$$\left|\left(\frac{\sqrt{n}}{h_n}\right)^{2k} P_n[(t-x)^{2k}\eta(t-x);x]\right|$$

$$\leqslant \left(\frac{\sqrt{n}}{h_n}\right)^{2k} \frac{\left(\int\limits_{x-\delta_n}^{x+\delta_n} + \int\limits_0^{x-\delta_n} + \int\limits_{x+\delta_n}^{h_n}\right)(t-x)^{2k}|\eta(t-x)|\left[1-\left(\frac{t-x}{h_n}\right)^2\right]^n dt}{2h_n\int\limits_0^1(1-t^2)^n dt}$$

$$\leqslant \frac{\sup\limits_{|u|\leqslant\delta_n}|\eta(u)|\int\limits_{-\delta_n\sqrt{n}/h_n}^{\delta_n\sqrt{n}/h_n} u^{2k}e^{-u^2}du}{2\sqrt{n}\int\limits_0^1(1-t^2)^n dt} +$$

$$+L(x)\frac{e^{mx^s}\int\limits_{\delta_n\sqrt{n}/h_n}^{x\sqrt{n}/h_n}u^{2k}e^{-u^2}du + \int\limits_{\delta_n\sqrt{n}/h_n}^{(1-x/h_n)\sqrt{n}} u^{2k}\exp[m(x+h_nu/\sqrt{n})^s-u^2]du}{2\sqrt{n}\int\limits_0^1(1-t^2)^n dt}$$

$$\leqslant \sup_{|u|\leqslant\delta_n}|\eta(u)|\int\limits_{-\infty}^{+\infty}u^{2k}e^{-u^2}du + L(x)\Bigg\{e^{mx^s}\int\limits_{\delta_n\sqrt{n}/h_n}^{x\sqrt{n}/h_n}u^{2k}e^{-u^2}du+$$

$$+\int\limits_{\delta_n\sqrt{n}/h_n}^{(1-x/h_n)\sqrt{n}} u^{2k}\exp\left\{u^2\left[m\left(\frac{x}{(\delta_n\sqrt{n}/h_n)^{2/s}}+\frac{h_n}{n^{1/s}}\left(1-\frac{x}{h_n}\right)^{1-2/s}\right)^s-1\right]\right\}du$$

for sufficiently large n. Since $\lim\limits_{u\to 0}\eta(u)=0$ and the expression in brackets is less than $-\frac{1}{2}$ for sufficiently large n, we have

$$\lim_{n\to\infty}\left(\frac{\sqrt{n}}{h_n}\right)^{2k}P_n[(t-x)^{2k}\eta(t-x);x]=0. \tag{9}$$

Applying (8) and (9) we obtain the theorem from (6).

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Entire functions in B_0-algebras

by

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A B_0-algebra is a completely metrizable locally convex topological algebra over the real or complex scalars. We shall also assume that the algebras in question possess unit elements.

The topology in a B_0-algebra R may be introduced by means of a denumerable sequence of pseudonorms satisfying

$$\|x\|_i \leqslant \|x\|_{i+1}, \quad i=1,2,\ldots, \tag{1}$$

and

$$\|xy\|_i \leqslant \|x\|_{i+1}\|y\|_{i+1} \tag{2}$$

(see [13], theorem 24). A sequence x_n tends to x_0 if and only if $\lim\limits_{n\to\infty}\|x_n-x_0\|_i=0$, $i=1,2,\ldots$ The basis of neighbourhoods of zero in R is of the form $\{K_i(1/n)\}$ $(i,n=1,2,\ldots)$, where $K_i(r)=\{x\in R\colon \|x\|_i<r\}$. Any subsequence of the sequence $\{\|x\|_i\}$ also satisfies (1) and (2) and gives in R the same topology.

A B_0-algebra R is called *m-convex* if there exists an equivalent system of pseudonorms satisfying

$$\|xy\|_i \leqslant \|x\|_i\|y\|_i, \quad i=1,2,\ldots \tag{3}$$

The concept of an m-convex B_0-algebra, first introduced by Arens [2], was then considered in detail by Michael in [7]. A B_0-algebra is m-convex if and only if there exists a fundamental system $\{U\}$ of neighbourhoods of 0 which are *idempodent* (i. e. such that $UU\subset U$, where $XY=\{z\in R\colon z=xy,\ x\in X,\ y\in Y\}$, X, Y — arbitrary subsets of R), or if there exists an equivalent system of pseudonorms such that multiplication is continuous with respect to each one [7]. In [7] it is also shown that if U is an idempotent subset of R, then so are its convex hull $\operatorname{conv} U$ and its closure $\overline{U}$.

If R is an m-convex B_0-algebra and $\varphi(z)=\sum\limits_{n=0}^{\infty}a_nz^n$ is an entire function of complex variable z, then for every $x\in R$ the series $\varphi(x)=\sum\limits_{0}^{\infty}a_nx^n$

is convergent in R. In this paper we shall prove that, conversely, if for a commutative B_0-algebra R and for every entire function $\varphi(z)$ the series $\varphi(x) = \sum_{0}^{\infty} a_n x^n$ is convergent for every $x \in R$, then R is an m-convex algebra. A similar question for non-commutative algebras is open. We thus give an answer to the question stated in [14]. We shall also construct, for a given entire function $\varphi(z)$, a non-m-convex B_0-algebra R_φ in which the series $\varphi(x) = \sum_{0}^{n} a_n x^n$ is convergent for every $x \in R$. We shall also give a negative answer to problem 6 stated in [13], and formulate some unsolved problems.

1. Theorem on entire functions

In this section we shall assume that the pseudonorms in the algebras in question satisfy (1) and (2).

Definition 1.1. Let R be a B_0-algebra, and X any subset of R. The *m-convex hull of* X is defined as $H(X) = \operatorname{conv} \bigcup_{n=1}^{\infty} X^n$, where $X^k = X^{k-1}X$. $H(X)$ is a convex idempotent set. If X is open, then so is $H(X)$. X is convex and idempotent if and only if $X = H(X)$.

LEMMA 1.2. *If for a B_0-algebra R there exists a matrix (C_n^i) of positive numbers such that*

$$\|x_1 x_2 \dots x_n\|_i \leqslant C_n^i \|x_1\|_{i+1} \|x_2\|_{i+1} \dots \|x_n\|_{i+1}, \tag{4}$$

$i, n = 1, 2, \dots$, for every finite sequence $x_1, x_2, \dots, x_n$ of elements of R, and if

$$\sup_n \sqrt[n]{C_n^i} = p_i < \infty, \quad i = 1, 2, \dots, \tag{5}$$

then R is an m-convex algebra.

Proof. Put $K_i(r) = \{x \in R: \|x\|_i < r\}$; by (4) and (5) we have

$$\|x_1 x_2 \dots x_n\|_i \leqslant p_i^n \|x_1\|_{i+1} \|x_2\|_{i+1} \dots \|x_n\|_{i+1}.$$

Consequently

$$K_{i+1}^n(1/p_i) \subset K_i(1), \quad n = 1, 2, \dots$$

It follows that

$$K_{i+1}(1/p_i) \subset U_i \subset K_i(1),$$

where $U_i = H(K_{i+1}(1/p_i))$, and $(1/n)U_i$ $(i, n = 1, 2, \dots)$ is a basis of idempotent neighbourhoods of 0, q. e. d.

LEMMA 1.3. *If for a commutative B_0-algebra R there exists a matrix of positive numbers (C_n^i) such that*

$$\|x^n\|_i \leqslant C_n^i \|x\|_{i+1}^n, \quad i, n = 1, 2, \dots, \ x \in R, \tag{6}$$

and (5) holds, then R is an m-convex algebra.

Proof. Let $x_1, x_2, \dots, x_n \in R$. Put

$$w_k^n(x_1, \dots, x_n) = \sum (x_{i_1} + x_{i_2} + \dots + x_{i_k})^n, \quad k \leqslant n, \tag{7}$$

where the summation is extended to all sequences $i_1 < i_2 < \dots < i_k$ of entire numbers satisfying $1 \leqslant i_l \leqslant n$, $l = 1, 2, \dots, k$.

We have

$$x_1 x_2 \dots x_n = \big((-1)^n/n!\big) \sum_{k=1}^{n} (-1)^k w_k^n. \tag{8}$$

In fact, the coefficient preceding the term $x_{n_1}^{p_1} \dots x_{n_r}^{p_n}$ in w_k^n is equal to

$$\frac{n!}{p_1! p_2! \dots p_r!} \binom{n-r}{k-r};$$

the first member is the coefficient in the k-nom of Newton, the second is the number of k-noms containing fixed elements $x_{n_1}, \dots, x_{n_r}$.

Calculating the right side of (8) we see that the coefficient preceding $x_{n_1}^{p_1} \dots x_{n_r}^{p_r}$ is equal to

$$\frac{n!}{p_1! p_2! \dots p_r!} \sum_{j=0}^{n-r} (-1)^j \binom{n-r}{n-j-r}.$$

It is equal to 0 if $r \neq n$, and to $n!$ if $r = n$, and so (8) holds.

It follows that

$$\begin{aligned} \|x_1 x_2 \dots x_n\|_i &= \|x_1\|_{i+1} \dots \|x_n\|_{i+1} \big\| x_1/\|x_1\|_{i+1} \dots x_n/\|x_n\|_{i+1} \big\|_i \\ &\leqslant \|x_1\|_{i+1} \dots \|x_n\|_{i+1} \frac{1}{n!} \sum_{k=1}^{n} \|w_k^n(\bar{x}_1, \dots, \bar{x}_n)\|_i, \end{aligned} \tag{9}$$

where $\bar{x}_p = x_p/\|x_p\|_{i+1}$.

By (6) and (7) we have the following estimation (note that $\|\bar{x}_p\|_i = \big\| x_p/\|x_p\|_{i+1} \big\|_i \leqslant 1$):

$$\|w_k^n(\bar{x}_1, \dots, \bar{x}_n)\|_i \leqslant \binom{n}{k} k^n C_n^i \leqslant \binom{n}{k} n^n C_n^i,$$

and by (9) we have

$$\|x_1 \dots x_n\|_i \leqslant C_n'^i \|x_1\|_{i+1} \dots \|x_n\|_{i+1},$$

where

$$C_n'^i = (1/n!) n^n C_n^i \sum_0^n \binom{n}{k} = (1/n!)(2n)^n C_n^i.$$

The desired conclusion follows from lemma 1.2 and from the equality $\lim_n \sqrt[n]{(1/n!)(2n)^n} = 2e$, q. e.d.

LEMMA 1.4. *If R is a B_0-algebra and for every $x \in R$*

(10) $$\sup_n \sqrt[n]{\|x^n\|_i} = p_i(x) < \infty,$$

then (6) holds for an equivalent system of pseudonorms, and, consequently, by 1.3, R is m-convex if it is commutative.

Proof. We have

$$p_i(x) = \lim_n \max_{k \leqslant n} \sqrt[k]{\|x^k\|_i},$$

whence $p_i(x)$ is a function of the first class of Baire defined on a complete metric space, and there exists an element x_0 in which $p_i(x)$ is continuous. Consequently, there exist a neighbourhood U of x_0 and a constant C such that

(11) $$p_i(x) < C$$

for every $x \in U$. Let $z = x - x_0$, $x \in U$. We have

$$z^n = (x - x_0)^n = \sum_0^n \binom{n}{k} x^{n-k} x_0^k,$$

and by (11) and (2)

$$\|z^n\|_{i-1} \leqslant \sum_0^n \binom{n}{k} \|x^{n-k}\|_i \|x_0^k\|_i \leqslant \sum_0^n \binom{n}{k} C^{n-k} C^k = (2C)^n.$$

Let $V = U - x_0 = \{z \in R \colon z = x - x_0, x \in U\}$. We can choose such $k(i) \geqslant i$ and $r_i > 0$ that $V \supset K_{k(i)}(r_i)$. Consequently,

$$\|z^n\|_{i-1} \leqslant (2C)^n (1/r_i^n) \|z\|_{k(i)}^n$$

and

$$\|x^n\|_i' \leqslant C_n^i \|x\|_{i+1}'^n \quad \text{for every} \quad x \in R,$$

where $C_n^i = (2C)^n (1/r_i^n)$, $\|x\|_i' = \|x\|_{t_i}$, $t_1 = 1$, $t_{i+1} = k(t_i + 1)$. The conclusion then follows from lemma 1.3, q. e. d.

LEMMA 1.5. *If R is a B_0-algebra and, for every entire function $\varphi(z) = \sum_0^\infty a_n z^n$, the series $\varphi(x) = \sum_0^\infty a_n x^n$ is convergent for every $x \in R$, then for every $x \in R$ formula (10) holds.*

Proof. If (10) does not hold, then in R there exist such an x_0 and an index i_0 that $\|x_0^{k_n}\|_{i_0} \geqslant n^{k_n}$ for a certain sequence (k_n) of integers. In this case there exists an entire function $\varphi(z) = \sum_n z^{k_n}/n^{k_n}$ for which the series $\varphi(x_0)$ diverges, q. e. d.

From the preceding lemmas and from the fact that in an m-convex B_0-algebra every entire function is defined, we have the following

THEOREM 1. *A commutative B_0-algebra R is m-convex if and only if, for every entire function $\varphi(z) = \sum_0^\infty a_n z^n$ and every element $x \in R$, the series $\varphi(x) = \sum_0^\infty a_n x^n$ is convergent.*

Remark. The assumption that the algebra R is complete is essential. In fact, in the Arens' algebra L^ω (1) (see [1]) the algebra $C(0,1)$ is a dense subalgebra for which all entire functions are defined. On the other hand, in L^ω no entire functions are defined but only polynomials. Indeed, if $x \in L^\omega$, then for every n we have $\|x^n\|_1 = \|x\|_n^n$. Now, by the lemma of Šilov ([3], p. 40), for every sequence of positive numbers M_n there exists in L^ω such an element x_0 that $\|x_0\|_n \geqslant M_n$. Hence for every entire function $\varphi(z) = \sum_0^\infty a_n z^n$ which is not a polynomial we can choose such an $x_0 \in L^\omega$ that $\|a_n x_0^n\|_1 \geqslant 1$ for every $a_n \neq 0$. Thus the following question arises: is a B_0-algebra R m-convex if there is defined at least one entire function which is not a polynomial? The negative answer is contained in the following

THEOREM 2. *For every entire function $\varphi(z) = \sum_0^\infty a_n z^n$ there exists a non-m-convex algebra R_φ such that the series $\varphi(x) = \sum_0^\infty a_n x^n$ in convergent for every $x \in R_\varphi$.*

The following section is devoted to the construction of such algebras.

2. Algebras R_φ

LEMMA 2.1. *For every continuous function $\Gamma(u) > 0$, $0 \leqslant u < \infty$, such that $\gamma(u) = \Gamma(u)/u \to \infty$ as $u \to \infty$ there exists a function $\Omega(u)$ such that*

1° *$\Omega(u)$ is a convex function,*

2° *$\omega(u) = \Omega(u)/u$ is increasing to infinity,*

3° *$\sup_t \big(\Omega(Nt) - 8N\Omega(t)\big) \leqslant \Gamma(N)$ for sufficiently great N.*

(1) L^ω consists of all functions $f(t)$ defined for $0 \leqslant t \leqslant 1$ and such that $\|f\|_k = \big(\int_0^1 |f(t)|^k dt\big)^{1/k} < \infty$, $k = 1, 2, \ldots$, with the pointwise multiplication.

Proof. It may easily be verified that there exists a continuous function $t(x)$, $1 \leqslant x < \infty$, strictly increasing to infinity and such that $t(1) = 0$ and $t(x) \leqslant \min\{2^{-1/2}\gamma^{1/2}(x^{1/2}), (x-1)^{1/2}\}$.

Let $T(t)$ be its inverse function. We have

$$T(t) \geqslant 1+t^2 \geqslant 2t. \tag{12}$$

Consequently, if we put $h_0 = 0$, $h_n = T(h_{n-1}) = T^n(0) = T^{n-1}(1)$, we have $h_n \geqslant 2^{n-1}$, $n = 1, 2, \ldots$ We put $\omega(h_n) = 2^{n-1}$, $\Omega(h_n) = 2^{n-1}h_n$, and between h_i and h_{i+1} we define Ω as a continuous linear function. We shall prove that Ω satisfies 1°, 2°, and 3°.

Ad 1°. It is sufficient to prove that for the nodes $(h_n, \Omega(h_n))$ we have

$$\Omega(h_n) \leqslant \lambda_n \Omega(h_{n-1}) + \mu_n \Omega(h_{n+1}), \tag{13}$$

where

$$\lambda_n = \frac{h_{n+1}-h_n}{h_{n+1}-h_{n-1}}, \quad \lambda_n + \mu_n = 1.$$

We write (13) in the form

$$2^{n-1}h_n \leqslant \frac{h_{n+1}-h_n}{h_{n+1}-h_{n-1}} \cdot h_{n-1} \cdot 2^{n-2} + \frac{h_n - h_{n-1}}{h_{n+1}-h_{n-1}} \cdot 2^n h_{n+1}, \tag{14}$$

or, setting $k_n = h_n/h_{n-1}$, in the form

$$2 \leqslant \frac{k_{n+1}-1}{k_{n+1}-k_n^{-1}} \cdot k_n^{-1} + 4 \cdot \frac{1-k_n^{-1}}{k_{n+1}-k_n^{-1}} \cdot k_{n+1}. \tag{15}$$

But if $n > 1$, then $T(h_n) \geqslant 2h_n$, and $k_n = T(h_{n-1})/h_{n-1} \geqslant 2$. This implies that for the second term of (13) (the first term being positive) we have

$$4 \cdot \frac{1-k_n^{-1}}{k_{n+1}-k_n^{-1}} \cdot k_{n+1} \geqslant 4 \cdot \frac{1-\frac{1}{2}}{k_{n+1}} \cdot k_{n+1} = 2,$$

and (13) holds for $n \geqslant 1$. For $n = 1$ formula (13) is obvious, because the nodes are $(0, 0)$, $(1, 1)$, $(h_2, 2h_2)$. We have thus proved 1°.

Ad 2°. We have $\omega(h_{n+1}) = 2\omega(h_n) > \omega(h_n)$ for $n \geqslant 1$, and for $h_n \leqslant t \leqslant h_{n+1}$

$$\omega(t) = (2^n h_{n+1} - 2^{n-1}h_n)/(h_{n+1}-h_n) - 1/t \cdot (2^{n-1}h_{n+1})/(h_{n+1}-h_n).$$

We also have $\omega(t) = 1$ for $0 \leqslant t \leqslant 1$, and so ω is non-decreasing.

Ad 3°. Let $h_{n-1} \leqslant u \leqslant h_n$, $n \geqslant 2$. We have $\omega(u) \leqslant \omega(h_n) = 2^{n-1}$, where

$$n = \min\{m : T^m(0) \geqslant u\} = \min\{m : T^{m-1}(0) \geqslant t(u)\} \leqslant \min\{m : 2^{m-2} \geqslant t(u)\} \leqslant 2 + \log_2 t(u).$$

We thus have

$$\omega(u) \leqslant 2^{n-1} \leqslant 2^{2+\log_2 t(u)-1} = 2t(u). \tag{16}$$

From (12) follows $T(t) \geqslant Nt$ for $t \geqslant N$, and $Nt \leqslant N^2$ for $t \leqslant N$; we thus have

$$\omega(Nt) \leqslant \omega(N^2) + \omega(T(t)).$$

But if $h_{n-1} \leqslant t \leqslant h_n$, then $h_n \leqslant T(t) \leqslant h_{n+1}$, and so

$$\omega(T(t)) \leqslant \omega(h_{n+1}) = 4\omega(h_{n-1}) \leqslant 4\omega(t), \tag{17}$$

and

$$\omega(Nt) \leqslant \omega(N^2) + 4\omega(t).$$

To obtain 3° we write

$$\begin{aligned} \Omega(Nt) - 8N\Omega(t) &= Nt\omega(Nt) - 8Nt\omega(t) \\ &\leqslant Nt(\omega(N^2) + 4\omega(t) - 8\omega(t)) = Nt(\omega(N^2) - 4\omega(t)). \end{aligned} \tag{18}$$

By (16), $\omega(N^2) - 4\omega(t) \leqslant 0$ for $t \geqslant t(N^2)$, and so by (17) and (15) we have

$$\sup_t (\Omega(Nt) - 8N\Omega(t)) \leqslant \sup_{t \leqslant t(N^2)} Nt(\omega(N^2) - 4\omega(t))$$

$$\leqslant \sup_{t \leqslant t(N^2)} Nt\omega(N^2) = Nt(N^2)\omega(N^2) \leqslant 2Nt^2(N^2) \leqslant N\gamma(N) = \Gamma(N), \quad \text{q. e. d.}$$

LEMMA 2.2. *For the function $\Omega(t)$ defined above we have*

$$\sup_{\{\xi_i\}} \left(\Omega\left(\sum_{k=1}^N \xi_k\right) - 8\sum_{k=1}^N \Omega(\xi_k)\right) \leqslant \Gamma(N), \tag{19}$$

where $\xi = \{\xi_i\}$ is any N-sequence of positive reals $\xi_1, \ldots, \xi_N$.

Proof. In fact, if we put $\sum_{k=1}^N \xi_k = Nu$, then, by the convexity of Ω, we have $\sum_{k=1}^N \Omega(\xi_k) \geqslant N\Omega(u)$, which, by lemma 2.1, implies

$$\Omega\left(\sum_{k=1}^N \xi_k\right) - 8\sum_{k=1}^N \Omega(\xi_k) \leqslant \Omega(Nu) - 8N\Omega(u) \leqslant \Gamma(N), \quad \text{q. e. d.}$$

Example 2.3 (see [4]). Let $(a_{n,p})_{n,p=0}^{\infty}$ be a matrix of reals satisfying the following conditions:

1° $0 < a_{n,p} \leqslant a_{n,p+1}$,

2° There exist such constants $C_p > 0$ that

$$a_{n+m,p} \leqslant C_p a_{n,p+1} a_{m,p+1} \quad \text{for every } n \text{ and } m.$$

We define $K = K(a_{n,p})$ as a B_0-algebra of all formal power series of the form

$$x = \sum_{n=0}^{\infty} \xi_n \lambda^n$$

such that

(20) $$\|x\|_p = \sum_{n=0}^{\infty} |\xi_n| \cdot a_{n,p} < \infty, \quad p = 1, 2, \ldots$$

Multiplication in K is defined as "pointwise multiplication" of elements, or convolution multiplication of coefficients. The continuity of multiplication follows from 2°, and so K is a B_0-algebra with pseudonorms (20). Such algebras were considered in papers [4] and [5].

In the sequel we shall consider algebras K for matrices of the form

$$a_{n,p} = e^{p\Omega_n},$$

where Ω_n is a suitable sequence of positive reals.

Algebras $K(e^{p\Omega_n})$, treated as topological linear spaces, were considered in papers [8], [9], [11] and [12].

Proposition 2.4. *If $\Omega_{n+m} \leqslant C(\Omega_n + \Omega_m)$, then $a_{n,p} = e^{p\Omega_n}$ satisfies 1° and 2° of example 2.3. Moreover, if $\Omega_n/n \to \infty$, then not all entire functions are defined in $K(e^{p\Omega_n})$, and so, by theorem 1, $K(e^{p\Omega_n})$ is a non-m-convex B_0-algebra.*

Proof. The first assertion is obvious. Now if we put

$$h(z) = \sum_{n=0}^{\infty} h_n z^n, \quad h_n = e^{-\sqrt{n\Omega_n}}, \quad n \geqslant 0,$$

then h is an entire function. If we take element x of K defined as $x = \sum_{n=0}^{\infty} \eta_n \lambda^n \equiv \lambda$, then the series $h(x)$ is divergent because the n-th term of the series $\sum_{0}^{\infty} h_n a_{n,p}$, equal to $\exp(-\sqrt{n\Omega_n} + p\Omega_n)$, tends to infinity, q. e. d.

The proof of theorem 2 is based upon the following

Proposition 2.5. *Let M_n, $n = 0, 1, 2, \ldots$, be a sequence of positive reals such that*

(21) $$M_n/n \to \infty \quad (n \to \infty),$$

then there exists such a non-m-convex B_0-algebra R that for each complex sequence $(b_n)_{n=0}$ satisfying

(22) $$\sum_{n=0}^{\infty} |b_n| e^{M_n} < \infty$$

and for each $x \in R$ the series

$$\sum_{n=0}^{\infty} b_n x^n$$

is convergent in R.

Proof. Put $\Gamma(n) = \sqrt{nM_n}$; we have

$$\Gamma(n)/M_n \to 0 \quad (n \to \infty) \quad \text{and} \quad \Gamma(n)/n \to \infty \quad (n \to \infty).$$

Thus if we define $\Gamma(u)$ linearly in the segments $[n, n+1]$, then it satisfies the assumptions of lemma 2.1 and we obtain a function $\Omega(u)$ satisfying 1°-3° of that lemma. We now put $R = K(e^{p\Omega(n)})$. $a_{n,p} = e^{p\Omega(n)}$ evidently satisfy the assumptions of example 2.3 and so R is a B_0-algebra. By 2° of lemma 2.1 and proposition 2.4 it is non-m-convex.

Now let $x = \sum_{n=0}^{\infty} \xi_n \lambda^n \in R$; we give an estimation of $\|x^n\|_p$. We have

$$\|x^n\|_p = \sum_{k=0}^{\infty} \Big| \sum_{i_1+\ldots+i_n=k} \xi_{i_1}\xi_{i_2}\ldots\xi_{i_n} \Big| e^{p\Omega(k)}.$$

We have $\sum_{j=1}^{n} i_j = k$, and so, by (19),

$$\Omega(k) \leqslant 8 \sum_{j=1}^{n} \Omega(i_j) + \Gamma(n)$$

and

$$\|x^n\|_p \leqslant \sum_{k=0}^{\infty} \sum_{\sum i = k} |\xi_{i_1}| \ldots |\xi_{i_n}| e^{8p\sum_{j=1}^{n}\Omega(i_j) + p\Gamma(n)}$$

$$= e^{p\Gamma(n)} \Big(\sum_{i=0}^{\infty} |\xi_i| e^{8p\Omega(i)} \Big)^n = e^{p\Gamma(n)} \|x\|_{8p}^n.$$

Now if (b_n) satisfies (22), then there exists such a C that $|b_n| \leqslant Ce^{-M_n}$ and for an arbitrary $x \in R$ and $p = 1, 2, \ldots$ we have

$$|b_n|\,\|x^n\|_p \leqslant C\exp(-M_n + p\Gamma(n) + n\log\|x\|_{8p}).$$

But for arbitrary reals A and B we have

$$(A\Gamma(n) + Bn)/M_n \to 0,$$

and so for $n > n_0(p, \log\|x\|_{8p})$ we have

$$-M_n + p\Gamma(n) + n\log\|x\|_{8p} \leqslant -\tfrac{1}{2}M_n,$$

and by (22)

$$\sum_{n=0}^{\infty} |b_n|\,\|x^n\|_p \leqslant \sum_{n=0}^{\infty} e^{-\frac{1}{2}M_n} < \infty, \quad \text{q. e. d.}$$

As a corollary we obtain the proof of Theorem 2. In fact, if $\varphi(z) = \sum_{n=0}^{\infty} b_n z^n$, then it is easy to construct such sequences of positive reals (M_n) that (21) and (22) hold. We can thus define algebra R_φ as algebra R constructed in proposition 2.5.

3. An example

In [13] it was posed the following question (Problem 6):

"Is a B_0-algebra m-convex if for every element $x \in R$ non-invertible in R there exists a non-zero multiplicative linear functional f such that $f(x) = 0$?"

Here we give an example of a non-m-convex B_0-algebra R_w which possesses a total family $\mathfrak{M}$ of multiplicative linear functionals, and has the "Wiener property" (i. e. $x^{-1} \in R$ if and only if $f(x) \neq 0$ for every $f \in \mathfrak{M}$). We thus give a negative answer to the question mentioned above.

We note that algebra R_w has also another property, namely its only invertible elements are scalar multiplicities of the unit.

Example 3.1. Let $M_p(r)$, $p = 0, 1, 2, \ldots$, be a sequence of continuous positive functions monotonely increasing to infinity and suppose that, for each p, there exist such a q and a positive constant $C_{p,q}$ that

$$M_p(r) \geqslant C_{p,q} M_q^2(r). \tag{23}$$

We define algebra R_w as the algebra of all entire functions $x(\lambda)$ such that

$$\|x\|_p = \sup |x(\lambda)|/M_p(|\lambda|) < \infty, \quad p = 1, 2, \ldots \tag{24}$$

Multiplication in R_M is defined as pointwise multiplication and its continuity in pseudonorms (24) follows from (23). In the same way as in Proposition 2.4 we prove that R_M is a non-m-convex B_0-algebra.

It may be shown that R_M is isomorphic with $K(e^{p\Omega_n})$, where $M_p(r) = M^p(r^{1/p})$, and $M(r) = \sup_{n \geqslant 0} r^n e^{-\Omega_n}$ (cf. [3], p. 203-206).

Proposition 3.2. *Every linear multiplicative functional f defined in algebra R_M is of the form*

$$f(x) = x(\lambda_0), \tag{25}$$

where λ_0 is a fixed complex number.

Proof. Let f be a non-zero multiplicative linear functional defined on R_M, and let $z = z(\lambda) \equiv \lambda$. Put $\lambda_0 = f(z)$. If $x \in R_M$, then it may easily be verified that $y(\lambda) = (x(\lambda) - x(\lambda_0))/(\lambda - \lambda_0) \in R_M$. Thus by the relation

$$x = x(\lambda_0)e + (z - \lambda_0 e)y,$$

where $e = e(\lambda) \equiv 1$, (25) holds, q. e. d.

Definition 3.3. We define algebra R_w as R_M, where

$$M_p(r) = \exp(r^\beta/p)$$

and $\beta > 0$ is a fixed real. This is an algebra of entire functions of order β and minimal type.

We shall prove that algebras R_w have the "Wiener property". The proof is based upon the following known ([6])

Lemma 3.4. *If $x(\lambda)$ is an entire function and*

$$|x(\lambda)| \geqslant A\exp(-C|\lambda|^\alpha), \quad \alpha > 0, \tag{26}$$

then there exist such constants A_1 and C_1 that

$$|x(\lambda)| \leqslant A_1 \exp(C_1|\lambda|^\alpha). \tag{27}$$

From this lemma we deduce

Proposition 3.5. *If $x \in R_w$ and $x(\lambda) \neq 0$, then $1/x(\lambda) \in R_w$.*

Proof. We have $x(\lambda) = e^{y(\lambda)}$, and by the definition of R_w we have

$$|e^{y(\lambda)}| \leqslant A\exp(C|\lambda|^\beta). \tag{28}$$

Hence

$$|e^{-y(\lambda)}| \geqslant A^{-1}\exp(-C|\lambda|^\beta)$$

and, by lemma 3.4,

$$|e^{-y(\lambda)}| \leqslant A_1\exp(C_1|\lambda|^\beta). \tag{29}$$

By (28) and (29) we have

$$|\mathrm{Re}\, y(\lambda)| \leqslant C_3|\lambda|^\beta + C_4$$

and $y(\lambda)$ is a polynomial. By (28) its degree $q \leqslant [\beta]$. Hence it follows $1/x = e^{-y(\lambda)} \in R_w$, q. e. d.

By propositions 3.2 and 3.5 we have a negative answer to the question under consideration.

Remark 3.6. If $0 < \beta < 1$, then only the invertible elements in R_w are constants.

Remark 3.7. In algebras R_w there are not entire functions but polynomials. In fact, for any $x \in R_w$ we have

$$\|x^n\|_p = \sup_\lambda |x^n(\lambda)e^{-|\lambda|^\beta/p}| = \sup_\lambda |x(\lambda)e^{-|\lambda|^\beta/np}|^n = \|x\|_{np}^n,$$

and next we continue the proof in the same way as for L^ω (see p. 295).

4. Another example

In Arens' example L^ω ([1]; see also the footnote on p. 295 of this paper) there are no multiplicative linear functionals and there are not entire functions but polynomials. We are interested in the following question: are there any connections between these properties of B_0-algebras? The answer is negative. By remark 3.7 we see that there exists a B_0-algebra having a total family of multiplicative linear functionals and having not entire functions but polynomials. In this section we shall construct a B_0-algebra R in which there are some transcendental entire functions, but in which there are no multiplicative linear functionals. This construction is a modification of Arens' example L^ω.

LEMMA 4.1. *Let $\{\gamma_n\}$ be any sequence of positive real numbers such that*

$$\sum_{n=1}^{\infty} e^{-\gamma_n} \leqslant 1.$$

Then there exists a matrix $A_{n,k}$ of positive reals, $0 \leqslant k < \infty$, $n \geqslant 1$, such that

(a) $$0 < A_{n,k} \leqslant 1,$$

(b) $$\lim_n A_{n,k} = 0, \quad k = 0, 1, 2, \ldots,$$

(c) $$A_{n,k}^{1/m} \leqslant e^{\gamma_m} A_{nm,k+1}.$$

Proof. The rows of $A_{n,k}$ will be constructed by induction. We set

$$A_{n,0} = e^{-\gamma_n}$$

and suppose that $A_{n,k}$ is constructed in such a way that (a)-(c) holds. We shall construct the row $A_{n,k+1}$. We put

$$B_{k,m}(nm) = A_{n,k}^{1/m}, \quad B_{k,m}(1) = 1,$$

and define $B_{k,m}(t)$ linearly on the segments $[mn, m(n+1)]$. It is clear that

$$0 < B_{k,m}(t) \leqslant 1 \quad \text{and} \quad \lim_{t\to\infty} B_{k,m}(t) = 0.$$

We define

$$A_{n,k+1} = \sum_{m=1}^{\infty} e^{-\gamma_m} B_{k,m}(n), \quad n = 1, 2, \ldots$$

We have

$$A_{nm,k+1} \leqslant e^{-\gamma_m} B_{k,m}(nm) = e^{-\gamma_m} A_{n,k}^{1/m},$$

which satisfies (c), and clearly satisfies (a) and (b), q. e. d.

Example 4.2. We define R as the algebra of all measurable functions on the interval $[0, 1]$ such that

(30) $$\|x(t)\|_k = \sup_n A_{n,k} \Big(\int_0^1 |x(t)|^n dt\Big)^{1/n} < \infty \quad (k = 1, 2, \ldots).$$

It is a B_0-algebra with pseudonorms (30). In fact, we have

(31) $$\|x^m\|_k = \sup_n A_{n,k} \Big(\int_0^1 |x(t)|^{mn} dt\Big)^{1/n} = \sup_n \Big[A_{n,k}^{1/m} \Big(\int_0^1 |x(t)|^{mn} dt\Big)^{1/mn}\Big]^m$$
$$\leqslant \Big[\sup_n e^{\gamma_m} A_{nm,k+1} \Big(\int_0^1 |x(t)|^{mn} dt\Big)^{1/mn}\Big]^m$$
$$\leqslant \Big[\sup_p e^{\gamma_m} A_{p,k+1} \Big(\int_0^1 |x(t)|^p dt\Big)^{1/p}\Big]^m = e^{m\gamma_m} \|x(t)\|_{k+1}^m.$$

It follows that the operation of taking the m-th power is defined and continuous in R. Thus multiplication is defined and continuous in R because

$$xy = \tfrac{1}{2}\big((x+y)^2 - x^2 - y^2\big).$$

Remark 4.3. In R some transcendental entire functions are defined, e. g., by (31), the function

$$f(z) = \sum_{m=1}^{\infty} e^{-m(\gamma_m + m)} z^m$$

is defined.

PROPOSITION 4.4. *In algebra R defined in 4.2 there are no multiplicative linear functionals.*

Proof. The algebra $C(0, 1)$ of all continuous functions defined for $0 \leqslant t \leqslant 1$ is a subalgebra of R. So, if there existed in R a non-zero multiplicative linear functional $F(x)$, it would be of the form

$$F(x) = x(t_0) \quad (0 \leqslant t_0 \leqslant 1)$$

for every $x \in C(0, 1)$. To obtain our conclusion it is enough to prove that for every $t_0 \in [0, 1]$ there exists a continuous function $x_0(t)$, $x_0(t_0) = 0$, which is invertible in R. Or, what is equivalent, we should prove that for every t_0 there exists in R a function $x(t)$, x being continuous and non--zero for $t \neq t_0$, and

$$\lim_{t=t_0} x(t) = \infty.$$

We may assume that $t_0 > 0$; otherwise we should apply an automorphism of $R: x(t) \to x(1-t)$.

Let

$$A_p = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} A_{p,k}.$$

We have $\lim A_p = 0$, and $0 < A_p \leqslant 1$. It is clear that if

$$\sup_p A_p \Big(\int_0^1 |x(t)|^p dt \Big)^{1/p} < \infty, \tag{32}$$

then $x \in R$.

We write

$$\delta_n = \inf_p A_p^{-p_n - p}.$$

By the fact of A_p tending to zero we have $\delta_n > 0$. We now put

$$\Delta_n = \min(1/2^{n+1}, \delta_n/2^n)$$

and choose λ in such a way that $\sum_{n=1}^{\infty} \lambda \Delta_n = t_0$. Now we set

$$x(t) = \begin{cases} 1 & \text{for} \quad 0 \leqslant t \leqslant \lambda\Delta_1, \\ n & \text{for} \quad t = \sum_{k=1}^{n} \lambda \Delta_k, \\ \text{is linear in the segments } \Big[\sum_{k=1}^{n} \lambda\Delta_k, \sum_{k=1}^{n+1} \lambda\Delta_k\Big]. \end{cases}$$

Moreover, if $t_0 < 1$, we put $x(t_0 + t) = x(t_0 - t)$ for $|t| < \varepsilon$, where $(t_0 - \varepsilon, t_0 + \varepsilon) \subset [0, 1]$, and $x(t) = x(t_0 + \varepsilon)$ for $t \geqslant t_0 + \varepsilon$.

We have

$$\Big(\int_0^1 |x(t)|^p dt \Big)^{1/p} = \Big(\int_0^{t_0} x^p + \int_{t_0}^{t_0+\varepsilon} x^p + \int_{t_0+\varepsilon}^{1} x^p \Big)^{1/p}$$

$$\leqslant 2 \Big(\int_0^{t_0} x^p \Big)^{1/p} + x(t_0+\varepsilon)(1 - t_0 - \varepsilon)^{1/p} = 2 \Big(\sum_1^{\infty} \int_{K_n}^{K_{n+1}} x^p \Big)^{1/p} + C$$

$$\leqslant 2 \Big(\sum_1^{\infty} \lambda n^p \Delta_n \Big)^{1/p} + C \leqslant 2\lambda^{1/p} \Big(\sum_{n=1}^{\infty} A_p^{-p}/2^n \Big)^{1/p} + C = 2\lambda^{1/p} A_p^{-1} + C,$$

where $C = x(t_0 + \varepsilon)(1 - t_0 - \varepsilon)^{1/p}$, and $K_n = \sum_{i=1}^{M} \lambda \Delta_i$. This estimation is also true in the case where $t_0 = 1$. Thus $x(t)$ is the desired function because by (32) it is a member of R, q. e. d.

5. Final remarks

By the considerations of section 4 we have seen that there exists a B_0-algebra in which there are no multiplicative linear functionals and there are some transcendental entire functions. We can give such a construction only for entire functions which are "slightly increasing", and we cannot give it for, say, $f(z) = e^z$. We thus pose the following

Problem 1. Let R be a commutative B_0-algebra with a unit, and let $\sum_{n=0}^{\infty} x^n/n!$ converge for each $x \in R$. Does there exist at least one non-zero multiplicative linear functional?

If R is a B_0-algebra in which e^x is defined for each $x \in R$, then we can prove that e^x is a continuous function of x. We can prove this also for every entire function $\varphi(z) = \sum a_n z^n$ such that $a_n \neq 0$, $n = 0, 1, 2, \ldots$, and $C_1 b_n \leqslant a_n^{-1} \leqslant C_2 b_n$, where $b_n = \max_{1 \leqslant k \leqslant n} \big(a_k^{-1}, (a_{n-k} a_k)^{-1}\big)$, but the following question is open:

Problem 2. Let R be a B_0-algebra with a unit. Let $\varphi(x) = \sum a_n x^n$ converge for each $x \in R$. Is $\varphi(x)$ a continuous function of x?

Here difficulties arise in the case of lacunary series.

We pose also

Problem 3. Is the statement of theorem 1 also true for non-commutative B_0-algebras?

We cannot answer also on the following questions:

Problem 4. Suppose f, g be two entire functions defined on the B_0-algebra R. Is the superposition $f \circ g$ an entire function defined on R?

Problem 4a. Suppose that f is an entire function defined on R. Is $f(z + z_0)$ defined on R?

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A stochastic dam process with non-homogeneous Poisson inputs

by

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0. Summary. This paper considers the distribution function (d. f.) for the content of a dam fed by non-homogeneous Poisson inputs of random size, and subject to a release at constant unit rate unless the dam is empty. The d. f. may be expressed quite generally in an integral form; if inputs are of unit size, an explicit solution is obtained to the difference-differential equation for the d. f. of the content.

1. Introduction. In two recent papers, Prabhu [4, 5] has extended some methods of storage theory to a queue for which the waiting-time $0 \leqslant Z(t) < \infty$ at time $t \geqslant 0$ satisfies the equation

$$Z(t+\delta t) = Z(t) + \delta X(t) - (1-\eta)\,\delta t. \tag{1.1}$$

In storage terminology $Z(t)$ represents the dam content; the input $0 \leqslant X(t) < \infty$ entering the dam in time t is such that the arrival times of single inputs form a Poisson process with parameter λ, the inputs (independent of arrival times) being identically and independently distributed with d. f. $H(u)$; $\eta\delta t$ $(0 \leqslant \eta \leqslant 1)$ indicates that part of the interval δt for which the dam is empty.

The content $Z(t)$ is a time-homogeneous Markov process whose transition d. f.

$$F(z_0, z, t) = \Pr\{Z(t) \leqslant z \,|\, Z(0) = z_0\} \qquad (0 \leqslant z_0,\ 0 \leqslant z < \infty)$$

satisfies Takács' [8] well-known integro-differential equation. For such a process, the probability of first emptiness $dG(z_0, t)$ of the dam at time $t \geqslant z_0$ was given by Kendall [3]. Prabhu obtains an integral of this as the probability of emptiness $F(z_0, 0, t)$ of the dam, and finds the d. f. $F(z_0, z, t)$ in an integral form involving the known input distribution and $F(z_0, 0, t)$.

In the argument some use is made of the additive nature of the input $X(t)$; in fact, the results apply equally to the non-additive input obtained when the Poisson process of arrival times is non-homogeneous with parameter $\lambda(t)$. Reich [6, 7] has studied this case, and reduced the