Now take a bounded measurable function $\varphi(y)$ of compact support. The uniform convergence of $T_i(k(\cdot, y))$ to $T(k(\cdot, y))$ implies

$$\int T_i(k(\cdot, y))\varphi(y)\,dy \to \int T(k(\cdot, y))\varphi(y)\,dy.$$

On the other hand,

$$T_i\left(\int k(\cdot, y)\varphi(y)\,dy\right) = T_i(K\varphi) \to T(K\varphi) = T\left(\int k(\cdot, y)\varphi(y)\,dy\right).$$

Thus (6) holds for every bounded measurable $\varphi(y)$ of compact support. However, both sides of formula (6) are linear functionals over $\mathscr{L}^*_{N_2}$ and the set of bounded measurable functions of compact support is dense in $\mathscr{L}^*_{N_2}$. Hence (6) holds for every $\varphi \epsilon \mathscr{L}^*_{N_2}$.

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A characterization of the class $\mathscr{L}$ of probability distributions

by

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1. Let us consider the sequence

$$\xi_{n1}+\xi_{n2}+\ldots+\xi_{nk_n}-A_n \quad (n = 1, 2, \ldots), \tag{1}$$

where $A_n = \text{const}$ and the random variables ξ_{nk} $(k = 1, 2, \ldots, k_n)$ are independent and uniformly asymptotically negligible, i. e. for every $\varepsilon > 0$

$$\lim_{n\to\infty} \max_{1\leqslant k\leqslant k_n} P(|\xi_{nk}| > \varepsilon) = 0.$$

It is known that the class of all possible limiting distributions of sums (1) is equal to the class of all infinitely divisible distributions. The class of infinitely divisible distributions can be characterized as follows (see Gnedenko and Kolomogorov [2], § 17, theorem 5):

(i) The class of infinitely divisible distributions is equal to the class of compositions of a finite number of Poisson distributions and of their limits (in the sense of weak convergence).

Let us now consider the cumulative sums of independent random variables

$$\frac{\xi_1+\xi_2+\ldots+\xi_n}{B_n}-A_n, \quad A_n, B_n = \text{const}, \quad B_n > 0, \tag{2}$$

where the random variables ξ_k/B_n $(k = 1, 2, \ldots, n)$ are uniformly asymptotically negligible. The class of limiting distributions of sums (2) is called class $\mathscr{L}$. The aim of this paper is to give a characterization of class $\mathscr{L}$ which would correspond to characterization (i) of the class of infinitely divisible distributions.

2. In the sequel we shall use the following two lemmas (see Kubik [4]):

LEMMA 1. *Let $f(x)$ be a continuous function defined in the interval $\langle a, b\rangle$. Let the right derivative $f'_+(x)$ and the left derivative $f'_-(x)$ exist at every* *point x of this interval. If $f(a) = f(b) = 0$, then there exists a number c $(a < c < b)$ such that*

$$f'_+(c)\cdot f'_-(c) \leqslant 0.$$

LEMMA 2. *Let $f(x)$ and $h(x)$ be continuous functions defined in the interval $\langle a, b\rangle$. Let the right derivatives $f'_+(x)$, $h'_+(x)$ and the left derivatives $f'_-(x)$, $h'_-(x)$ exist at every point x of this interval. If $h(a) \neq h(b)$, then there exists a number c $(a < c < b)$ such that*

$$\left(\frac{f(b)-f(a)}{h(b)-h(a)}h'_+(c)-f'_+(c)\right)\left(\frac{f(b)-f(a)}{h(b)-h(a)}h'_-(c)-f'_-(c)\right) \leqslant 0. \tag{3}$$

If (3) holds, then either

$$\frac{f(b)-f(a)}{h(b)-h(a)}h'_+(c)-f'_+(c) \leqslant 0, \quad \frac{f(b)-f(a)}{h(b)-h(a)}h'_-(c)-f'_-(c) \geqslant 0, \tag{3'}$$

or

$$\frac{f(b)-f(a)}{h(b)-h(a)}h'_-(c)-f'_-(c) \leqslant 0, \quad \frac{f(b)-f(a)}{h(b)-h(a)}h'_+(c)-f'_+(c) \geqslant 0. \tag{3''}$$

In the sequel we shall use the following notation:

$$*f'(c) = \begin{cases} f'_+(c) \text{ if (3') holds,} \\ f'_-(c) \text{ if (3'') holds,} \end{cases} \qquad **f'(c) = \begin{cases} f'_-(c) \text{ if (3') holds,} \\ f'_+(c) \text{ if (3'') holds.} \end{cases}$$

3. We shall now give the characterization of class $\mathscr{L}$, mentioned in section 1. If X has the distribution from class $\mathscr{L}$, then the logarithm of the characteristic function $\varphi(t)$ of X can be written in the Lévy-Khintchine form

$$\log\varphi(t) = i\gamma t + \int_{-\infty}^{+\infty}\left(e^{itu}-1-\frac{itu}{1+u^2}\right)\frac{1+u^2}{u^2}\,dG(u); \tag{4}$$

γ is real number and $G(u)$ is a non-decreasing bounded function $(G(-\infty) = 0)$ having a right derivative and a left derivative, denoted indifferently by $G'(u)$, at every point $u \neq 0$, and such that

$$\frac{1+u^2}{u}G'(u)$$

is a non-increasing function for $u < 0$ and for $u > 0$ (Gnedenko and Groshev [1], p. 522). The conditions given above are also sufficient for X to belong to class $\mathscr{L}$.

Let us consider distributions with function $G(u)$ of the form

$$G(u) = \begin{cases} 0 & \text{for} \quad u < A, \\ a\log\dfrac{1+A^2}{1+u^2} & \text{for} \quad A \leqslant u \leqslant 0, \\ a\log(1+A^2) & \text{for} \quad u > 0, \end{cases} \quad (a \geqslant 0) \tag{5}$$

or

$$G(u) = \begin{cases} 0 & \text{for} \quad u \leqslant 0, \\ b\log(1+u^2) & \text{for} \quad 0 < u \leqslant B, \\ b\log(1+B^2) & \text{for} \quad u > B, \end{cases} \quad (b \geqslant 0) \tag{5̄}$$

or

$$G(u) = \begin{cases} 0 & \text{for} \quad u \leqslant 0, \\ c & \text{for} \quad u > 0 \end{cases} \quad (c \geqslant 0). \tag{5*}$$

Let us denote by $\mathscr{G}$ the class of all such distributions. A distribution from the class $\mathscr{G}$ is, for class $\mathscr{L}$, an analogon of the Poisson distribution for the class of infinitely divisible distributions; namely the following theorem holds:

THEOREM (¹). *Class $\mathscr{L}$ of distributions is equal to the class of compositions of a finite number of distributions from class $\mathscr{G}$ and of their limits (in the sense of weak convergence).*

Before we present the proof of this theorem we shall give two simple auxiliary propositions.

PROPOSITION 1. *A composition of two distributions from class $\mathscr{L}$ belongs to class $\mathscr{L}$.*

Proof. Let $\varphi_1(t)$ and $\varphi_2(t)$ be characteristic functions of X_1 and X_2 from class $\mathscr{L}$ Then (see Gnedenko and Kolmogorov [2], § 29, theorem 1) for every α $(0 < \alpha < 1)$

$$\varphi_k(t) = \varphi_k(\alpha t)\cdot\varphi_{k\alpha}(t) \quad (k = 1, 2),$$

$\varphi_{k\alpha}(t)$ being a characteristic function. (It is also a sufficient condition for a distribution to belong to class $\mathscr{L}$.) The characteristic function $\psi(t)$ of the composition of X_1 and X_2 is

$$\psi(t) = \varphi_1(t)\cdot\varphi_2(t) = [\varphi_1(\alpha t)\cdot\varphi_2(\alpha t)][\varphi_{1\alpha}(t)\cdot\varphi_{2\alpha}(t)] = \psi(\alpha t)\cdot\psi_\alpha(t),$$

where

$$\psi_\alpha(t) = \varphi_{1\alpha}(t)\cdot\varphi_{2\alpha}(t),$$

(¹) This theorem has been published without proof in [5].

whence $\varphi_a(t)$ is a characteristic function. We see that $\psi(t)$ is the characteristic function of a distribution from class $\mathscr{L}$.

PROPOSITION 2. *The limit of distributions from class $\mathscr{L}$ belong to class $\mathscr{L}$.*

Proof. Let $\varphi_n(t)$ $(n = 1, 2, \ldots)$ be a sequence of characteristic functions from class $\mathscr{L}$. Thus we have

$$\varphi_n(t) = \varphi_n(at)\cdot\varphi_{na}(t).$$

We assume that

$$\lim_{n\to\infty}\varphi_n(t) = \varphi(t)$$

uniformly in every finite interval $|t| \leqslant T$. Hence for every a $(0 < a < 1)$ and t

$$\lim_{n\to\infty}\varphi_{na}(t) = \lim_{n\to\infty}\frac{\varphi_n(t)}{\varphi_n(at)} = \frac{\varphi(t)}{\varphi(at)} \overset{\text{def}}{=} \varphi_a(t)$$

and $\varphi_a(t)$ is a characteristic function. Thus the limiting distribution belongs to class $\mathscr{L}$.

Proof of the theorem. Every distribution from class $\mathscr{G}$ belongs to class $\mathscr{L}$, since $\frac{1+u^2}{u}G'(u)$ is a non-increasing function for $u < 0$ and for $u > 0$. Compositions of a finite number of distributions from class $\mathscr{G}$ and their limits belong, according to propositions 1 and 2, to class $\mathscr{L}$.

We shall now prove that every distribution from class $\mathscr{L}$ is a limit of compositions of distributions from class $\mathscr{G}$. Let us consider an arbitrary distribution from class $\mathscr{L}$. The logarithm of the characteristic function of this distribution is given by formula (4). Let us define the functions $\bar{G}(u)$, $\tilde{G}(u)$ and $G^*(u)$ as follows:

$$\bar{G}(u) = \begin{cases} G(u) & \text{for} \quad u \leqslant 0, \\ G(-0) & \text{for} \quad u > 0, \end{cases}$$

$$\tilde{G}(u) = \begin{cases} 0 & \text{for} \quad u \leqslant 0, \\ G(u) - G(+0) & \text{for} \quad u > 0, \end{cases}$$

$$G^*(u) = \begin{cases} 0 & \text{for} \quad u \leqslant 0, \\ G(+0) - G(-0) & \text{for} \quad u > 0. \end{cases}$$

Thus

$$G(u) = \bar{G}(u) + \tilde{G}(u) + G^*(u). \tag{6}$$

The functions $\bar{G}(u)$ and $\tilde{G}(u)$ are continuous.

Let us consider the sequence of functions

$$\tilde{G}_n(u) = \begin{cases} 0 & \text{for} \quad u < 0, \\ \tilde{G}_n^{(k)}(u) = b_{nk}\log(1+u^2)+d_{nk} & \text{for} \quad \dfrac{k-1}{2^n} \leqslant u \leqslant \dfrac{k}{2^n} \quad (k = 1, 2, \ldots, n\cdot 2^n), \\ \tilde{G}(n) & \text{for} \quad u > n, \end{cases} \tag{7}$$

where b_{nk} and d_{nk} are chosen in such a way that

$$\tilde{G}_n^{(k)}\left(\frac{k-1}{2^n}\right) = \tilde{G}\left(\frac{k-1}{2^n}\right), \quad \tilde{G}_n^{(k)}\left(\frac{k}{2^n}\right) = \tilde{G}\left(\frac{k}{2^n}\right) \quad (k = 1, 2, \ldots, n\cdot 2^n). \tag{8}$$

Thus we have

$$b_{nk} = \frac{\tilde{G}\left(\frac{k}{2^n}\right) - \tilde{G}\left(\frac{k-1}{2^n}\right)}{\log\left[1+\left(\frac{k}{2^n}\right)^2\right] - \log\left[1+\left(\frac{k-1}{2^n}\right)^2\right]}, \quad (k = 1, 2, \ldots, n\cdot 2^n)$$

$$d_{nk} = \tilde{G}\left(\frac{k}{2^n}\right) - b_{nk}\log\left[1+\left(\frac{k}{2^n}\right)^2\right].$$

Since $\tilde{G}(+\infty) < +\infty$, for every $\varepsilon > 0$ there exists an n_0 such that for $u \geqslant n_0$ we have $\tilde{G}(+\infty) - \tilde{G}(u) < \varepsilon$. The function $\tilde{G}(u)$ is uniformly continuous, whence there exists an n_1 such that for $n > n_1$ we have

$$\tilde{G}\left(\frac{k}{2^n}\right) - \tilde{G}\left(\frac{k-1}{2^n}\right) < \varepsilon \quad \text{for} \quad k = 1, 2, \ldots, n\cdot 2^n.$$

Since $\tilde{G}_n(u)$ is a non-decreasing function and since (8) holds, we have for $n > \max(n_0, n_1)$ and for every u $(-\infty \leqslant u \leqslant +\infty)$

$$|\tilde{G}(u) - \tilde{G}_n(u)| < \varepsilon.$$

Thus

$$\lim_{n\to\infty}\tilde{G}_n(u) = \tilde{G}(u) \quad (-\infty \leqslant u \leqslant +\infty). \tag{9}$$

We shall now show that

$$b_{n,k+1} \leqslant b_{nk} \quad (k = 1, 2, \ldots, n\cdot 2^n - 1). \tag{10}$$

Let us observe that the functions $\tilde{G}(u)$ and $\log(1+u^2)$ $(u>0)$ satisfy the assumptions of lemma 2. Therefore for every k $(k=1,2,\ldots,n\cdot 2^n-1)$ there exists a u_k such that $(k-1)/2^n < u_k < k/2^n$ and

$$(11)\qquad \left\{\frac{\tilde{G}(k/2^n)-\tilde{G}\big((k-1)/2^n\big)}{\log[1+(k/2^n)^2]-\log\big[1+\big((k-1)/2^n\big)^2\big]}\cdot\frac{2u_k}{1+u_k^2}-\tilde{G}'_+(u_k)\right\}\times$$
$$\times\left\{\frac{\tilde{G}(k/2^n)-\tilde{G}\big((k-1)/2^n\big)}{\log\big[1+(k/2^n)^2\big]-\log\big[1+\big((k-1)/2^n\big)^2\big]}\cdot\frac{2u_k}{1+u_k^2}-\tilde{G}'_-(u_k)\right\}\leqslant 0.$$

Since $\frac{1+u^2}{u}\tilde{G}'(u)$ is for $u>0$ a non-increasing function we have

$$(12)\qquad \frac{\tilde{G}\big((k+1)/2^n\big)-\tilde{G}(k/2^n)}{\log[1+\big((k+1)/2^n\big)^2]-\log[1+(k/2^n)^2]}\leqslant\frac{1+u_{k+1}^2}{2u_{k+1}}{}_*\tilde{G}'(u_{k+1})$$
$$\leqslant\frac{1+u_k^2}{2u_k}{}_{**}\tilde{G}'(u_k)\leqslant\frac{\tilde{G}(k/2^n)-\tilde{G}\big((k-1)/2^n\big)}{\log\big[1+(k/2^n)^2\big]-\log\big[1+\big((k-1)/2^n\big)^2\big]}.$$

Hence we have (10).

Let us now find the difference $d_{n,k+1}-d_{nk}$:

$$(13)\qquad d_{n,k+1}-d_{nk}=\tilde{G}\left(\frac{k+1}{2^n}\right)-b_{n,k+1}\log\left[1+\left(\frac{k+1}{2^n}\right)^2\right]-$$
$$-\tilde{G}\left(\frac{k}{2^n}\right)-b_{nk}\log\left[1+\left(\frac{k}{2^n}\right)^2\right]$$
$$=b_{nk}\log\left[1+\left(\frac{k}{2^n}\right)^2\right]+b_{n,k+1}\left\{\log\left[1+\left(\frac{k+1}{2^n}\right)^2\right]-\log\left[1+\left(\frac{k}{2^n}\right)^2\right]\right\}-$$
$$-b_{n,k+1}\log\left[1+\left(\frac{k+1}{2^n}\right)^2\right]=(b_{nk}-b_{n,k+1})\log\left[1+\left(\frac{k}{2^n}\right)^2\right]$$
$$(k=1,2,\ldots,n\cdot 2^n-1).$$

From (10) and (13) it follows that the function $\tilde{G}_n(u)$ defined by (7) represents a distribution which is a composition of a finite number of distributions from class $\mathscr{G}$. Namely, we shall prove that

$$(14)\qquad \tilde{G}_n(u)=\sum_{k=1}^{n\cdot 2^n}\tilde{G}_{nk}(u),$$

where

$$(15)\qquad \tilde{G}_{nk}(u)=\begin{cases}0 & \text{for}\quad u\leqslant 0,\\ (b_{nk}-b_{n,k+1})\log(1+u^2) & \text{for}\quad 0<u\leqslant k/2^n,\\ (b_{nk}-b_{n,k+1})\log[1+(k/2^n)^2] & \text{for}\quad u>k/2^n\end{cases}$$
$$(k=1,2,\ldots,n\cdot 2^n),$$

where we put $b_{n,n\cdot 2^n+1}=0$. Distribution (15) belongs, of course, to class $\mathscr{G}$. Let us take a fixed i $(1\leqslant i\leqslant n\cdot 2^n)$. For $(i-1)/2^n\leqslant u\leqslant i/2^n$ we have

$$\sum_{k=1}^{n\cdot 2^n}\tilde{G}_{nk}(u)=\sum_{k=i}^{n\cdot 2^n}(b_{nk}-b_{n,k+1})\log(1+u^2)+\sum_{k=1}^{i-1}(b_{nk}-b_{n,k+1})\log\left[1+\left(\frac{k}{2^n}\right)^2\right]$$
$$=b_{ni}\log(1+u^2)+\sum_{k=1}^{i-1}(d_{n,k+1}-d_{nk})=b_{ni}\log(1+u^2)+d_{ni}=\tilde{G}_n(u).$$

For $u<0$ equality (14) is obvious. For $u>n$ equality (14) also holds, since

$$\sum_{k=1}^{n\cdot 2^n}\tilde{G}_{nk}(u)=\sum_{k=1}^{n\cdot 2^n-1}(d_{n,k+1}-d_{nk})+b_{n,n\cdot 2^n}\log(1+n^2)$$
$$=d_{n,n\cdot 2^n}+b_{n,n\cdot 2^n}\log(1+n^2)=\tilde{G}(n)=\tilde{G}_n(u),$$

which completes the proof of (14).

Quite analogically we prove that

$$(16)\qquad \bar{G}(u)=\lim_{n\to\infty}\bar{G}_n(u)\qquad(-\infty\leqslant u\leqslant+\infty),$$

where $\bar{G}_n(u)$ represents a distribution which is a composition of a finite number of distributions from class $\mathscr{G}$.

Let us write

$$G_n(u)=\bar{G}_n(u)+\tilde{G}_n(u)+G^*(u).$$

According to (6), (9), (16) we have

$$(17)\qquad \lim_{n\to\infty}G_n(u)=G(u)\qquad(-\infty\leqslant u\leqslant+\infty).$$

Let us consider the sequence of distributions with characteristic functions $\varphi_n(t)$ $(n=1,2,\ldots)$ given by the formula

$$\log\varphi_n(t)=i\gamma t+\int_{-\infty}^{+\infty}\left(e^{itu}-1-\frac{itu}{1+u^2}\right)\frac{1+u^2}{u^2}\,dG_n(u).$$

In virtue of (17) and of the theorem of Gnedenko (see Gnedenko and Kolmogorov [2], § 19, theorem 1) we have

$$\lim_{n\to\infty} \varphi_n(t) = \varphi(t)$$

uniformly in every finite interval. Since every $\varphi_n(t)$ is a characteristic function of a composition of a finite number of distributions from class $\mathscr{G}$, theorem 1 is proved.

4. Let us also observe that the analogy between the Poisson distribution and the distribution from class $\mathscr{G}$ goes further. It is known that the Poisson distribution is the limiting distribution of sums (1), where ξ_{nk} are suitably chosen two-valued random variables. Similarly every distribution from class $\mathscr{G}$ is the limiting distribution of sums (2), where ξ_k are suitably chosen two-valued random variables. This follows immediately from paper [3] and from the relation between $G(u)$ and $K(u)$, where $G(u)$ is the function in Lévy-Khintchine formula and $K(u)$ is the function in Kolmogorov formula.

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