D'après (12) et (13),

$$\left|\frac{1}{Y}\int_X^{X+Y} f(x)\,e^{iux}\,dx\right| \leqslant \frac{k\varepsilon_k+2}{k-2},$$

donc $b_f(\lambda)$ existe pour tout λ, et $f\in R_0$. Cela achève la démonstration du théorème 2.

Démonstration du théorème 3. Désignons par C l'ensemble des fonctions uniformément continues et bornées. Soit Δ_1 et Δ_2 deux fonctions continues, périodiques et de période 1, respectivement nulles en 0 et 1/2, telles que $\Delta_1^2+\Delta_2^2=1$. Toute fonction $h\in C$ s'écrit $h=g^2$, $g\in C$. Posons $g_1=g\Delta_1$ et $g_2=g\Delta_2$. D'après le théorème 2, il existe une fonction $f_1\in R_0\cap C$ telle que $f_1^2=g_1^2$ (f_1 est construit à partir de g_1 comme, dans le théorème, f à partir de g); de même il existe $f_2\in R_0\cap C$ telle que $f_2^2=g_2^2$. Ainsi $h=g^2=g_1^2+g_2^2=f_1^2+f_2^2$, soit $h=(f_1+if_2)(f_1-if_2)$, et le théorème 3 est démontré.

Remarquons qu'au lieu de C, on aurait pu considérer, par exemple, la classe D des fonctions indéfiniment dérivables et bornées ainsi que toutes leurs dérivées. Il est alors faux que toute $h\in D$ s'écrive $h=g^2$, $g\in D$. Mais il est facile de montrer que toute $h\in D$ est la somme des carrés de quatre fonctions $\in R_0\cap D$.

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Reçu par la Rédaction le 2. 5. 1961

STUDIA MATHEMATICA T. XXI. (1962)

On some spaces of functions and distributions (II)

Integral transforms in $\mathscr{D}_M$ and $\mathscr{D}'_M$

by

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1. Preliminaries. Let $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_m)$ and let $k(x,y)$ be a measurable function of $n+m$ variables. We shall write [1]

$$K\varphi(x)=\int k(x,y)\varphi(y)\,dy \tag{1}$$

for every function $\varphi(y)$ such that the above integral exists for almost every x; the integral is taken over the whole m-dimensional space. $K\varphi$ is called the *integral transform of the function* $\varphi(y)$ *generated by the kernel* $k(x,y)$. Assume that $k(x,y)$ is such that K is a linear operator from $\mathscr{L}^*_{N_2}$ to $\mathscr{D}_{N_1}$. Then we denote by K^* the adjoint of K, i. e. an operator over $\mathscr{D}'_{M_1}$ defined by

$$(K^*T)(\varphi)=T(K\varphi), \tag{2}$$

where $\varphi\in\mathscr{L}^*_{N_2}$. Assuming that $N_2(u)$ satisfies condition (Δ_2) for all u, it is obvious that K^*T is a linear functional over $\mathscr{L}^*_{N_2}$, i. e. $K^*T\in\mathscr{L}^*_{M_2}$. We shall call K^*T the *integral transform of the distribution* T *generated by the kernel* $k(x,y)$.

The following assumption concerning the kernel $k(x,y)$ will be made

(As) *The function* $k(x,y)$ *is measurable in* $R^n\times R^m$, *belongs to* $\mathscr{E}(R^n)$ *for almost every* $y\in R^m$ *and satisfies the following three conditions:*

1° $D_x^p k(x,y)$ *are equicontinuous for* $x\in R^n$ *in every bounded set of* $y\in R^m$, p *being fixed,*

2° $k_p(x)=\|D_x^p k(x,\cdot)\|_{M_2}$ *is bounded for every* p *separately,*

3° *the double-norm* $|||D_x^p k|||_{M_2N_1}=\|k_p\|_{N_1}$ *is finite for every* p.

For instance, the function $k(x,y)=\exp(-|x|^2-|y|^2)$ satisfies assumption (As).

As is well known, the following two notions of convergence and boundedness in a dual space $\mathscr{X}'$ of a B_0-space $\mathscr{X}$ are considered: on one hand, convergence and boundedness defined by the strong topology in

[1] We apply here the same notation as in [10] and [9].

$\mathscr{X}'$ (cf. [2], vol. II, Chap. IV, 3.1, or [5], 21.1) and, on the other hand, convergence and boundedness in the sense of Mazur and Orlicz ([8], p. 164). A sequence $\{x_i'\} \subset \mathscr{X}'$ is called *MO-convergent* to 0 in $\mathscr{X}'$ if $x_i'(x) \to 0$ uniformly in a neighbourhood of zero in $\mathscr{X}$. A set $A' \subset \mathscr{X}'$ is called *MO-bounded* in $\mathscr{X}'$ if there exist a positive number ε and a positive integer m such that every $x \in \mathscr{X}$ satisfying the condition $\|x\|_i \leqslant \varepsilon$ for $i \leqslant m$ satisfies the inequality $|x'(x)| \leqslant 1$ for any $x' \in A'$ ([8], p. 164 and 166). Obviously, every MO-convergent sequence $\{x_i'\} \subset \mathscr{X}'$ is strongly convergent (²) and every MO-bounded set in $\mathscr{X}'$ is strongly bounded. Conversely, there are strongly convergent sequences in spaces $\mathscr{X}'$ which are not MO-convergent (cf. [8], 2.81). However, the notions of MO-boundedness and strong boundedness in $\mathscr{X}'$ are equivalent; this follows easily from [8], 2.882 (cf. [2], Vol. II, Chap. IV, 3.2 Prop. 2 and Chap. III, 1.1, Coroll.).

Now, an additive homogeneous operator K^* over a space $\mathscr{X}'$ dual to a B_0-space $\mathscr{X}$ with values in a B-space is called *compact* if it maps strongly bounded sets in $\mathscr{X}'$ onto relatively compact sets. Since strong boundedness and MO-boundedness in $\mathscr{X}'$ are equivalent, the operator K^* is compact if and only if it maps MO-bounded sets onto relatively compact sets (³).

2. Continuity and compactness of integral operators.

We shall prove the following theorem on the compactness of the operator K and the compactness of the operator K^*:

THEOREM 1. *Let $M_1(u)$, $M_2(u)$, $N_1(u)$ and $N_2(u)$ satisfy condition (Δ_2) for all u. Assume that the kernel $k(x, y)$ satisfies condition* (As). *Then the integral operators K and K^* generated by the kernel $k(x, y)$ are linear operators from $\mathscr{L}^*_{N_2}$ to $\mathscr{D}_{N_1}$ and from $\mathscr{D}'_{M_1}$ to $\mathscr{L}^*_{M_2}$, respectively, the operators K and K^* being compact; moreover, the three conditions: $K = 0$, $K^* = 0$, $k(x, y) = 0$ almost everywhere in $R^n \times R^m$, are equivalent.*

Proof. In the proof of the linearity of K we assume only that $M_1(u)$ and $N_2(u)$ satisfy (Δ_2). First, let us see that $\int D_x^p k(x, y)\varphi(y)\,dy$ is a function belonging to $\mathscr{L}^*_{N_1}$. This is obtained by applying the following inequality (cf. [13], Chapter 7, § 15):

$$\int \Big\{ \int |D_x^p k(x, y)| |\varphi(y)| dy \Big\} |\psi(x)| dx \leqslant |||D_x^p k|||_{M_2 N_1} \|\varphi\|_{N_2} \|\psi\|_{M_1}. \tag{3}$$

(²) It is known that $\mathscr{L}^*_M$ is linearly isometric with a part A of $\mathscr{D}'_M$ and the topology defined in A by $\mathscr{L}^*_M$ is stronger than the topology induced by $\mathscr{D}'_M$ ([9], 2.3 (b) and 3.3). It is trivial that every sequence convergent in A with respect to the norm defined in $\mathscr{L}^*_M$ is also MO-convergent in $\mathscr{D}'_M$ to the same limit, i.e. convergence in $\mathscr{L}^*_M$ is stronger than MO-convergence in $\mathscr{D}'_M$.

(³) Compact transformations of a locally convex linear topological space into the same space and their adjoints are considered in [1].

Now we have

$$D^p K\varphi(x) = \int D_x^p k(x, y)\varphi(y)\,dy. \tag{4}$$

Indeed, it suffices to show that the integral on the right-hand side of this equation is uniformly convergent. Take $\varepsilon > 0$ and let $\sup_x k_p(x) = k_p$. Assume $i = E[\log_2(k_p \varepsilon^{-1} \|\varphi\|_{N_2})] + 1$. As $\int N_2(\varphi(y)/\|\varphi\|_{N_2})\,dy \leqslant 1$, there is a set A in R^m such that $R^m \setminus A$ is of finite measure and $\int_A N_2(\varphi(y)/\|\varphi\|_{N_2})\,dy < \varkappa^{-i}$, where $\varkappa > 0$ is a constant satisfying the condition $N_2(2u) \leqslant \varkappa N_2(u)$ for all u. Hence $\int_A N_2(2^i \varphi(y)/\|\varphi\|_{N_2})\,dy \leqslant 1$, whence, writing $\chi_A(x)$ for the characteristic function of the set A, we have $\|\varphi\chi_A\|_{N_2} \leqslant 2^{-i}\|\varphi\|_{N_2} \leqslant \varepsilon k_p^{-1}$. Hence $\int_A |D_x^p k(x, y)\varphi(y)|\,dy \leqslant \|D^p k(x, \cdot)\|_{M_2} \|\varphi\chi_A\|_{N_2} \leqslant \varepsilon$ and thus we have proved uniform convergence, i. e. (4) holds. However, (4) implies $D^p(K\varphi) \in \mathscr{L}^*_{N_1}$, i. e. $K\varphi \in \mathscr{D}_{N_1}$.

Now, (3) implies

$$\|D^p(K\varphi)\|_{N_1} = \sup \Big\{ \int D^p K\varphi(x)\psi(x)\,dx : \|\psi\|_{M_1} = 1 \Big\} \leqslant |||D_x^p k|||_{M_2 N_1} \|\varphi\|_{N_2},$$

whence K is continuous.

Now, assuming (Δ_2) to be satisfied by all functions $M_1(u)$, $M_2(u)$, $N_1(u)$, $N_2(u)$, we prove K to be compact, arguing as in [13], p. 320. Assuming that A is a bounded set in $\mathscr{L}^*_{N_2}$, there is a sequence $\{\varphi_i\} \subset A$ and $\varphi_0 \in \mathscr{L}^*_{N_2}$ such that $\int D_x^p k(x, y)\varphi_i(y)\,dy \to \int D_x^p k(x, y)\varphi_0(y)\,dy$ for every x, i. e. $D^p K\varphi_i(x) \to D^p K\varphi_0(x)$, whence $K\varphi_i \to K\varphi_0$ in $\mathscr{D}_{N_1}$, by the bounded convergence theorem applied to $N_1[\varkappa D^p K(\varphi_i - \varphi_0)(x)]$, $\varkappa$ being arbitrary.

The fact that K^* is continuous is obvious. The compactness of K^* results from the following general lemma: *If K is a compact operator from a B-space $\mathscr{X}$ into a B_0-space $\mathscr{Y}$, then the adjoint K^* of K is also compact.* For the compactness of K implies the compactness of the operators K_i from $\mathscr{X}$ to the quotient space $\mathscr{Y}/\|\ \|_i$, defined by means of the formula $K_i(x) = [K(x)]$, $x \in \mathscr{X}$, $y \in [y] \in \mathscr{Y}/\|\ \|_i$, and applying Schauder's theorem to K_i we obtain the compactness of the adjoint K_i^* of K_i; hence it easily follows that K^* is compact. The compactness of K^* may be proved also without application of Schauder's theorem; if A' is bounded in $\mathscr{D}'_{M_1}$, the set $\{K^*T : T \in A'\}$ is bounded in $\mathscr{L}^*_{M_2}$, whence A' contains a sequence $\{T_i\}$ such that K^*T_i converges weakly in $\mathscr{L}^*_{M_2}$ to a $\chi \in \mathscr{L}^*_{M_2}$. However, every sequence $\{\varphi_i\}$ from the unit sphere in $\mathscr{L}^*_{N_2}$ contains a subsequence $\{\varphi_{i_j}\}$ such that $K^*T(\varphi_{i_j} - \varphi_0) \to 0$ uniformly in A', where $\varphi_0 \in \mathscr{L}^*_{N_2}$. Hence $K^*T_i \to \chi$ strongly in $\mathscr{L}^*_{M_2}$.

Finally, the fact that $k(x, y) = 0$ almost everywhere is equivalent to $K = 0$ is well known (cf. [13]). Obviously, $K = 0$ is equivalent to $K^* = 0$. Thus Theorem 1 is proved completely.

We remark that, assuming the support of $k(x, y)$ in $R^n \times R^m$ to be of finite measure, condition 2° in Theorem 1 is unnecessary. In this case, other theorems may also be proved by applying the results of [6], § 15.

Let us also remark that if the assumptions of Theorem 1 hold and if $T \in \mathcal{L}^*_{M_1}$, then K^*T is the integral transform of the function $T(x)$, generated by the kernel $k(x, y)$. Indeed, we then have (cf. [9], 3.3):

$$K^*T(\varphi) = \int T(x) K\varphi(x)\,dx = \int \Big\{ \int k(x, y) T(x)\,dx \Big\} \varphi(y)\,dy,$$

whence

$$K^*T(y) = \int k(x, y) T(x)\,dx. \tag{5}$$

In the last section we shall generalize this formula.

As another example, take $T = \delta_x$ or $T = D^p \delta_x$, where δ_x is the Dirac distribution $\delta_x(\varphi) = \varphi(x)$. Then $(K^*\delta_x)(y) = k(x, y)$ and $(K^*D^p\delta_x)(y) = D^p_x k(x, y)$. In general, a linear operator K^* from D'_{M_1} to $\mathcal{L}^*_{M_2}$ may be differentiated as follows: $(D^p K^*)T = (-1)^{|p|} K^*(D^p T)$. It is easily seen that if K^* is the integral operator generated by a kernel $k(x, y)$ satisfying the assumption (As), then $D^p K^*$ is the integral operator generated by the kernel $D^p_x k(x, y)$.

3. The range of integral operators. We shall consider the ranges of the integral transforms $K\varphi = \psi$ and $K^*T = \chi$. For this purpose, the following lemma will be of importance:

LEMMA 1. *If $M(u)$ and $N(u)$ satisfy condition (Δ_2) for all u, then the space $\mathcal{D}_N$ is reflexive* (4).

Proof. By [5], 24.2(7) and 24.3(9), it is sufficient to prove that every bounded set A in $\mathcal{D}_N$ is relatively sequentially weakly compact, i. e. that every sequence $\{\varphi_i\} \subset A$ contains a subsequence $\{\varphi_{i_j}\}$ weakly convergent in $\mathcal{D}_N$ to an element $\varphi_0 \in \mathcal{D}_N$. Assume $\|D^p\varphi_i\|_N \leqslant \lambda_p$ for $i = 1, 2, \ldots$ and every p and arrange all systems p in a sequence $p^1, p^2, \ldots$ so that $|p^i| \leqslant |p^j|$ for $i < j$. Choose a subsequence $\{\varphi_{1i}\}$ of the sequence $\{\varphi_i\}$ so that $\{\varphi_{1i}\} = \{D^{p^1}\varphi_{1i}\}$ is weakly convergent to a ψ^1 in $\mathcal{L}^*_N$; $\mathcal{L}^*_N$ being reflexive ([13], Th. 6, p. 154), such a subsequence exists. Then extract a sequence $\{\varphi_{2i}\}$ from $\{\varphi_{1i}\}$ so that $\{D^{p^2}\varphi_{2i}\}$ is weakly convergent to ψ^2 in $\mathcal{L}^*_N$, etc. Now, the diagonal sequence $\{\varphi_{ii}\}$ is such that $D^{p^j}\varphi_{ii} \to \psi^j$ weakly in $\mathcal{L}^*_N$. Write $\psi^1 = \varphi_0$. It is easily seen that $\psi^j = D^{p^j}\varphi_0$ for any j. Indeed, assuming this to be true for $|p| < m$, taking $p = p^j = (p_1, p_2, \ldots, p_n)$, $p_1 > 0$, $|p| = m$ and writing $p' = (p_1 - 1, p_2, \ldots, p_n)$, we have

$$\int D^p \varphi_{ii}(x) \alpha(x)\,dx = -\int D^{p'}\varphi_{ii}(x) \frac{\partial \alpha}{\partial x_1}(x)\,dx$$

(4) In the case of $\mathcal{D}_{L^p}$ this lemma is stated in [10], Vol. II, p. 56.

for every $\alpha \in \mathcal{D}$. Since

$$\int D^p\varphi_{ii}(x)\alpha(x)\,dx \to \int \psi^j(x)\alpha(x)\,dx = \psi^j(\alpha)$$

and

$$\int D^{p'}\varphi_{ii}(x) \frac{\partial \alpha}{\partial x_1}(x)\,dx \to \int D^{p'}\varphi_0(x) \frac{\partial \alpha}{\partial x_1}(x)\,dx = (D^{p'}\varphi_0)\Big(\frac{\partial \alpha}{\partial x_1}\Big)$$

$$= -\Big(\frac{\partial}{\partial x_1} D^{p'}\varphi_0\Big)(\alpha) = -(D^p\varphi_0)(\alpha),$$

this implies $\psi^j = D^p\varphi_0$. Since $D^p\varphi_0 \in \mathcal{L}^*_N$ for all p, we have $\varphi_0 \in \mathcal{D}_N$ (cf. [10], Vol. II, Th. XIX). We have thus proved $D^p\varphi_{ii} \to D^p\varphi_0$ weakly in $\mathcal{L}^*_N$ for every p. Let $T \in \mathcal{D}'_M$. Then there exists functions $f_1, f_2, \ldots, f_k \in \mathcal{L}^*_M$ such that $T = \sum_1^k D^{p^j} f_j$ ([9], 2.3(d)). Hence

$$T(\varphi_{ii}) = \sum_{j=1}^k (-1)^{|p^j|} f_j(D^{p^j}\varphi_{ii}) \to \sum_{j=1}^k (-1)^{|p^j|} f_j(D^{p^j}\varphi_0) = T(\varphi_0)$$

as $i \to \infty$. Thus, $\varphi_{ii} \to \varphi_0$ weakly in $\mathcal{D}_N$.

We are now able to prove the following theorem on the ranges of the integral operators K and K^*:

THEOREM 2. *Assume that $M_1(u), M_2(u), N_1(u), N_2(u)$ satisfy the condition (Δ_2) for all u and that the kernel $k(x, y)$ satisfies the condition* (As). *Let Ψ and X be the ranges of the operatos K and K^*, respectively. Then*

(α) *if Φ is a bounded set in $\mathcal{L}^*_{N_2}$, sequentially weakly closed in $\mathcal{L}^*_{N_2}$, then the set $K\Phi$ is sequentially weakly closed in $\mathcal{D}_{N_1}$, whence also closed in $\mathcal{D}_{N_1}$;*

(β) *if Γ is a strongly bounded set in $\mathcal{D}'_{M_1}$, weakly closed in $\mathcal{D}'_{M_1}$, then the set K^*T is sequentially weakly closed in $\mathcal{L}^*_{M_2}$, whence also closed in $\mathcal{L}^*_{M_2}$;*

(γ) *the set Ψ is a linear subspace of $\mathcal{D}_{N_1}$ of the first category in $\mathcal{D}_{N_1}$ and the set X is a linear subspace of $\mathcal{L}^*_{M_2}$ of the first category in $\mathcal{L}^*_{M_2}$.*

Proof. Since $\mathcal{L}^*_{N_2}$ is reflexive, Φ is sequentially weakly compact. However, K maps weakly convergent sequences in $\mathcal{L}^*_{N_2}$ in weakly convergent sequences in $\mathcal{D}_{N_1}$. Thus $K\Phi$ is sequentially weakly compact in $\mathcal{D}_{N_1}$ (cf. e. g. [7], p. 92), whence (α). Now, by lemma 1 and [5], 23.5(5), 23.3(1), Γ is weakly compact in $\mathcal{D}'_{M_1}$ (cf. e. g. [4], p. 141) and this implies (β). In order to prove the first part of (γ) observe that the spheres $\Phi_r = \{\varphi \in \mathcal{L}^*_{N_2} : \|\varphi\|_{N_2} \leqslant r\}$, $r = 1, 2, \ldots$, satisfy the assumptions of (α), whence $K\Phi_r$ are closed in $\mathcal{D}_{N_1}$. Moreover, $\Psi = \bigcup_1^\infty K\Phi_r$. Assuming Ψ

to be of the second ctaegory in $\mathscr{D}_{N_1}$ it is easily seen by the linearity of Ψ that $\Psi = \mathscr{D}_{N_1}$. However, this is impossible. Indeed, if Ψ were equal to $\mathscr{D}_{N_1}$, one of the sets $K\Phi_r$ would contain a closed neighbourhood of zero in $\mathscr{D}_{N_1}$, say

$$U = \{\psi \in \mathscr{D}_{N_1}\colon \|D^p\psi\|_{N_1} \leqslant \varepsilon \quad \text{for} \quad |p| \leqslant m\}.$$

Since K is compact, U is a compact set. Now, $\|\psi\| = \sup\{\|D^p\psi\|_{N_1}\colon |p| \leqslant m\}$ is a B-norm in $\mathscr{D}_{N_1}$ and $U = \{\psi \in \mathscr{D}_{N_1}\colon \|\psi\| \leqslant \varepsilon\}$ is compact in the norm $\| \ \|$, but this is impossible ([3], IV.3.5). The second part of (γ) is obtained by the following arguments. Take the fundamental system of neighbourhoods of zero in $\mathscr{D}_{N_1}$, $U_r = \{\psi \in \mathscr{D}_{N_1}\colon \|D^p\psi\|_{N_1} \leqslant 1/r \text{ for } |p| \leqslant r\}$ and let $\Gamma_r = U_r^0$ be the polar set of U_r. Then Γ_r are strongly bounded in $\mathscr{D}'_{M_1}$ and $\bigcup_1^\infty \Gamma_r = \mathscr{D}'_{M_1}$ ([5], 29.1(6)). Let us note that Γ_r are weakly closed in $\mathscr{D}'_{M_1}$. Obviously, $X = \bigcup_1^\infty K^*\Gamma_r$. Assuming X to be of the second category in $\mathscr{L}^*_{M_2}$, we should have $X = \mathscr{L}^*_{M_2}$, by the linearity of X. But this is impossible, for $K^*\Gamma_r$ are compact in $\mathscr{L}^*_{M_2}$ and assuming one of the sets $K^*\Gamma_r$ to contain a sphere we should get a contradiction.

4. Representation of integral transforms. Formula (5) may be chosen as a starting point of another definition of an integral transform of a distribution by means of integrals of distributions $S \in \mathscr{D}'_{L^1}$ defined by the formula $\int S_x dx = 1(S)$ (cf. [11], 21, Déf. 4) (⁵). Namely, an integral transform may be defined as $\int k(x,y)T_x dx$, where the integral is understood in the sense defined above. We shall prove the equivalence of the two definitions under some additional assumptions concerning the kernel. First, we prove the following representation theorem for distributions belonging to $\mathscr{D}'_M$:

LEMMA 2. *If $M(u)$ and $N(u)$ satisfy condition (Δ_2) for all u, $\psi \in \mathscr{D}_N$ and $T \in \mathscr{D}'_M$, then $\psi T \in \mathscr{D}'_{L^1}$ and*

$$\int \psi(x)T_x dx = T(\psi).$$

Proof. Let us take any p, q and any $\alpha \in \mathscr{B}$, and let us choose $\varkappa'_q > 0$ so that $\int N(\varkappa'_q D^q\psi(x))\,dx < \infty$; then we have $\int N(\varkappa_{p,q} D^p\alpha(x) D^q\psi(x))\,dx < \infty$, where $\varkappa_{p,q} = \varkappa'_q(\sup|D^p\alpha(x)|)^{-1}$, whence $D^p\alpha D^q\psi \in \mathscr{L}^*_N$. Now, given any $\varepsilon > 0$ and $\varkappa > 0$, choose a number η so that $0 < \eta < \varkappa'_q\varkappa^{-1}$, and a set A in R^n such that $R^n \setminus A$ is of finite measure and $\int_A N(\varkappa'_q D^q\psi(x))\,dx < \frac{1}{2}\varepsilon$.

(⁵) A definition of integral transforms of distributions may also be obtained by applying the theory of integrals of distributions developed recently by R. Sikorski [12].

Then if $\alpha_i \to 0$ in $\mathscr{B}$, $|D^p\alpha_i(x)| \leqslant \eta$ for sufficiently large i and

$$\int N(\varkappa D^p\alpha_i(x) D^q\psi(x))\,dx \leqslant \int_{R^n \setminus A} N(\varkappa'_q \eta D^q\psi(x))\,dx + \tfrac{1}{2}\varepsilon < \varepsilon$$

$$\text{for} \quad \eta < \frac{1}{\varkappa'_p \sup|D^q\psi(x)|} N_{-1}\left(\frac{\varepsilon}{2\mu(R^n \setminus A)}\right).$$

Thus $D^p\alpha_i D^q\psi \to 0$ in $\mathscr{L}^*_N$. Consequently, the formula

$$\int N[\varkappa D^p(\psi(x)\alpha(x))]\,dx = \sum_{\nu_1=0}^{p_1} \cdots \sum_{\nu_n=0}^{p_n} \binom{p_1}{\nu_1} \cdots \binom{p_n}{\nu_n} \int N(\varkappa D^{p(\nu)}\alpha(x) D^{q(\nu)}\psi(x))\,dx,$$

where $p(\nu) = (\nu_1, \ldots, \nu_n)$, $q(\nu) = (p_1-\nu_1, \ldots, p_n-\nu_n)$, implies $\psi\alpha \in \mathscr{D}_N$ for $\alpha \in \mathscr{B}$ and $\psi\alpha_i \to 0$ in $\mathscr{D}_N$ if $\alpha_i \to 0$ in $\mathscr{B}$. Hence $\psi T(\alpha) = T(\psi\alpha)$ is a linear functional over $\mathscr{B}$, i. e. $\psi T \in \mathscr{D}'_{L^1}$, and $\int \psi(x)T_x dx = 1(\psi T) = T(\psi)$.

LEMMA 3. *If $M(u)$ and $N(u)$ satisfy condition (Δ_2) for all u, then $\mathscr{D}$ is dense in $\mathscr{D}'_M$.*

Proof of the density of $\mathscr{D}$ in $\mathscr{D}'_M$ is analogous to the proof of the density of $\mathscr{D}$ in $\mathscr{D}'$ ([10], Vol. I, Chap. III, Theorem 15).

THEOREM 3. *Let $M_1(u), M_2(u), N_1(u)$ and $N_2(u)$ satisfy condition (Δ_2) for all u. Assume $k(x,y)$ satisfies assumption* (As) *and let $\|D_x^p k(\cdot, y)\|_{N_1}$ be a bounded function of y for each p separately. Finally, let the support of $k(x,y)$ be contained in a strip $\{(x,y)\colon y \in A\}$, A being a fixed set of finite measure in R^m. Then*

$$K^*T(y) = \int k(x,y)T_x dx$$

for every $T \in \mathscr{D}'_{M_1}$.

Proof. By lemma 2, $T(k(\cdot, y)) = \int k(x,y)T_x dx$. Thus it is to be proved that $\int K^*T(y)\varphi(y)\,dy = \int T(k(\cdot, y))\varphi(y)\,dy$ for every $\varphi \in \mathscr{L}^*_{N_2}$, i. e. that

$$T\left(\int k(\cdot, y)\varphi(y)\,dy\right) = \int T(k(\cdot, y))\varphi(y)\,dy. \tag{6}$$

If $T \in \mathscr{D}$, formula (6) is evident and the function $T(k(\cdot, y))$ belongs to $\mathscr{L}^*_{M_2}$. If $T \notin \mathscr{D}$, by lemma 3 there is a sequence $\{T_i\} \subset \mathscr{D}$, $T_i \to T$ in $\mathscr{D}'_{M_1}$. Since the set of all $k(\cdot, y) \in \mathscr{D}_{N_1}$ with $y \in R^m$ is bounded in $\mathscr{D}_{N_1}$, we have $T_i(k(\cdot, y)) \to T(k(\cdot, y))$ uniformly in y; the support of $T(k(\cdot, y))$ being of finite measure, this implies that the function $T(k(\cdot, y))$ belongs to $\mathscr{L}^*_{M_2}$.

Now take a bounded measurable function $\varphi(y)$ of compact support. The uniform convergence of $T_i(k(\cdot, y))$ to $T(k(\cdot, y))$ implies

$$\int T_i(k(\cdot, y))\varphi(y)\,dy \to \int T(k(\cdot, y))\varphi(y)\,dy.$$

On the other hand,

$$T_i\Big(\int k(\cdot, y)\varphi(y)\,dy\Big) = T_i(K\varphi) \to T(K\varphi) = T\Big(\int k(\cdot, y)\varphi(y)\,dy\Big).$$

Thus (6) holds for every bounded measurable $\varphi(y)$ of compact support. However, both sides of formula (6) are linear functionals over $\mathscr{L}^*_{N_2}$ and the set of bounded measurable functions of compact support is dense in $\mathscr{L}^*_{N_2}$. Hence (6) holds for every $\varphi \in \mathscr{L}^*_{N_2}$.

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Reçu par la Rédaction le 16. 5. 1961

STUDIA MATHEMATICA, T. XXI. (1962)

A characterization of the class $\mathscr{L}$ of probability distributions

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1. Let us consider the sequence

$$\xi_{n1} + \xi_{n2} + \ldots + \xi_{nk_n} - A_n \quad (n = 1, 2, \ldots), \tag{1}$$

where $A_n = \text{const}$ and the random variables ξ_{nk} $(k = 1, 2, \ldots, k_n)$ are independent and uniformly asymptotically negligible, i. e. for every $\varepsilon > 0$

$$\lim_{n\to\infty} \max_{1\leqslant k\leqslant k_n} P(|\xi_{nk}| > \varepsilon) = 0.$$

It is known that the class of all possible limiting distributions of sums (1) is equal to the class of all infinitely divisible distributions. The class of infinitely divisible distributions can be characterized as follows (see Gnedenko and Kolomogorov [2], § 17, theorem 5):

(i) The class of infinitely divisible distributions is equal to the class of compositions of a finite number of Poisson distributions and of their limits (in the sense of weak convergence).

Let us now consider the cumulative sums of independent random variables

$$\frac{\xi_1 + \xi_2 + \ldots + \xi_n}{B_n} - A_n, \quad A_n, B_n = \text{const}, \quad B_n > 0, \tag{2}$$

where the random variables ξ_k/B_n $(k = 1, 2, \ldots, n)$ are uniformly asymptotically negligible. The class of limiting distributions of sums (2) is called class $\mathscr{L}$. The aim of this paper is to give a characterization of class $\mathscr{L}$ which would correspond to characterization (i) of the class of infinitely divisible distributions.

2. In the sequel we shall use the following two lemmas (see Kubik [4]):

LEMMA 1. *Let $f(x)$ be a continuous function defined in the interval $\langle a, b\rangle$. Let the right derivative $f'_+(x)$ and the left derivative $f'_-(x)$ exist at every*