Entire functions in $B_k$-algebras

by

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$A_B$-algebras are a completely metrizable locally convex topological algebra over the real or complex scalars. We shall also assume that the algebra in question possess unit elements.

The topology in a $B_k$-algebra $R$ may be introduced by means of a denumerable sequence of pseudonorms satisfying

\[ \|x\|_i \leq \|x\|_{i+1}, \quad i = 1, 2, \ldots, \]

and

\[ \|xy\| \leq \|x\|_{i+1} \|y\|_{i+1} \]

(see [13], theorem 24). A sequence $x_n$ tends to $x$ if and only if $\lim_{n} \|x_n - x\|_i = 0, i = 1, 2, \ldots$. The basis of neighbourhoods of zero in $R$ is of the form $(K_i(1))_i (i, n = 1, 2, \ldots)$, where $K_i(x) = \{x \in R : \|x\|_i < r\}$. Any subsequence of the sequence $(\|\cdot\|_i)$ also satisfies (1) and (2) and gives in $R$ the same topology.

A $B_k$-algebra $R$ is called $m$-convex if there exists an equivalent system of pseudonorms satisfying

\[ \|xy\| \leq \|x\|_i \|y\|_i, \quad i = 1, 2, \ldots \]

The concept of an $m$-convex $B_k$-algebra, first introduced by Arens [2], was then considered in detail by Michael [7]. A $B_k$-algebra is $m$-convex if and only if there exists a fundamental system $(U)\text{ of}

\begin{align*}
\text{neighbourhoods of zero which are idempotent (i.e. such that } \bigcup U U \in U, \text{ where} \bigcup U \in U, \text{ and there exists an equivalent system of pseudonorms such that multiplication is continuous with respect to each one } (7). \text{ In } [7] \text{ it is also shown that if } U \text{ is an idempotent subset of } R, \text{ then so are its convex hull conv } U \text{ and its closure } \overline{U}. \end{align*}

If $R$ is an $m$-convex $B_k$-algebra and $\varphi(x) = \sum_{n=0}^{m} a_n x^n$ is an entire function of complex variable $z$, then for every $x \in R$ the series $\varphi(x) = \sum_{n=0}^{m} a_n x^n$.

References


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is convergent in $R$. In this paper we shall prove that, conversely, if for a
commutative $B_*$-algebra $E$ and for every entire function $f$ the series
$f(z) = \sum a_n z^n$ is convergent for every $x \in E$, then $E$ is an $m$-convex algebra.

A similar question for non-commutative algebras is open. We thus give an answer to the question stated in [14]. We shall also construct, for a given entire function $f(z)$, a non-$m$-convex $B_*$-algebra $B_*$ in which the series $f(z) = \sum a_n z^n$ is convergent for every $x \in E$. We shall also give a negative answer to problem 6 stated in [15], and formulate some unsolved problems.

1. Theorem on entire functions

In this section we shall assume that the pseudonorms in the algebras in question satisfy (1) and (2).

Definition 1.1. Let $R$ be a $B_*$-algebra, and $X$ any subset of $E$.

The $m$-convex hull of $X$ is defined as $H(X) = \bigcap_{n=1}^{\infty} X^n$, where $X^n = X^{n-1}X$. $H(X)$ is a convex idempotent set. If $X$ is open, then so is $H(X)$. $X$ is convex and idempotent if and only if $X = H(X)$.

Lemma 1.2. If for a $B_*$-algebra $E$ there exists a matrix $(C_{i,j})$ of positive numbers such that

$$
||a_1a_2...a_n|| \leq C_{i,j}||a_1||_{p_1}||a_2||_{p_2}...||a_n||_{p_n},
$$

$i, n = 1, 2, ..., \text{ for every } n \geq 1,$

and if

$$
\sup_{n} \frac{1}{n!} \frac{n!}{p_1p_2...p_n} = p_i < \infty, \quad i = 1, 2, ...
$$

then $E$ is an $m$-convex algebra.

Proof. Put $E_{i,j}(r) = \{x \in E : ||x||_{p_i} < r\}$; by (4) and (5) we have

$$
||a_1a_2...a_n|| \leq C_{i,j}||a_1||_{p_1}||a_2||_{p_2}...||a_n||_{p_n},
$$

Consequently

$$
K_{i,j}(1/p_i) \subset E_{i,j}(1),
$$

It follows that

$$
K_{i,j}(1/p_i) \subset U_i \subset K_{i,j}(1),
$$

where $U_i = H(K_{i,j}(1/p_i))$, and $(1/n)U_i$ is a basis of idempotent neighbourhoods of $0$, q. e. d.

Lemma 1.3. If for a commutative $B_*$-algebra $E$ there exists a matrix

$$
\begin{align*}
\tag{6} ||a||^m &\leq C_{i,j}||a||^m, \\
\tag{7} ||a||^m &\leq C_{i,j}||a||^m,
\end{align*}
$$

and (5) holds, then $E$ is an $m$-convex algebra.

Proof. Let $x_1, x_2, ..., x_n \in E$. Put

$$
\sum_{k=1}^{n} (x_1 + x_2 + ... + x_n)^k, \quad k \leq n,
$$

where the summation is extended to all sequences $i_1 < i_2 < ... < i_k$ of elements of $E$ satisfying $1 \leq i_1 < i_2 < ... < i_n$. We have

$$
\prod_{k=1}^{n} (x_1 + x_2 + ... + x_n)^k = \sum_{k=1}^{n} (-1)^{n-k} \sum_{i_1 < i_2 < ... < i_k} \prod_{k=1}^{n} (x_1 + x_2 + ... + x_n)^k.
$$

In fact, the coefficient preceding the term $x_1^{i_1}x_2^{i_2}...x_n^{i_n}$ in $w_1$ is equal to

$$
\frac{n!}{p_1p_2...p_n} \prod_{i=1}^{n} \frac{n-r}{n-i},
$$

where

$$
\sum_{i=1}^{n} (-1)^{n-i} \prod_{i=1}^{n} \frac{n-r}{n-i}.
$$

It is equal to 0 if $r \neq n$, and to $n!$ if $r = n$, and so (5) holds.

It follows that

$$
\prod_{k=1}^{n} (x_1 + x_2 + ... + x_n)^k = \sum_{k=1}^{n} (-1)^{n-k} \sum_{i_1 < i_2 < ... < i_k} \prod_{k=1}^{n} (x_1 + x_2 + ... + x_n)^k.
$$

where $x_1 = x_2 = ... = x_n = 1$. By (6) and (7) we have

$$
\prod_{k=1}^{n} (x_1 + x_2 + ... + x_n)^k \leq C_{i,j} \prod_{k=1}^{n} (x_1 + x_2 + ... + x_n)^k.
$$

and by (9) we have

$$
\prod_{k=1}^{n} (x_1 + x_2 + ... + x_n)^k \leq C_{i,j} \prod_{k=1}^{n} (x_1 + x_2 + ... + x_n)^k.
$$

and (5) holds, then $E$ is an $m$-convex algebra.
where
\[ C'_n = (1/n!)n^nC_n \sum_{k=0}^{n-1} \binom{n}{k} = (1/n!(2n)^n)C_n. \]

The desired conclusion follows from lemma 1.2 and from the equality
\[ \lim_{n \to \infty} (1/n!(2n)^n) = 2e, \ q. e. d. \]

**Lemma 1.4.** If \( R \) is a \( B_\alpha \)-algebra and for every \( x \in R \)
\[ \sup_n \|x^n\| = p_1(x) < \infty, \]
then (6) holds for an equivalent system of pseudonorms, and, consequently, by 1.3, \( R \) is \( m \)-convex if it is commutative.

**Proof.** We have
\[ p_1(x) = \lim_n \sup \|x^n\| \]
whence \( p_1(x) \) is a function of the first class of Baire defined on a complete metric space, and there exists an element \( x_0 \) in \( R \) such \( p_1(x) \) is continuous. Consequently, there exist a neighbourhood \( U \) of \( x_0 \) and a constant \( C \) such that
\[ p_1(x) < C \]
for every \( x \in U \). Let \( z = x - x_0, \ x \in U \). We have
\[ \|z^n\| = \|(x - x_0)^n\| = \sum_{k=0}^{n-1} \binom{n}{k} \|x^{n-k} - x_0^{n-k}\|^k \]
and by (11) and (2)
\[ \|z^n\|_{\infty} \leq \|z^n\|_{\infty} \|x^n\|_{\infty} \leq \sum_{k=1}^{n-1} \|z^{n-k}\| \|z^n\|_{\infty} \|z^n\|_{\infty} = (2\delta)^n. \]

Let \( V = U - x_0 = \{x \in R; \ z \neq x - x_0, \ x \in U\} \). We can choose such \( \delta \) that \( \delta > 0 \) that \( V \supset K_{\delta}(x_0) \). Consequently,
\[ \|z^n\|_{\infty} \leq (2\delta)^n(1/\delta) \|z^n\|_{\infty} \]
and
\[ \|z^n\|_{\infty} \leq C'_n \|z^n\|_{\infty} \]
for every \( x \in V \).

The conclusion then follows from lemma 1.3, q. e. d. 

**Lemma 1.5.** If \( R \) is a \( B_\alpha \)-algebra and, for every entire function \( \varphi(z) = \sum a_n z^n \), the series \( \varphi(z) = \sum a_n z^n \) is convergent for every \( z \in R \), then for every \( z \in R \), formula (10) holds.

**Proof.** If (10) does not hold, then in \( R \) there exist such an \( x_0 \) and an index \( j_0 \) that \( \|z^{j_0}\|_{\infty} \geq 1 \) for a certain sequence \( \{k_n\} \) of integers. In this case there exists an entire function \( \varphi(z) = \sum \alpha_n z^n \) for which the series \( \varphi(z) \) diverges, q. e. d. 

From the preceding lemmas and from the fact that in an \( m \)-convex \( B_\alpha \)-algebra every function is defined, we have the following

**Theorem 1.** A commutative \( B_\alpha \)-algebra \( R \) is \( m \)-convex if and only if, for every entire function \( \varphi(z) = \sum a_n z^n \) and every element \( x \in R \), the series \( \varphi(x) = \sum a_n x^n \) is convergent.

**Remark.** The assumption that the \( B_\alpha \)-algebra \( R \) is complete is essential. In fact, in the Arens’ algebra \( ^* \bar{L}^* \) (1) the algebra \( C_0(0, 1) \) is a dense subalgebra for which all entire functions are defined. On the other hand, in \( L^* \) no entire functions are defined but only polynomials. Indeed, if \( x \in L^* \), then for every \( n \) we have \( \|x^n\| = \|x\|^n \). Now, by the lemma of Shilov ([2], p. 40), for every sequence of positive numbers \( M_n \) there exists in \( L^* \) such an element \( x_n \) that \( \|x_n\| > M_n \). Hence, for every entire function \( \varphi(z) = \sum a_n z^n \) which is not a polynomial, we can choose such an \( x \in L^* \) that \( \|a_n x^n\| > M_n \) for every \( a_n \neq 0 \). Thus the following question arises: is a \( B_\alpha \)-algebra \( R \) \( m \)-convex if there is defined at least one entire function which is not a polynomial? The positive answer is contained in the following

**Theorem 2.** For every entire function \( \varphi(z) = \sum a_n z^n \) there exists a non-\( m \)-convex algebra \( R_\alpha \) such that the series \( \varphi(x) = \sum a_n x^n \) is convergent for every \( x \in R_\alpha \).

The following section is devoted to the construction of such algebras.

**2. Algebras \( \alpha \).**

**Lemma 2.1.** For every continuous function \( \Gamma(u) > 0, \ u \leq u < \infty \), such that \( \Gamma(u) \downarrow \Gamma(u)/u \) as \( u \to \infty \) there exists a function \( \Omega(u) \) such that
\[ \Gamma(u) = \Omega(u)/u \]
which is increasing to infinity, such that \( \Omega(u) \) is a convex function,
\[ \Omega(u) = \Omega(u)/u \] is increasing to infinity,
\[ \Omega(u) \leq \Omega(u)/u \] for sufficiently great \( u \).

\[ \Gamma(u) \leq \Omega(u)/u \] for sufficiently great \( u \).

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(1) \( L^* \) consists of all functions \( f(t) \) defined for \( 0 < t < 1 \) and such that
\[ \int_0^1 \Gamma(t) f(t) \|dt\| < \infty, \ t = 1, 2, \ldots, \] with the pointwise multiplication.
Proof. It may easily be verified that there exists a continuous function \( t(x) \), \( 1 \leq x < \infty \), strictly increasing to infinity and such that
\[
t(0) = 0 \quad \text{and} \quad t(x) = \min(2^{x-1}, 2^{2^{x-1}}(x-1)^2).
\]
Let \( T(t) \) be its inverse function. We have
\[
T(t) \geq 1 + P \geq 2.
\]
Consequently, if we put \( h_0 = 0 \), \( h_n = T(h_{n-1}) = T^n(0) = T^{n-1}(1) \), we have \( h_n \geq 2^{n-1} \), \( n = 1, 2, \ldots \). We put \( \omega(h_n) = 2^{n-1} \), \( \Omega(h_n) = 2^n - h_n \), and between \( h_n \) and \( h_{n+1} \) we define \( \Omega \) as a continuous linear function.

We shall prove that \( \Omega \) satisfying 1\(^{\circ} \), 2\(^{\circ} \), and 3\(^{\circ} \).

Ad 1\(^{\circ} \). It is sufficient to prove that for the nodes \( h_n \), \( \Omega(h_n) \) we have
\[
\Omega(h_n) \leq \lambda_n \Omega(h_{n-1}) + \mu_n \Omega(h_{n+1}),
\]
where
\[
\lambda_n = \frac{h_{n+1} - h_n}{h_{n+1} - h_{n-1}}, \quad \lambda_n + \mu_n = 1.
\]
We write (13) in the form
\[
2^{n-3} \frac{h_n - h_{n-1}}{h_{n+1} - h_{n-1}} h_{n-1} \leq \frac{h_n}{h_{n+1} - h_{n-1}} \frac{h_{n-1}}{h_{n+1} - h_{n-1}} - \frac{h_{n-1}}{h_{n+1} - h_{n-1}} 2^{n-1} h_{n+1},
\]
or, setting \( h_n = h_n h_{n+1} \), in the form
\[
2 \leq \frac{h_{n+1} - 1}{h_{n+1} - h_n} h_n + 4 \cdot \frac{1 - h_n}{h_{n+1} - h_n} h_{n+1} + 2 \cdot \frac{1 - h_{n+1}}{h_{n+1} - h_n} h_n.
\]
But if \( n > 1 \), then \( T(h_n) \geq 2 h_n \), and \( h_n = T(h_{n-1})/h_{n-1} > 2 \). This implies that for the second term of (13) (the first term being positive) we have
\[
4 \cdot \frac{1 - h_n}{h_{n+1} - h_n} h_n \geq 4 \cdot \frac{1 - h_{n+1}}{h_{n+1} - h_n} h_n = 2,
\]
and (13) holds for \( n > 1 \). For \( n = 1 \) formula (13) is obvious, because the nodes are \( (0, 1), (1, 1), (h_2, 2 h_2) \). We have thus proved 1\(^{\circ} \).

Ad 2\(^{\circ} \). We have \( \omega(h_{n+1}) = 2 \omega(h_n) > \omega(h_n) \) for \( n \geq 1 \), and for \( h_n \leq t < h_{n+1} \)
\[
\omega(t) = \frac{(2^n h_0 + 2^{n-1} h_n h_{n+1})/(h_{n+1} - h_n) - 1}{(2^n h_0 + 2^{n-1} h_n h_{n+1})/(h_{n+1} - h_n)}.
\]
We also have \( \omega(t) = 1 \) for \( 0 \leq t \leq 1 \), and so \( \omega \) is non-decreasing.

Ad 3\(^{\circ} \). Let \( h_{n-1} \leq u \leq h_n \), \( n \geq 2 \). We have \( \omega(u) \leq \omega(h_n) = 2^{n-1} \), where
\[
u = \min(m: T^m(0) \geq u) = \min(m: T^{m-1}(0) \geq t(u)) \leq \min(m: 2^{m-2} \geq t(u)) \leq 2 + \log_2 t(u).
\]
We thus have
\[
\omega(u) \leq 2^{n-1} \leq 2^{n+1 + \log_2(t(u) - 1)} = 2 t(u).
\]
From (12) follows \( T(t) \geq N t \) for \( t \geq N \), and \( N t \leq N^2 \) for \( t < N \); we thus have
\[
\omega(N t) \leq \omega(N^2) + \omega(T(t))
\]
But if \( h_{n-1} \leq t \leq h_n \), then \( h_n = T(t) \leq h_{n+1} \), and so
\[
\omega(T(t)) \leq \omega(h_{n+1}) = 4 \omega(h_n) \leq 4 \omega(t),
\]
and
\[
\omega(N t) \leq \omega(N^2) + 4 \omega(t).
\]
To obtain 3\(^{\circ} \) we write
\[
\Omega(N t) = 3 \Omega(t) = N \omega(N t) - 8 \omega(t) \leq N t \omega(N t) - 8 \omega(t)
\]
\[
\leq N t \omega(N t) - 8 \omega(t) \leq N t \omega(N t) - 8 \omega(t)
\]
By (16), \( \omega(N t) - 4 \omega(t) \leq 0 \) for \( t \geq t(N^2) \), and so by (17) and (15) we have
\[
\sup_{t \leq t(N^2)} \Omega(N t) \leq \sup_{t \leq t(N^2)} N t \omega(N t) - 4 \omega(t)
\]
\[
\leq \sup_{t \leq t(N^2)} N t \omega(N t) \leq 2 t(N^2) \leq N t \omega(N t) \leq N t \omega(N t) \leq N t \omega(N t) = N t \omega(N t)
\]
where \( t = (t) \) is any \( N \)-sequence of positive reals \( t_1, \ldots, t_N \).

**Lemma 2.2.** For the function \( \Omega(t) \) defined above we have
\[
\sup_{t \leq t(N^2)} \Omega(t) = 8 \sum_{k=1}^{N} \Omega(t_k) \leq \Omega(N t) = \Omega(N t),
\]
where \( \xi = (t_k) \) is any \( N \)-sequence of positive reals \( t_1, \ldots, t_N \).

Proof. In fact, if we put \( \sum_{k=1}^{N} \xi_k = N \mu \), then, by the convexity of \( \Omega(t) \),
we have \( \sum_{k=1}^{N} \Omega(t_k) \geq N \Omega(\mu) \), which, by lemma 2.1, implies
\[
\Omega(\sum_{k=1}^{N} \xi_k) = \Omega(N \mu) \geq \sum_{k=1}^{N} \Omega(t_k) \leq \Omega(N t) \leq \Omega(N t)
\]
where \( \xi = \sum_{k=1}^{N} \xi_k \).

**Example 2.3** (see [4]). Let \( a_{n,p}, b_{n,p} \) be a matrix of real numbers satisfying the following conditions:
\[
a_{n,p} \leq a_{n,p+1} \leq b_{n,p} \leq b_{n,p+1}
\]
where \( a_{n,p} \leq a_{n,p+1} \), 2\(^{\circ} \) there exist such constants \( c_{n,p} \) that for every \( n \) and \( m \),
\[
a_{n,m,p} \leq c_{n,m,p} a_{n,p+1} b_{n,p+1}
\]
We define $K = K(a_{m,n})$ as a $B_2$-algebra of all formal power series of the form
\[ x = \sum_{n=0}^{\infty} \xi_n x^n \]
such that
\[ ||x||_p = \sum_{n=0}^{\infty} |\xi_n| \cdot a_{n,p} < \infty, \quad p = 1, 2, \ldots \]

Multiplication in $K$ is defined as “pointwise multiplication” of elements, or convolution multiplication of coefficients. The continuity of multiplication follows from 2o, and so $K$ is a $B_2$-algebra with pseudonorms (20). Such algebras were considered in papers [4] and [5].

In the sequel we shall consider algebras $K$ for matrices of the form
\[ a_{n,p} = e^{i\alpha_n}, \]
where $\Omega_n$ is a suitable sequence of positive reals.

Algebras $K(e^{\alpha_n})$, treated as topological linear spaces, were considered in papers [8], [9], [11] and [12].

**Proposition 2.5.** If $\Omega_n \ni a_n \leq C(\Omega_n + 2\Omega_n)$, then $a_{n,p} = e^{i\alpha_n}$ satisfies 1o and 2o of example 2.3. Moreover, if $\Omega_n \to \infty$, then not all entire functions are defined in $K(e^{\alpha_n})$, and so, by theorem 1, $K(e^{\alpha_n})$ is a non-$m$-convex $B_2$-algebra.

**Proof.** The first assertion is obvious. Now if we put
\[ h(z) = \sum_{n=0}^{\infty} h_n z^n, \quad h_n = e^{i\alpha_n}, \quad n \geq 0, \]
then $h$ is an entire function. If we take element $x$ of $K$ defined as $x = \sum_{n=0}^{\infty} \xi_n x^n = \lambda$, then the series $h(x)$ is divergent because the $n$-th term of the series $\sum_{n=0}^{\infty} h_n a_{n,p}$ equal to $e^{\sum_{\lambda=0}^{n} \alpha_n}$, tends to infinity, q.e.d.

The proof of theorem 2 is based upon the following proposition.

**Proposition 2.5.** Let $M_n$, $n = 0, 1, 2, \ldots$, be a sequence of positive reals such that
\[ M_n/n \to \infty \quad (n \to \infty), \]
then there exists such a non-$m$-convex $B_2$-algebra $R$ that for each complex sequence $(h_n)_{n=0}^{\infty}$ satisfying
\[ \sum_{n=0}^{\infty} |h_n| e^{M_n} < \infty \]
and for each $x \in R$ the series
\[ \sum_{n=0}^{\infty} b_n x^n \]
is convergent in $R$.

**Proof.** Put $\Gamma(n) = \sum_{n=0}^{\infty} M_n$; we have
\[ \Gamma(n)/M_n \to 0 \quad (n \to \infty) \quad \text{and} \quad \Gamma(n)/n \to \infty \quad (n \to \infty). \]

Thus if we define $R(x)$ linearly in the segments $[a, a+1)$, then it satisfies the assumptions of lemma 2.1 and we obtain a function $\Omega(x)$ satisfying 1o-3o of that lemma. We now put $R = K(e^{\alpha_{n,p}})$, $a_{n,p} = e^{i\alpha_n}$ evidently satisfy the assumptions of example 2.3 and so $R$ is a $B_2$-algebra. By 2o of lemma 2.1 and proposition 2.4 it is non-$m$-convex.

Now let $x = \sum_{n=0}^{\infty} \xi_n x^n \in R$; we give an estimation of $||x||_p$. We have
\[ ||x||_p \leq \sum_{n=0}^{\infty} \sum_{\lambda=0}^{n} \xi_\lambda \widetilde{\xi}_\lambda \cdots \widetilde{\xi}_n e^{i\alpha_{n,p}}. \]
We have $\sum_{\lambda=0}^{n} \xi_\lambda = k$, and so, by (18),
\[ \Omega(k) \leq \sum_{\lambda=0}^{n} \Omega(\xi_\lambda) + \Omega(n) \]
and
\[ ||x||_p \leq \sum_{n=0}^{\infty} \sum_{\lambda=0}^{n} \xi_\lambda \cdots \widetilde{\xi}_n e^{i\alpha_{n,p}} \sum_{\lambda=0}^{n} \Omega(n) + \Omega(n) \]
\[ \leq e^{\Gamma(n)} \sum_{n=0}^{\infty} |\xi_n| e^{M_n} \cdot e^{\Gamma(n)} \cdot e^{\Gamma(n)} \cdot e^{M_n} \cdot e^{\Gamma(n)} ||x||_p. \]

Now if $(h_n)$ satisfies (22), then there exists such a $C$ that $|b_n| \leq C e^{-M_n}$ and for an arbitrary $x \in R$ and $p = 1, 2, \ldots$ we have
\[ ||b_n||_p \leq C e^{-M_n} \quad (p \to \infty) \quad (n \to \infty), \]
and so for $n > \alpha_p$$\log||x||_p$ we have
\[ -M_n + p\Gamma(n) + n \log||x||_p \leq -\frac{1}{2} M_n, \]
and by (22)
\[ \sum_{n=0}^{\infty} |b_n| e^{M_n} \leq \sum_{n=0}^{\infty} e^{-\frac{1}{2} M_n} < \infty, \quad \text{q.e.d.} \]
As a corollary we obtain the proof of Theorem 2. In fact, if \( \varphi(\alpha) = \sum_{i=0}^{\infty} a_i \alpha^i \), then it is easy to construct such sequences of positive reals \((M_\alpha)\) that (21) and (22) hold. We can thus define algebra \( R_\alpha \) as algebra \( R \) constructed in proposition 2.5.

3. An example

In (13) it was posed the following question (Problem 6):

"Is a \( B_\alpha \)-algebra \( m \)-convex if for every element \( x \in R \) non-invertible in \( R \) there exists a non-zero multiplicative linear functional \( f \) such that \( f(x) = 0 \)?"

Here we give an example of a non-\( m \)-convex \( B_\alpha \)-algebra \( R_\alpha \) which possesses a total family \( \mathcal{M} \) of multiplicative linear functionals, and has the "Wiener property" (i.e., \( m^{-1} \epsilon R \) if and only if \( f(x) \neq 0 \) for every \( f \in \mathcal{M} \)). We thus give a negative answer to the question mentioned above.

We note that algebra \( R_\alpha \) has another property, namely its only invertible elements are scalar multiples of the unit.

Example 3.1. Let \( M_p(r), p = 0, 1, 2, \ldots \), be a sequence of continuous positive functions monotonically increasing to infinity and suppose that, for each \( p \), there exist such a \( q \) and a positive constant \( C_{pq} \) that

\[
M_p(r) \geq C_{pq} M_q(r).
\]

We define algebra \( R_\alpha \) as the algebra of all entire functions \( \alpha(z) \) such that

\[
|\alpha|_p = \sup |\alpha(z)|/M_p(|z|) < \infty, \quad p = 1, 2, \ldots
\]

Multiplication in \( R_\alpha \) is defined as pointwise multiplication and its continuity in pseudonorms (24) follows from (23). In the same way as in Proposition 2.4 we prove that \( R_\alpha \) is a non-\( m \)-convex \( B_\alpha \)-algebra.

It may be shown that \( R_\alpha \) is isomorphic with \( K(\epsilon^{\{\alpha\}}) \), where \( M_p(r) = m^{-1}(r^{np}), \) and \( M(r) = \text{sup}_{p \geq 0} r^{-p} \) (cf. [3], p. 203-206).

Proposition 3.2. Every linear multiplicative functional \( f \) defined in algebra \( R_\alpha \) is of the form

\[
f(\alpha) = \alpha(\lambda),
\]

where \( \lambda \) is a fixed complex number.

Proof. Let \( f \) be a non-zero multiplicative linear functional defined on \( R_\alpha \), and let \( x = \alpha(z) \). Put \( \lambda = f(x) \). If \( x \in R_\alpha \), then it may easily be verified that \( y(z) = |\alpha(z)|^{-1}(\alpha(z)/\lambda - \lambda) \in R_\alpha \). Thus by the relation

\[
x = \alpha(z) e^{-z} (e^{-z} - \lambda) y,
\]

where \( e = e(z) = 1, (23) \) holds, q. e. d.

Definition 3.3. We define algebra \( R_\alpha \) as \( R_M \), where

\[
M_p(r) = \exp(r^p/p)
\]

and \( \beta > 0 \) is a fixed real. This is an algebra of entire functions of order \( \beta \) and minimal type.

We shall prove that algebras \( R_\alpha \) have the "Wiener property". The proof is based upon the following known ([6]).

Lemma 3.4. If \( \alpha(\lambda) \) is an entire function and

\[
|\alpha(\lambda)| \geq A \exp(-C|\lambda|^p), \quad A > 0,
\]

then there exist such constants \( A_2 \) and \( C_1 \) that

\[
|\alpha(\lambda)| \leq A_2 \exp(C_1|\lambda|^p).
\]

From this lemma we deduce

Proposition 3.5. If \( \alpha \in R_\alpha \) and \( \alpha(z) \neq 0 \), then \( 1/\alpha(z) \in R_\alpha \).

Proof. We have \( \alpha(\lambda) = \exp(p), \) and by the definition of \( R_\alpha \) we have

\[
|\exp(p)| \leq A \exp(C|\lambda|^p).
\]

Hence

\[
|e^{-p}| \geq A^{-1} \exp(-C|\lambda|^p)
\]

and, by lemma 3.4,

\[
|e^{-p}| \leq A_2 \exp(C_1|\lambda|^p).
\]

By (28) and (29) we have

\[
|\exp(p)| \leq |\exp(C_1|\lambda|^p)| + C_1
\]

and \( y(\lambda) \) is a polynomial. By (28) its degree \( q \leq \beta \). Hence it follows

\[
1/\alpha(z) \in R_\alpha \quad q. e. d.
\]

By propositions 3.2 and 3.5 we have a negative answer to the question under consideration.

Remark 3.6. If \( 0 < \beta < 1 \), then only the invertible elements in \( R_\alpha \) are constants.

Remark 3.7. In algebras \( R_\alpha \) there are not entire functions but polynomials. In fact, for any \( \alpha \in R_\alpha \) we have

\[
|\alpha|_p = \sup_{\lambda} |\alpha(\lambda)e^{\lambda n}| = \sup_{\lambda} |\alpha(\lambda)e^{-\lambda n}| = |\alpha|_p
\]

and next we continue the proof in the same way as for \( L^p \) (see p. 295).

4. Another example

In Arens' example \( L^p \) ([1]; see also the footnote on p. 295 of this paper) there are no multiplicative linear functionals and there are not entire functions but polynomials. We are interested in the following ques-
tion: are there any connections between these properties of $B_0$-algebras? The answer is negative. By remark 3.7 we see that there exists a $B_0$-algebra having a total family of multiplicative linear functionals and having not entire functions but polynomials. In this section we shall construct a $B_0$-algebra $E$ in which there are some transcendental entire functions, but in which there are no multiplicative linear functionals. This construction is a modification of Arens’ example $E^*$. 

**Lemma 4.1.** Let $\{a_n\}$ be any sequence of positive real numbers such that

$$\sum_{n=1}^{\infty} e^{-a_n} < 1.$$ 

Then there exists a matrix $A_{k,n}$ of positive reals, $0 \leq k < \infty$, $n \geq 1$, such that

(a) $0 < A_{k,n} \leq 1,$

(b) $\lim_{n \to \infty} A_{k,n} = 0,$ $k = 0, 1, 2, \ldots,$

(c) $A_{k,n} \leq e^{-a_n A_{0,n+1}}.$

**Proof.** The rows of $A_{k,n}$ will be constructed by induction. We set

$A_{k,0} = e^{-a_n}$

and suppose that $A_{k,n}$ is constructed in such a way that (a)-(c) holds. We shall construct the row $A_{k,n+1}$. We put

$B_{k,n}(m) = A_{k,n}^{(m)}$, $B_{k,n}(1) = 1,$

and define $B_{k,n}(t)$ linearly on the segments $[m,n,m(n+1)]$. It is clear that

$0 < B_{k,n}(t) \leq 1$ and $\lim_{t \to \infty} B_{k,n}(t) = 0.$

We have

$A_{k,n+1} = \sum_{n=1}^{\infty} e^{-a_n} B_{k,n}(n)$, $n = 1, 2, \ldots$

which satisfies (c), and clearly satisfies (a) and (b), q.e.d.

**Example 4.2.** We define $E$ as the algebra of all measurable functions on the interval $[0,1]$ such that

$$|\sigma(t)|_a = \sup_{n} A_{k,n} \left( \int_0^1 |\sigma(t)|^n dt \right)^{1/n} < \infty \quad (k = 1, 2, \ldots).$$

It is a $B_0$-algebra with pseudonorms (30). In fact, we have

$$|\sigma^{(m)}|_a = \sup_{n} A_{k,n} \left( \int_0^1 |\sigma^{(m)}(t)|^n dt \right)^{1/n} \leq \left( \sup_{n} e^{-a_n A_{0,n+1}} \left( \int_0^1 |\sigma(t)|^n dt \right)^{1/n} \right)^{1/m} \leq \left( \sup_{n} e^{-a_n A_{0,n+1}} \left( \int_0^1 |\sigma(t)|^n dt \right)^{1/n} \right)^{1/m} = e^{-a_n} |\sigma(t)|_a.$$  

It follows that the operation of taking the $m$-th power is defined and continuous in $E$. Thus multiplication is defined and continuous in $E$ because

$$xy = \frac{1}{2}((x+y)^2-x^2-y^2).$$

**Remark 4.3.** In $E$ some transcendental entire functions are defined, e.g., by (31), the function

$$f(z) = \sum_{n=1}^{\infty} e^{-anm} a_n z^n$$

is defined.

**Proposition 4.4.** In algebra $E$ defined in 4.2 there are no multiplicative linear functionals.

**Proof.** The algebra $C(0,1)$ of all continuous functions defined for $0 \leq t \leq 1$ is a subalgebra of $E$. So, if there existed in $E$ a non-zero multiplicative linear functional $F_0(x)$, it would be of the form

$$F(x) = x(t_0) \quad (0 \leq t_0 \leq 1)$$

for every $x \in C(0,1)$. To obtain our conclusion it is enough to prove that for every $t_0 \in (0,1)$ there exists a continuous function $x(t)$, $x(t_0) = 0$, which is invertible in $E$. Or, what is equivalent, we should prove that for every $t_0$ there exists in $E$ a function $x(t)$, $x$ being continuous and non-zero for $t \neq t_0$, and

$$\lim_{t \to t_0} x(t) = \infty.$$

We may assume that $t_0 > 0$; otherwise we should apply an automorphism of $R$; $x(t) \to x(t-1)$. Let

$$A_{n} = \sum_{k=1}^{n} \frac{1}{k} A_{n,k}. $$
We have \( \lim A_p = 0 \), and \( 0 < A_p < 1 \). It is clear that if

\[
\sup_p A_p \left( \int_0^1 |x(t)|^p \, dt \right)^{1/p} < \infty,
\]

then \( x \in L^p \).

We write

\[
\delta_n = \inf A_n^{-1/p}.
\]

By the fact of \( A_p \) tending to zero we have \( \delta_n > 0 \). We now put

\[
A_n = \min\{1, 2^{3/2} \delta_n, \delta_n^2\}
\]

and choose \( \lambda \) in such a way that \( \sum \lambda A_n = \lambda \). Now we set

\[
x(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq \lambda A_n, \\ n & \text{for } t = \sum_{k=1}^n \lambda A_k, \\ \text{is linear in the segments } \sum_{k=1}^n \lambda A_k, \sum_{k=1}^{n+1} \lambda A_k. 
\end{cases}
\]

Moreover, if \( t_0 = 1 \), we put \( x(t_0 + t) = x(t_0 - t) \) for \( |t| < \epsilon \), where \( (t_0 - \epsilon, t_0 + \epsilon) \subset (0, 1) \), and \( x(t) = x(t_0 + t) \) for \( t \geq t_0 + \epsilon \).

We have

\[
\int_0^1 |x(t)|^p \, dt = \int_0^{t_0} a_0^{1/p} + \int_{t_0}^{t_0 + s} a_s^{1/p} + \int_{t_0 + s}^{t_0 + 2s} a_s^{1/p} + \cdots + \int_{t_0 + \epsilon}^{t_0 + 2s} a_s^{1/p} + \cdots + \int_0^1 a_0^{1/p} + C
\]

\[
\leq 2 \left( \int_0^{t_0} a_0^{1/p} \right) + 2 \left( \int_{t_0}^{t_0 + s} a_s^{1/p} \right) + 2 \left( \int_{t_0 + s}^{t_0 + 2s} a_s^{1/p} \right) + \cdots + 2 \left( \int_0^{t_0 + \epsilon} a_0^{1/p} \right) + \cdots + 2 \left( \int_0^1 a_0^{1/p} \right) + C,
\]

where \( C = a_0^{1/p} + a_0^{1/p} + a_0^{1/p} + \cdots + a_0^{1/p} + C \), and \( K_n = \sum_{k=1}^n \lambda A_k \). This estimation is also true in the case where \( t_0 = 1 \). Thus \( x(t) \) is the desired function because by (32) it is a member of \( L^p, q, e, d \).

5. Final remarks

By the considerations of section 4 we have seen that there exists a \( B_p \)-algebra in which there are no multiplicative linear functionals and there are some transcendental entire functions. We can give such a construction only for entire functions which are "slightly increasing", and we cannot give it for, say, \( f(x) = e^x \). We thus pose the following

Problem 1. Let \( R \) be a commutative \( B_p \)-algebra with a unit, and let \( \sum a_n x^n \) converge for each \( x \in R \). Does there exist at least one non-zero multiplicative linear functional?

If \( R \) is a \( B_p \)-algebra in which \( e^x \) is defined for each \( x \in R \), then we can prove that \( e^x \) is a continuous function of \( x \). We can prove this also for every entire function \( \varphi(x) = \sum a_n x^n \) such that \( a_n \neq 0, n = 0, 1, 2, \ldots \), and \( C_0 b_n \leq a_n \leq C_1 b_n \), where \( b_n = \max \{ \delta_n^{1/p}, (n+1)^{-1/p} \} \), but the following question is open:

Problem 2. Let \( R \) be a \( B_p \)-algebra with a unit. Let \( \varphi(x) = \sum a_n x^n \) converge for each \( x \in R \). Is \( \varphi(x) \) a continuous function of \( x \)?

Here difficulties arise in the case of lacunary series.

We pose also

Problem 3. Is the statement of theorem 1 also true for non-commutative \( B_p \)-algebras?

We cannot answer also on the following questions:

Problem 4. Suppose \( f, g \) be two entire functions defined on the \( B_p \)-algebra \( R \). Is the superposition \( f \circ g \) an entire function defined on \( R \)?

Problem 4a. Suppose that \( f \) is an entire function defined on \( R \). Is \( f(x + z) \) defined on \( R \)?

References

A stochastic dam process with non-homogeneous Poisson inputs

by

J. Gani (Canberra)

0. Summary. This paper considers the distribution function (d.f.) for the content of a dam fed by non-homogeneous Poisson inputs of random size, and subject to a release at constant unit rate unless the dam is empty. The d.f. may be expressed quite generally in an integral form; if inputs are of unit size, an explicit solution is obtained to the difference-differential equation for the d.f. of the content.

1. Introduction. In two recent papers, Prabhu [4, 5] has extended some methods of storage theory to a queue for which the waiting-time $0 \leq Z(t) < \infty$ at time $t \geq 0$ satisfies the equation

$$Z(t+dt) = Z(t) + \delta X(t) - (1-\eta)dt.$$  

(1.1)

In storage terminology $Z(t)$ represents the dam content; the input $0 \leq X(t) < \infty$ entering the dam in time $t$ is such that the arrival times of single inputs form a Poisson process with parameter $\lambda$, the inputs (independent of arrival times) being identically and independently distributed with d.f. $H(w); \eta dt$ ($0 \leq \eta \leq 1$) indicates that part of the interval $dt$ for which the dam is empty.

The content $Z(t)$ is a time-homogeneous Markov process whose transition d.f.

$$F(z, z', t) = \Pr \{Z(t) \leq z | Z(0) = z'\} \quad (0 \leq z, 0 \leq z < \infty)$$

satisfies Takacs' [8] well-known integro-differential equation. For such a process, the probability of first emptiness $\delta t \bar{G}(z, t)$ of the dam at time $t \geq z_0$ was given by Kendall [3]. Prabhu obtains an integral of this as the probability of emptiness $F(z, 0, t)$ of the dam, and finds the d.f. $F(z, z', t)$ in an integral form involving the known input distribution and $F(z, 0, t)$.

In the argument some use is made of the additive nature of the input $X(t)$; in fact, the results apply equally to the non-additive input obtained when the Poisson process of arrival times is non-homogeneous with parameter $\lambda(t)$. Reich [6, 7] has studied this case, and reduced the