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A stochastic dam process with non-homogeneous Poisson inputs

by

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0. Summary. This paper considers the distribution function (d. f.) for the content of a dam fed by non-homogeneous Poisson inputs of random size, and subject to a release at constant unit rate unless the dam is empty. The d. f. may be expressed quite generally in an integral form; if inputs are of unit size, an explicit solution is obtained to the difference-differential equation for the d. f. of the content.

1. Introduction. In two recent papers, Prabhu [4, 5] has extended some methods of storage theory to a queue for which the waiting-time $0 \leqslant Z(t) < \infty$ at time $t \geqslant 0$ satisfies the equation

$$Z(t+\delta t) = Z(t) + \delta X(t) - (1-\eta)\,\delta t. \tag{1.1}$$

In storage terminology $Z(t)$ represents the dam content; the input $0 \leqslant X(t) < \infty$ entering the dam in time t is such that the arrival times of single inputs form a Poisson process with parameter λ, the inputs (independent of arrival times) being identically and independently distributed with d. f. $H(u)$; $\eta\delta t$ $(0 \leqslant \eta \leqslant 1)$ indicates that part of the interval δt for which the dam is empty.

The content $Z(t)$ is a time-homogeneous Markov process whose transition d. f.

$$F(z_0, z, t) = \Pr\{Z(t) \leqslant z \mid Z(0) = z_0\} \qquad (0 \leqslant z_0,\ 0 \leqslant z < \infty)$$

satisfies Takács' [8] well-known integro-differential equation. For such a process, the probability of first emptiness $dG(z_0, t)$ of the dam at time $t \geqslant z_0$ was given by Kendall [3]. Prabhu obtains an integral of this as the probability of emptiness $F(z_0, 0, t)$ of the dam, and finds the d. f. $F(z_0, z, t)$ in an integral form involving the known input distribution and $F(z_0, 0, t)$.

In the argument some use is made of the additive nature of the input $X(t)$; in fact, the results apply equally to the non-additive input obtained when the Poisson process of arrival times is non-homogeneous with parameter $\lambda(t)$. Reich [6, 7] has studied this case, and reduced the finding of the d. f. of $Z(t)$ to the solution of a Volterra equation of the second kind. However, the methods outlined in this paper appear simpler: we show that the probability of first emptiness satisfies Kendall's integral equation, but that it can be obtained explicitly only when inputs are of constant (unit) size. The probability of emptiness is again a simple function of this, and the d. f. of $Z(t)$ satisfies an integral relation of the same type as that for the additive input. Explicit results for the d. f. extending those of Gani and Prabhu [2] for homogeneous Poisson inputs are obtained in the case where the inputs are of constant unit size.

Let us denote by $K(x;\tau,t)$ the d. f. of the non-additive input $X(t)-X(\tau)$ in the time interval (τ,t) $(t\geqslant\tau\geqslant 0)$; if $H(x)$ is the d. f. of a single input $(x>0)$, then

$$(1.2)\qquad K(x;\tau,t)=\sum_{j=0}^{\infty}e^{-\{\varrho(t)-\varrho(\tau)\}}\{\varrho(t)-\varrho(\tau)\}^{j}H_j(x)/j!,$$

where $H_j(x)$ indicates the j^{th} convolution of $H(x)$, $H_0(x)=1$ for $x\geqslant 0$, and $\varrho(u)=\int_0^u\lambda(v)\,dv$. The Laplace-Stieltjes transform of $K(x;0,t)$

$$(1.3)\qquad \int_0^\infty e^{-\theta x}dK(x;0,t)=\sum_{j=0}^{\infty}e^{-\varrho(t)}\{\varrho(t)\}^j\int_0^\infty e^{-\theta x}dH_j(x)/j! = e^{-\varrho(t)\{1-\psi(\theta)\}}\qquad (R(\theta)>0),$$

where $\psi(\theta)=\int_0^\infty e^{-\theta x}dH(x)$, shows the non-additive nature of this distribution.

Consider the transition d. f. of the dam content for the interval $(0,t)$, which we write

$$F(z_0,z;0,t)=\Pr\{Z(t)=z\,|\,Z(0)=z_0\}\qquad (0\leqslant z_0,\ 0\leqslant z<\infty);$$

this satisfies Takács' [8] integro-differential equation

$$(1.4)\qquad \frac{\partial F}{\partial t}-\frac{\partial F}{\partial z}=-\lambda(t)\Big[F(z_0,z;0,t)-\int_{0-}^{z}F(z_0,z-u;0,t)\,dH(u)\Big]\qquad (z\geqslant\max(0,z_0-t)).$$

From its transform, which is readily obtained from (1.4) as

$$(1.5)\qquad \Phi(\theta;0,t\,|\,z_0)=\int_0^\infty e^{-\theta z}dF(z_0,z;0,t) = e^{-\varrho(t)\{1-\psi(\theta)\}+\theta(t-z_0)}+\theta\int_0^t e^{-\{\varrho(t)-\varrho(u)\}\{1-\psi(\theta)\}+\theta(t-u)}F(z_0,0;0,u)\,du\qquad (R(\theta)>0),$$

we see that the solution for $F(z_0,z;0,t)$ hinges on the probability of emptiness $F(z_0,0;0,t)$.

2. Probabilities of first emptiness. For full generality, suppose we start with a dam content $u\geqslant 0$ at time $\tau>0$; the distribution of first emptiness at time $t+\tau>0$ may then be written as $dG(u;\tau,\tau+t)$ where it is understood that $dG(u;\tau,\tau+t)=0$ for $t<u$, and $dG(0;\tau,\tau)=1$. We note that if single inputs taking values $1,2,\ldots$ follow a discrete distribution, then $dG(u;\tau,\tau+t)$ will assume the discrete form:

$$dG(u;\tau,\tau+t)=\begin{cases}g(u;\tau,\tau+u+n) & \text{for}\quad t=u+n\ (n=0,1,2,\ldots),\\ 0 & \text{otherwise},\end{cases}$$

where $g(u;\tau,\tau+u)=e^{-\{\varrho(\tau+u)-\varrho(\tau)\}}$. If, however, individual inputs follow a continuous distribution $H(x)$ $(x>0)$, then

$$dG(u;\tau,\tau+t)=\begin{cases}e^{-\{\varrho(\tau+u)-\varrho(\tau)\}} & \text{for}\quad t=u,\\ g(u;\tau,\tau+t)\,dt & \text{for}\quad t>u\end{cases}$$

with a continuous probability for $t>u$, but a discrete concentration at $t=u$.

An immediate property of the first emptiness distribution for $v\geqslant 0$ is

$$(2.1)\qquad dG(u+v;\tau,\tau+t)=\int_{s=u-}^{t-v}dG(u;\tau,\tau+s)\,dG(v;\tau+s,\tau+t).$$

When inputs are continuous, for example, this gives for $t\geqslant u+v$

$$g(u+v;\tau,\tau+t)=e^{-\{\varrho(\tau+u)-\varrho(\tau)\}}g(v;\tau+u,\tau+t)+\int_{s=u}^{t-v}g(u;\tau,\tau+s)\,g(v;\tau+s,\tau+t)\,ds+g(u;\tau,\tau+t-v)\,e^{-\{\varrho(\tau+t)-\varrho(\tau+t-v)\}}.$$

Using arguments similar to Kendall's [3] we also obtain that, for $t\geqslant u$,

$$(2.2)\qquad dG(u;\tau,\tau+t)=\int_{v=0-}^{t-u}dG(v;\tau+u,\tau+t)\,dK(v;\tau,\tau+u).$$

If we define the Laplace-Stieltjes transform of this distribution by

$$\int_u^\infty e^{-\theta(\tau+t)}dG(u;\tau,\tau+t)=\alpha(\theta;\tau\,|\,u)\qquad (R(\theta)>0),$$

we find from (2.2) that

$$(2.3)\quad \alpha(\theta;\tau|u) = \int_{u}^{\infty} e^{-\theta(\tau+t)} \int_{0-}^{t-u} dG(v;\tau+u,\tau+t)\,dK(v,\tau,\tau+u)$$

$$= \int_{0-}^{\infty} dK(v;\tau,\tau+u) \int_{v+u}^{\infty} e^{-\theta(\tau+t)} dG(v;\tau+u,\tau+t)$$

$$= \int_{0-}^{\infty} dK(v;\tau,\tau+u)\,\alpha(\theta;\tau+u|v)$$

$$= \sum_{j=0}^{\infty} \frac{e^{-\{\varrho(\tau+u)-\varrho(\tau)\}}\{\varrho(\tau+u)-\varrho(\tau)\}^j}{j!} \int_{0-}^{\infty} \alpha(\theta;\tau+u|v)\,dH_j(v).$$

With $u = z_0$ and $\tau = 0$, the results (2.2) and (2.3) refer to the dam initially considered.

When all inputs are of unit size, so that

$$H(x) = \begin{cases} 0 & \text{for} \quad x < 1, \\ 1 & \text{for} \quad x \geqslant 1, \end{cases}$$

equation (2.2) provides a recurrence relation for the distribution of first emptiness, while (2.3) is somewhat simplified. Thus (2.2) becomes

$$(2.4)\quad g(u;\tau,\tau+u+n) = \sum_{j=0}^{n} e^{-\{\varrho(\tau+u)-\varrho(\tau)\}} \frac{\{\varrho(\tau+u)-\varrho(\tau)\}^j}{j!} g(j;\tau+u,\tau+u+n),$$

where $g(0;\tau,\tau) = 1$ for any τ; and (2.3) reduces to

$$(2.5)\quad \alpha(\theta;\tau|u) = \sum_{j=0}^{\infty} e^{-\{\varrho(\tau+u)-\varrho(\tau)\}} \frac{\{\varrho(\tau+u)-\varrho(\tau)\}^j}{j!} \alpha(\theta;\tau+u|j).$$

It is possible to obtain explicit values for the probabilities of first emptiness recursively from (2.4); for example if $u = z_0$, $\tau = 0$, we readily obtain

$$(2.6)\quad \begin{cases} g(z_0;0,z_0) = e^{-\varrho(z_0)}, \\ g(z_0;0,z_0+1) = e^{-\varrho(z_0)}\varrho(z_0)g(1;z_0,z_0+1) = e^{-\varrho(z_0+1)}\varrho(z_0), \\ g(z_0;0,z_0+2) = e^{-\varrho(z_0)}\{\varrho(z_0)g(1;z_0,z_0+2) + \dfrac{\varrho^2(z_0)}{2!} g(2;z_0,z_0+2)\} \\ \qquad = e^{-\varrho(z_0+2)} \dfrac{\{2\varrho(z_0)\varrho(z_0+1)-\varrho^2(z_0)\}}{2!}, \end{cases}$$

and so on. The method of truncated polynomials (cf. Gani [1]) will also give these probabilities in a simple systematic fashion.

3. Probabilities of emptiness and general representation for $F(z_0, z; 0, t)$. Correcting a minor error in Prabhu's equivalent result, we now show that, for non-additive inputs, the probability of emptiness $F(u, 0; \tau, \tau+t)$ is given by

$$(3.1)\quad F(u,0;\tau,\tau+t) = \sum_{j=0}^{[t-u]} g(t-j;\tau,\tau+t)$$

when inputs are discrete, and

$$(3.2)\quad F(u,0;\tau,\tau+t) = e^{-\{\varrho(\tau+t)-\varrho(\tau)\}} + \int_{u}^{t} g(v;\tau,\tau+t)\,dv$$

when they are continuous. We shall prove the result for continuous inputs only; the proof in the discrete case is analogous except that integrals are replaced by summations.

We note first that the probability of emptiness satisfies the relation

$$(3.3)\quad F(u,0;\tau,\tau+t) = \int_{s=u-}^{t} dG(u;\tau,\tau+s)F(0,0;\tau+s,\tau+t)$$

which in the continuous case takes the form

$$(3.4)\quad F(u,0;\tau,\tau+t) = e^{-\{\varrho(\tau+u)-\varrho(\tau)\}}F(0,0;\tau+u,\tau+t) + \int_{u}^{t} g(u;\tau,\tau+s)F(0,0;\tau+s,\tau+t)\,ds.$$

Suppose now, from (3.2), that we take

$$F(0,0;\tau+s,\tau+t) = e^{-\{\varrho(\tau+t)-\varrho(\tau+s)\}} + \int_{0}^{t-s} g(v;\tau+s,\tau+t)\,dv;$$

substituting in (3.4) we obtain

$$F(u,0;\tau,\tau+t) = e^{-\{\varrho(\tau+t)-\varrho(\tau)\}} + e^{-\{\varrho(\tau+u)-\varrho(\tau)\}} \int_{0}^{t-u} g(v;\tau+u,\tau+t)\,dv + \int_{u}^{t} g(u;\tau,\tau+s)e^{-\{\varrho(\tau+t)-\varrho(\tau+s)\}}ds + \int_{u}^{t} ds\, g(u;\tau,\tau+s) \int_{0}^{t-s} g(v;\tau+s,\tau+t)\,dv$$

and after changing variables in the second and third terms, and reversing the order of integration in the last term, we obtain

$$F(u,0;\tau,\tau+t) = e^{-\{\varrho(\tau+t)-\varrho(\tau)\}}+$$

$$+\int_{v=u}^{t} e^{-\{\varrho(\tau+u)-\varrho(\tau)\}} g(v-u;\tau+u,\tau+t)\,dv+$$

$$+\int_{v=u}^{t} g(u;\tau,\tau+t-v+u)\,e^{-\{\varrho(\tau+t)-\varrho(\tau+t-v+u)\}}\,dv+$$

$$+\int_{v=u}^{t} dv \int_{s=u}^{t-v+u} g(u;\tau,\tau+s)\,g(v-u;\tau+s,\tau+t)\,ds.$$

From (2.1) we recognize this as

$$F(u,0;\tau,\tau+t) = e^{-\{\varrho(\tau+t)-\varrho(\tau)\}}+\int_{u}^{t} g(v;\tau,\tau+t)\,dv,$$

thus proving our assertion (3.2).

We now show that the transform of $F(z_0,0;\tau,\tau+t)$ is an integral of the transform $\alpha(\theta,\tau|u)$. Again we prove the result for the case of continuous inputs; for the discrete case, the proof follows exactly the same steps. Let us define

$$\beta(\theta;\tau|u) = \int_{u}^{\infty} F(u,0;\tau,\tau+t)\,e^{-\theta(\tau+t)}\,dt;$$

then this is

$$\text{(3.5)}\quad \beta(\theta;\tau|u) = \int_{u}^{\infty}\Big\{e^{-\{\varrho(\tau+t)-\varrho(\tau)\}}+\int_{u}^{t} g(v;\tau,\tau+t)\,dv\Big\}e^{-\theta(\tau+t)}\,dt$$

$$= \int_{u}^{\infty} e^{-\{\varrho(\tau+t)-\varrho(\tau)\}-\theta(\tau+t)}\,dt+\int_{u}^{\infty} dv\int_{v}^{\infty} g(v;\tau,\tau+t)\,e^{-\theta(\tau+t)}\,dt$$

$$= \int_{u}^{\infty}\Big\{e^{-\{\varrho(\tau+v)-\varrho(\tau)\}-\theta(\tau+v)}+\int_{v}^{\infty} g(v;\tau,\tau+t)\,e^{-\theta(\tau+t)}\,dt\Big\}\,dv$$

$$= \int_{u}^{\infty} \alpha(\theta;\tau|v)\,dv.$$

Precisely as in Prabhu [5], transform (1.5) for the d. f. $F(z_0,z;0,t)$ may be inverted to yield the integral representation

$$\text{(3.6)}\quad F(z_0,z;0,t) = K(t+z-z_0,t)-\int_{0}^{t} F(z_0,0;0,t-\tau)\,dK(\tau+z;t-\tau,t).$$

The formal steps in the proof are identical with Prabhu's, and it is thus unnecessary to repeat them here. In general, this relation does not yield explicit solutions, although for the case of discrete inputs of unit size we obtain directly that

$$\text{(3.7)}\quad F(z_0,z;0,t) = \sum_{j=0}^{n} e^{-\varrho(t)}\varrho^j(t)/j! - \sum_{k=[z+1]}^{n} F(z_0,0;0,t+z-k)\,e^{-\gamma_k}\gamma_k^k/k!,$$

where $n=[t+z-z_0]$, $\gamma_k=\varrho(t)-\varrho(t+z-k)$, and the probability of emptiness is of the form (3.1). An equivalent result for the d. f. can also be found if we resort to the direct solution of the integro-differential equation (1.4).

4. A direct solution for discrete inputs of unit size. In the case of discrete inputs of unit size, equation (1.4) reduces to

$$\text{(4.1)}\quad \frac{\partial F}{\partial t}-\frac{\partial F}{\partial z} = \lambda(t)\{F(z_0,z;0,t)-F(z_0,z-1;0,t)\}\qquad (z\geqslant \max(0,z_0-t)).$$

For $t\leqslant z_0$ the solution is trivial and we therefore only consider $t>z_0$. Extending the methods used in the case where λ was a constant (Gani and Prabhu [2]), we may perform the transformations

$$u=z+t,\qquad \varrho(t)=\int_{0}^{t}\lambda(v)\,dv,$$

so that

$$F(z_0,z;0,t) = G\big(u_0,u;0,\varrho(t)\big).$$

We then find that (4.1) reduces to

$$\text{(4.2)}\quad \frac{\partial G}{\partial \varrho} = -G(u_0,u;0,\varrho)+G(u_0,u-1;0,\varrho)\qquad (u\geqslant t).$$

The d. f. $G(u_0,u;0,\varrho)$ is zero for all $u<t$ $(0\leqslant t<\infty)$ and satisfies the condition that $0<G(u_0,u;0,\varrho)\leqslant 1$ for all $u\geqslant t$. $\lim_{u\to\infty} G(u_0,u;0,\varrho)=1$, and $G(u_0,u;0,\varrho)$ is everywhere continuous in $t<u<\infty$ but has a discontinuity at $u=t$.

Solving (4.2) systematically in consecutive ranges starting with $t \leqslant u < t+1$, we obtain for the range $t+n \leqslant u < t+n+1$

$$(4.3)\quad G(u_0, u; 0, \varrho(t)) = \sum_{r=0}^{n} \frac{\{\varrho(t)-\varrho(u-r)\}^r}{r!} e^{-\{\varrho(t)-\varrho(u-r)\}} F(z_0, 0; 0, u-r),$$

where, by (3.1),

$$F(z_0, 0; 0, t) = \sum_{j=0}^{[t-z_0]} g(t-j; 0, t),$$

and the $g(t-j; 0, t)$ are of the form (2.6) with z_0 replaced by $t-j$.

We verify by induction that (4.3) is the required solution. For, assuming (4.3) in $t+n \leqslant u < t+n+1$, then, in the consecutive range $t+n+1 \leqslant u < t+n+2$, we have from (4.2)

$$\frac{\partial G}{\partial \varrho} + G(u_0, u; 0, \varrho) = G(u_0, u-1; 0, \varrho)$$

or

$$e^{\varrho} G = \int \sum_{r=0}^{n} \frac{\{\varrho(t)-\varrho(u-r)\}^r e^{\varrho(u-r)}}{r!} F(z_0, 0; 0, u-r-1)\, d\varrho(t) + A_{n+2}(u)$$

so that

$$G(u_0, u; 0, \varrho) = \sum_{r=0}^{n} \frac{\{\varrho(t)-\varrho(u-r)\}^{r+1}}{(r+1)!} e^{-\{\varrho(t)-\varrho(u-r)\}} F(z_0, 0; 0, u-r-1) + e^{-\varrho(t)} A_{n+2}(u).$$

Using the continuity of $G(u_0, u; 0, \varrho)$ at the point $u = t+n+1$, we find the value of $A_{n+2}(u)$, and so finally obtain that for $t+n+1 \leqslant u < t+n+2$

$$G(u_0, u; 0, \varrho) = \sum_{r=0}^{n+1} \frac{\{\varrho(t)-\varrho(u+r)\}^r}{r!} e^{-\{\varrho(t)-\varrho(u-r)\}} F(z_0, 0; 0, u-r)$$

thus verifying our equation (4.3). We may finally express $F(z_0, z; 0, t)$ in the form

$$(4.4)\quad F(z_0, z; 0, t) = \sum_{r=0}^{[z]} \frac{\{\varrho(t)-\varrho(z+t-r)\}^r}{r!} e^{-\{\varrho(t)-\varrho(z+t-r)\}} F(z_0, 0; 0, z+t-r),$$

where $F(z_0, 0; 0, z+t-r)$ has the form previously given. This, in fact, is also found to hold for $0 < t < z_0$.

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