varphi} = 1 \). Since \( \varphi_1(\lambda) = \varphi_2(\lambda) = \infty \) for \( 0 < \lambda < 1 \), the equation \( r_{\varphi} = \infty \) is obvious.

To prove (b) let us note that, by (2.3 a), \( p \) is slowly varying for \( r_{\varphi} = 0 \). Hence \( \varphi_1(w)/\varphi(w) \to 1 \), and (\( \ast \)) implies \( \varphi_1^*(w)/\varphi_1^*(w) \to 1 \) as \( v \to \infty \). According to (2.4 a), \( \varphi_1^*(u)/\varphi^*(u) \to \infty \) as \( u \to \infty \), if \( \lambda > 1 \). Inequality (\( \ast \ast \)) yields

\[
\frac{\varphi^*(u)}{\varphi_1^*(u)} \geq \frac{1}{\lambda - 1} \varphi_1^*\left(\lambda \frac{1}{\lambda} - 1\right) \quad \text{as} \quad u \to \infty.
\]

But given \( \mu > 1 \) we may choose \( \lambda > 1 \) so that \( (\lambda - 1)\mu/\lambda > 1 \), whence \( \varphi^*(\lambda u)/\varphi^*(u) \to \infty \) as \( u \to \infty \) for \( \lambda > 1 \), and, by 2.31, \( \varphi^* \) is slowly varying.

33. If \( a > 1 \), we denote by \( \beta \) the conjugate exponent, \( 1/a + 1/\beta = 1 \). A regularly increasing function \( \varphi \) of index \( a \) may be written in the form

\[
\varphi(u) = \frac{u^a}{\alpha} \gamma(u),
\]

where \( \gamma \) is slowly varying. Hence, by 3.2,

\[
\varphi^*(u) = \frac{u^a}{\beta} \gamma^*(u),
\]

where \( \gamma^* \) is also a slowly varying function. Under suitable assumptions regarding \( \gamma \), additional information on the asymptotic behaviour of \( \varphi^* \) for large \( u \) can be obtained.

3.4. Let

\[ \varphi(u) = \frac{u^a}{\alpha} \gamma(u), \quad a > 1, \]

where \( \gamma(u) = o(\varphi\log(1+u)) \) and \( \omega \) is a regularly increasing or slowly varying function of index \( r_\omega = s \). Then

\[ \varphi^*(u) \sim \frac{1}{\beta} \omega^s (\varphi^*(u))^{-\lambda s}, \quad \text{where} \quad c = (\beta/a)^{\lambda s}. \]

Let \( p(u), \varphi_1(u) \) have the same meaning as in 3.2. According to 2.3 b), \( \gamma \) is slowly varying. Hence \( r_{\varphi^*} = o, \varphi(u) = (u^{(a-1)/\alpha})\gamma(u), r_\omega = o \). Consequently, \( r_{\varphi^*} = 1/(a-1) = \beta/a - 1, \) i.e. \( \varphi^* \) is regularly increasing of the form \( (u^{a-1/\alpha}) = (u^{(a-1)/\alpha}}\gamma(u^{(a-1)/\alpha}) \), where \( \lambda(u) \) is a slowly varying function. Thus we have

\[ u = p(u^{a-1/\alpha}) = \omega^{(a-1)/(a-1)} \gamma(u^{a-1/\alpha}) \]

and since \( (a-1)/(a-1) = 1 \), we have

\[ a \lambda(u) / \omega(u) \to \infty \quad \text{as} \quad u \to \infty. \]

Let \( r_\omega = s \); then \( \omega(\lambda(u))/\omega(u) \to \infty \) as \( u \to \infty \), this convergence being uniform in each finite interval \( (\lambda(u), \lambda(u)) \) above. Hence \( \gamma(u^{-1}) / \gamma(u) = o(\log(1+u)) / o(\log(1+u)) = 1 \) as \( u \to \infty \), \( \gamma(u^{-1}) / \omega(u) = o(\omega(u)) \). But

\[ \frac{(a-1)}{a} \log u + \log \lambda(u) \to \infty \quad \text{as} \quad u \to \infty, \]

for 2.2 b) implies \( \log \lambda(u) + \log u = \infty \) as \( u \to \infty \). Thus

\[ \gamma(u^{(a-1)/\alpha}) \sim \frac{c}{\omega(u)} (\log u + \log \lambda(u)) \quad \text{as} \quad u \to \infty. \]

It follows from (\( \ast \)) that \( (\varphi^*)^{-\lambda s} (\varphi^*) / (\varphi^*(u))^{-\lambda s} \to (\frac{1}{\beta} \gamma(u^{-1}))^{-\lambda s} \) as \( u \to \infty \), where \( \varphi = (\beta/a)^{\lambda s} \neq 0 \). However, \( (\varphi^*)^{-s} \sim \varphi^*(u) \), \( (\varphi^*)^{-s} \sim \omega(u) \), as follows from the proof of 3.2, and since \( \varphi^* \) is regularly increasing, we have \( \omega(u) / \omega(u) \sim \beta/a \) as \( u \to \infty \), i.e.

\[ 1/\beta \cdot \omega(u) \sim \omega(u). \]

Moreover, \( \varphi_1^*(u) / \varphi^*(u) \to (1/\alpha) \). As \( u \to \infty \), \( \varphi_1^*(u) / \omega(u) \sim \omega(u) \). Finally, we obtain \( \varphi^*(u) \sim \frac{c}{\gamma(u^{-1})} \frac{1}{\beta} (\varphi^*(u))^{-\lambda s} \), where \( c = \omega(u) \).

Theorem 3.4 is a strengthened form of a theorem of Krasnosieleckii and Rutickii [5], who obtain \( \lambda(u) \) in place of \( \varphi \), the constant being unspecified, and who make a little more restrictive assumption regarding \( \gamma \). If \( a = 1 \), then \( \gamma = 1, r_\omega = 0 \), and \( \varphi(u) = u^a/a, \varphi^*(u) = u^\beta/a \), while 3.4 gives only \( \varphi(u) \sim u^\beta/a \).
On the analytic functions in p-normed algebras

by

W. ŻELAZKO (Warszawa)

A p-normed algebra is a complete metric algebra in which topology is given by the meaning of a p-homogeneous submultiplicative norm ||z||:

(1) \[ ||az|| = ||a||^p ||z||, \]

(2) \[ ||ay|| \leq ||a|| ||y||, \]

where a is a scalar, p — fixed real number satisfying 0 < p ≤ 1.

It is known that every complete locally bounded algebra is a p-normed algebra. These algebras were considered in papers [4], [5], and [6]. The greater part of Gelfand’s theory on commutative complex Banach algebras is also true for p-normed algebras. In this paper we give an extension of Gelfand’s theory of analytic functions in Banach algebras onto p-normed algebras [2]. We note that the classical method based upon the concept of abstract Riemann integral cannot be applied here, because the algebras in question are not locally convex (cf. [3]).

Let A be a commutative complex p-normed algebra with a unit designed by e. Let MN be the compact space of its multiplicative linear functionals (= maximal ideals). The spectrum of an element \( a \in A \) is defined as

(3) \[ \sigma(a) = \{ f(a) : f \in MN \}. \]

It is a compact subset of the complex plane. Here we give the positive answer to the following question stated in [5]:

“Let \( \Phi(z) \) be a holomorphic function defined in the neighbourhood \( U \) of spectrum \( \sigma(z) \) of an element \( a \in A \). Does there exist a \( y \in A \) such that for every \( f \in MN \)

(4) \[ f(y) = \Phi(f(a)) \]

We shall give a step by step construction of such an element \( y \). It is natural to write \( y = \Phi(a) \). So we give a natural definition of \( \Phi(a) \) in locally bounded algebras.

As a corollary we obtain the generalization of the theorem of Lévy [2] on trigonometrical series.