Proof. We have

1. $|f(x)| \leqslant \|x\|$ for every $x \in R$, $f \in \mathfrak{M}$, whence $|f(x)| = \sqrt[n]{|f(x^n)|} \leqslant \sqrt[n]{\|x^n\|}$, and $\max |f(x)| \leqslant \underline{\lim} \sqrt[n]{\|x^n\|}$.

2. Let F be a linear functional defined on R; then $g_F(z) = F((e-zx)^{-1})$ is a holomorphic function defined for $|z| < 1/a$, where $a = \max_{f \in \mathfrak{M}} |f(x)|$, and x is a fixed element of R. It may easily be seen that $g_F(z) = \sum_{n=0}^{\infty} F(x^n) z^n = \sum_{n=0}^{\infty} F(x^n z^n)$. Consequently the sequence $x^n z^n$ is bounded, being weakly convergent to 0. We have $\|x^n z^n\| \leqslant M_z$, and $\sqrt[n]{\|x^n\|} \leqslant \sqrt[n]{M_z}/|z|$. Consequently $\overline{\lim} \sqrt[n]{\|x^n\|} \leqslant 1/|z|$ for every $|z| < 1/a$, and $\overline{\lim} \sqrt[n]{\|x^n\|} \leqslant a$.

By 1 and 2 we have $a = \lim \sqrt[n]{\|x^n\|}$, q. e. d.

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On Banach *-semialgebras

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1. Preliminaries

We shall use the term *halfring* in the sense given by the author in a previous paper [3]. We repeat that a *halfring* is a semiring which is embeddable in a ring. Since addition in our semiring S is commutative, a necessary and sufficient condition for S to be a halfring is that the additive semigroup of S be cancellative. Following [3], we construct the ring in which H is embedded. The product set $H \times H$ again forms a halfring according to the laws of addition and multiplication: $(s_1, s_2) + (t_1, t_2) = (s_1+t_1, s_2+t_2)$, $(s_1, s_2)(t_1, t_2) = (s_1t_1+s_2t_2, \; s_1t_2+s_2t_1)$. The diagonal $\Delta = \{(x, x) | \epsilon H\}$ is an ideal in $H \times H$. We say that $(s_1, s_2) \equiv (u_1, u_2)(\Delta)$ if and only if there exist elements (x, x) and (y, y) in Δ such that $(s_1, s_2) + (x, x) = (t_1, t_2) + (y, y)$. The quotient ring $\mathfrak{R} = H \times H/\Delta$ is called the *ring generated by H*. Let ν denote the natural homomorphism of $H \times H$ onto $\mathfrak{R}$, then the halfring H is embedded in the ring $\mathfrak{R}$, for the mapping $h \leftrightarrow \nu(h+a, a)$, for any a, is an isomorphism of H into $\mathfrak{R}$. We designate by $\nu(H)$ this isomorphic map of H in $\mathfrak{R}$ and by $\nu(s_1, s_2)$ the equivalence class of (s_1, s_2). A *division semiring* is a semiring, in which the elements $\neq 0$, form a multiplicative group. A semifield is a commutative division semiring. A halffield is a semifield which is embeddable in a field.

In a recent paper [4], we introduced the concept of a *normed semialgebra*. For the sake of completeness we repeat:

Definition 1. A semiring S is said to be a *semialgebra over a commutative semiring Σ with unit*, if a law of composition $(\sigma, s) = \sigma s$ of the product $\Sigma \times S$ is defined such that

(i) $(S, +)$ is a unital left Σ-semimodule relative to the composition $(\sigma, s) = \sigma s$,

(ii) for all $\sigma \epsilon \Sigma$ and $s, t \epsilon S$, $\sigma(s, t) = (\sigma s)t = s(\sigma t)$.

Definition 2. A *semivector space* is a semialgebra over a semifield.

Definition 3. A metric for a semilinear space S is said to be *invariant* if and only if $d(s+x, t+x) = d(s, t)$ for all $s, t, x \epsilon S$.

Definition 4. A *norm* for a semilinear space S, over the halffield of nonnegative reals $\boldsymbol{R}^+$, is a nonnegative real-valued function $\|s\|$ satisfying for $s, t \in S$ and $\varrho \in \boldsymbol{R}^+$

(i) $\|s\| \geqslant 0$,

(ii) $\|s\| = 0$ if and only if $s = 0$,

(iii) $\|\varrho s\| = \varrho \|s\|$,

(iv) $\|s+t\| \leqslant \|s\| + \|t\|$.

Definition 5. A set S of elements $s, t, \ldots$ is a *normed semiring* if and only if

(1) S is a semialgebra over the halffield of nonnegative reals $\boldsymbol{R}^+$,

(2) S is a semilinear space with an invariant metric $d(s, t)$,

(3) $\|s\| = d(s, 0)$ is a norm for the space S and $\|st\| \leqslant \|s\| \cdot \|t\|$ for $s, t \in S$,

(4) If S contains a unit e, then $\|e\| = 1$.

Definition 6. A *Banach semiring* is a complete normed semiring.

If in definition 6, the semiring is a halfring H, we shall refer to it as a *Banach halfring.*

Lemma 1. *If H is a Banach halfring, then the halfring $H \times H$ is a Banach halfring over $\boldsymbol{R}$ with invariant metric $D((s_1, s_2), (t_1, t_2)) = d(s_1, t_1) + d(s_2, t_2)$ and $\|(s_1, s_2)\| = \|s_1\| + \|s_2\|$.*

Proof. Immediate verification.

The ideal Δ in $H \times H$ is a closed set in the product topology [5].

Lemma 2. *The ring $\mathfrak{R}$ generated by the Banach halfring H is a Banach ring over the real field $\boldsymbol{R}$ with norm*

$$\|\nu(s_1, s_2)\| = \inf_{(u,v) \in \nu^{-1}\nu(s_1, s_2)} \|(u, v)\|.$$

Proof. In [4], we showed that $\mathfrak{R}$ is a normed ring over the real field $\boldsymbol{R}$. There remains to be proven that the completeness of H implies the completeness of $\mathfrak{R}$.

Kelley [8] proves the following theorem: *Let f be a continuous uniformly open map of a complete pseudo-metrizable space into a Haussdorff uniform space. Then the range of the map f is complete.* Now *a map of a uniform space $(X, \mathscr{U})$ into a uniform space $(Y, \mathscr{V})$ is uniformly open iff for each U in $\mathscr{U}$ there is V in $\mathscr{V}$ such that $f[U(x)] \supset V[f(x)]$ for each x in X* [8].

In [4], we showed that the continuous homomorphism ν of the topological halfring $H \times H$ onto the topological ring $\mathfrak{R}$ is an open mapping. We proceed to show that ν is uniformly open. We recall that $H \times H$ has the invariant metric $D((s_1, s_2), (t_1, t_2)) = d(s_1, t_1) + d(s_2, t_2)$ while $\mathfrak{R}$ the invariant metric $D(\nu(s_1, s_2), \nu(t_1, t_2)) = \|\nu(s_1, s_2) - \nu(t_1, t_2)\|$. Hence, a uniformity for $H \times H$ is defined by neighborhoods $U_\varepsilon[(s_1, s_2)] = \{(x, y) \mid D((x, y), (s_1, s_2)) < \varepsilon\}$ and a uniformity for $\mathfrak{R}$ by neighborhoods $V_\delta[\nu(t_1, t_2)] = \{\nu(w, z) \mid D(\nu(w, z), \nu(t_1, t_2)) < \delta\}$. Let $U_\varepsilon[(0, 0)] = \{(\xi, \eta) \mid \|(\xi, \eta)\| < \varepsilon\}$, then $(s_1, s_2) + U_\varepsilon[(0, 0)] = \{(s_1 + \xi, s_2 + \eta) \mid \|(\xi, \eta)\| < \varepsilon\}$. Now $D((s_1 + \xi, s_2 + \eta), (s_1, s_2)) = d(s_1 + \xi, s_1) + d(s_2 + \eta, \xi_2) = d(\xi, 0) + d(\eta, 0) = \|(\xi, \eta)\| < \varepsilon$. Therefore, $U_\varepsilon[(s_1, s_2)] \supset (s_1, s_2) + U_\varepsilon[(0, 0)]$. Since ν is open, then $\nu[U_\varepsilon[0, 0]] \supset V_\delta[\Delta]$. Hence, $\nu[U_\varepsilon[(s_1, s_2)]] \supset \nu(s_1, s_2) + \nu[U_\varepsilon[0, 0]] \supset \nu(s_1, s_2) + V_\delta[\Delta] = V_\delta[\nu(s, s_2)]$, which is a δ-neighborhood of $\nu(s_1, s_2)$ for $\mathfrak{R}$ is a topological ring [4]. Hence, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $U_\varepsilon[(s_1, s_2)]$ there is a $V_\delta[\nu(s_1, s_2)]$ for which $\nu[U_\varepsilon[(s_1, s_2)]] \supset V_\delta[\nu(s_1, s_2)]$ for each (s_1, s_2) in $H \times H$. ν is uniformly open and this implies that $\mathfrak{R}$ is a complete normed ring.

As a consequence of lemmas 1 and 2, we have

Theorem 1. *A Banach halfring H over the nonnegative reals $\boldsymbol{R}^+$ is embeddable in the Banach ring $\mathfrak{R}$ over the reals $\boldsymbol{R}$.*

2. Introduction

Gelfand and Naimark [6] proved the following structure theorem: *Suppose $\mathfrak{R}$ is a commutative Banach ring with identity and that an involution is defined in $\mathfrak{R}$ satisfying the usual algebraic conditions $(\lambda x + \mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^*$, $x^{**} = x$, $(xy)^* = y^* x^*$ and, furthermore, the condition $\|x^* x\| = \|x^*\| \|x\|$. Then the ring $\mathfrak{R}$ is completely isomorphic to the ring $C(\mathfrak{M})$ of all continuous functions $x(M)$ in the space $\mathfrak{M}$ of maximal ideals of the ring $\mathfrak{R}$.*

Arens [1] extended this theorem to read: *Let A be a commutative Banach *-algebra satisfying $k\|f\|\|f^*\| \leqslant \|ff^*\|$ but having no unit. Then there exists a locally compact Hausdorff space X such that A is the class of all continuous complex-valued functions on X which vanish at infinity. Also $f^*(x) = \overline{f(x)}$.*

Arens and Kaplansky [2] then obtained a theorem of this type for a *real Banach *-algebra* where there is defined an operation* which satisfies $(\lambda f + \mu g)^* = \lambda f^* + \mu g^*$, $(fg)^* = g^* f^*$, $f^{**} = f$. *Let A be a commutative real Banach *-algebra, with [without] unit, satisfying $\|f\|^2 \leqslant k\|f^* f + g^* g\|$, for f and g in A, k constant. Then there exists a [locally] compact Hausdorff space X having an involutory homeomorphism σ such that A is isomorphic to the ring of all continuous complex-valued functions on X [vanishing at infinity] which satisfy $f[\sigma(x)] = \overline{f(x)}$ for $x \in X$. Furthermore, if $\|\cdot\|$ is the norm in A, then $\|f\| = \sup_{x \in X} |f(x)|$.*

In section 3 we introduce the concept of a normed semialgebra with an involution and show that it is embeddable in a real Banach *-algebra. In section 4 we extend the Arens-Kaplansky theorem [2] to read that *if H is a Banach halfring with an involution, such that $\|(s_1, s_2)\|^2 \leqslant k\|(s_1, s_2)^*(s_1, s_2)+(t_1, t_2)^*(t_1, t_2)\|$ for all (s_1, s_2) and (t_1, t_2) in $H \times H$, where k is a constant. Then there exists a compact space $\mathfrak{M}_H$ with an involutory homeomorphism σ such that H is *-isomorphic to the halfring $C^+(\mathfrak{M})$ of all continuous complex-valued functions on $\mathfrak{M}_H$, for which $f(\sigma(M^+)) = f(M^+)$.*

3. Banach halfrings with involution

Definition 7. An *involution* in a normed semialgebra S is a mapping $s \to s^*$ of S onto itself satisfying for $s, t \in S$ and $\varrho \in \boldsymbol{R}^+$

(i) $(s^*)^* = s$,

(ii) $(s+t)^* = s^*+t^*$,

(iii) $(st)^* = t^*s^*$,

(iv) $(\varrho s)^* = \varrho s^*$.

LEMMA 3. *If H is a normed halfring with an involution $s \to s^*$, then $H \times H$ is a normed halfring with involution $(s_1, s_2) \to (s_1^*, s_2^*)$.*

Proof. Immediate verification.

If $(s_1, s_2) \equiv (u_1, u_2)(\Delta)$, then $(s_1, s_2)+(x, x) = (u_1, u_2)+(y, y)$ and $(s_1^*, s_2^*)+(x^*, x^*) = (u_1^*, u_2^*)+(y^*, y^*)$. This implies that $(s_1^*, s_2^*) \equiv (u_1^*, u_2^*)(\Delta)$, the mapping $\nu(s_1, s_2) \to \nu^*(s_1, s_2) = \nu(s_1^*, s_2^*)$ defines an involution in the ring $\mathfrak{R}$ and

LEMMA 4. *The ring $\mathfrak{R}$ generated by the normed halfring H with involution $s \to s^*$ is a normed ring over the real field $\boldsymbol{R}$ with involution $\nu(s_1, s_2) \to \nu^*(s_1, s_2)$.*

As a consequence of lemma 4 and theorem 1, we have

THEOREM 2. *A Banach halfring H over the nonnegative reals $\boldsymbol{R}^+$ with involution $s \to s^*$ is embeddable in the Banach ring $\mathfrak{R}$ over the reals $\boldsymbol{R}$ with involution $\nu(s_1, s_2) \to \nu^*(s_1, s_2) = \nu(s_1^*, s_2^*)$.*

We assume that H is commutative and possesses a unit e.

Following Słowikowski and Zawadowski [12], we state

Definition 8. A commutative semiring S is *positive* if and only if S possesses a unit e and $e+s$ has an inverse in S, for every s in S.

We recall that *the quotient ring of a commutative real normed ring by a maximal ideal is isomorphic to the real or complex field*. [7]. Let $\mathfrak{M}_{\mathfrak{R}}$ be a set of the maximal ideals of the ring $\mathfrak{R}$. We denote the natural homomorphism of $\mathfrak{R}$ onto $\mathfrak{R}/M$, $M \in \mathfrak{M}_{\mathfrak{R}}$, by Φ_M. If we hold s fixed and let M vary over $\mathfrak{M}_{\mathfrak{R}}$ we obtain a function $f_s(M) = \Phi_M(s)$, defined on $\mathfrak{M}_{\mathfrak{R}}$. We suppose that H is a positive Banach halfring which is isomorphically and homeomorphically embedded in the Banach ring $\mathfrak{R}$. If s is an element of H, then the positive nature of H implies that $1+f_s(M) \neq 0$, for $\mathfrak{R}/M$ is the complex field $\boldsymbol{C}$. If $f_s(M) = -\varrho$, ϱ a positive real number, then $f_{s/\varrho}(M) = -1$, a contradiction. If $f_s(M) = \lambda i$, λ a real number, then $f_{s^2}(M) = [f_s(M)]^2 = -\lambda^2$, a contradiction. If $f_s(M) = -\varrho+\lambda i$, ϱ a positive real number, then $f_{\varrho+s}(M) = f_\varrho(M)+f_s(M) = \varrho-\varrho+\lambda i = \lambda i$, a contradiction. Suppose $f_s(M) = \varrho+\lambda i$, $\varrho \in \boldsymbol{R}^+$, $\lambda \in \boldsymbol{R}$, then a power of $f_s(M)$ would be equal to $-\sigma+\mu i$, again a contradiction. Hence, if $s \in H$, $f_s(M)$ must be a non-negative number.

For each $M \in \mathfrak{M}_{\mathfrak{R}}$ the restriction of Φ_M to H defines a proper homomorphism of H into the halffield $\boldsymbol{R}^+$ of non-negative real numbers, which in turn determines a maximal ideal $M^+ \in \mathfrak{M}_H$, the set of maximal ideals of the halfring, such that $f_s(M^+) = f_s(M)$, for $s \in H$ [4]. If $s \in M^+$, then $0 = f_s(M^+) = f_s(M)$, which implies that $s \in M$. Hence, $M^+ = H \cap M$.

Let $M^+ \in \mathfrak{M}_H$ and M be the ideal generated by M^+. M consists of all differences $m_1 - m_2$, with $m_1, m_2 \in M^+$. M is a maximal ideal of $\mathfrak{R}$ for the mapping which associates to each element $s_1 - s_2 \in \mathfrak{R}$ the number $f_{s_1-s_2}(M^+) = f_{s_1}(M^+)-f_{s_2}(M^+)$ defines a proper homomorphism of $\mathfrak{R}$ into $\boldsymbol{R}$, with M as kernel. Hence $H \cap M = M^+$. Since M is the minimal such ideal of $\mathfrak{R}$, M^+ is contained in no other ideal of $\mathfrak{M}_{\mathfrak{R}}$.

We have set up a 1-1 correspondence between the sets $\mathfrak{M}_H$ and $\mathfrak{M}_{\mathfrak{R}}$ such that $f_s(M^+) = f_s(M)$ for any $s \in H$. Since $f_s(M) = \varrho$, $\varrho \in \boldsymbol{R}^+$, $s \in H$, the quotient semiring H/M^+ is the halffield of non-negative real numbers $\boldsymbol{R}^+$.

We have just proved the basic result which is an extension of the Mazur theorem [9]:

THEOREM 3. *If H is a positive Banach halfring, then the quotient semiring of H by a maximal ideal is the halffield of non-negative real numbers.*

We topologize $\mathfrak{M}_H$ after the manner of Gelfand [7]. It is the weakest topology in which the functions $f_s(M^+)$ are continuous and $\mathfrak{M}_H$ is a compact Hausdorff space. Since the halfring H generates the ring $\mathfrak{R}$, $\mathfrak{M}_H \leftrightarrow \mathfrak{M}_{\mathfrak{R}}$ and $f_s(M^+) = f_s(M)$, $M^+ \leftrightarrow M$, $s \in H$, the topology of $\mathfrak{M}_H$ is the same as that of $\mathfrak{M}_{\mathfrak{R}}$. Let $\boldsymbol{R}^+(\mathfrak{M}_H)$ denote the halfring of continuous functions $f_s(M^+)$ on space $\mathfrak{M}_H$ of maximal ideals of the halfring H. Then we have an extension of Gelfand's theorem [7]:

THEOREM 4. *If H is a positive Banach halfring, then there exists* *a homomorphism of H into the halfring $\mathbf{R}^+(\mathfrak{M}_H)$ of all continuous functions on a compact Hausdorff space.*

4. Structure of Banach *-halfrings

In accord with Naĭmark [10] we give

Definition 9. A mapping $s \to s'$ of the *-semialgebra S into the *-semialgebra S' is a **-homomorphism* if and only if (1) $s \to s'$ is a homomorphism, (2) $s \to s'$ implies that $s^* \to s'^*$.

Definition 10. A *complete isomorphism* of the *-semialgebra S onto *-semialgebra S' is an isometric mapping of the space S onto the space S' which is a *-isomorphism of the *-semialgebra S onto the *-semialgebra S'.

As in [11], by embedding the Banach *-halfalgebra H into a real Banach *-algebra whose structure is given by Arens and Kaplansky [2], we obtain

THEOREM 5. *Let H be a Banach halfring with an involution such that $\|(s_1, s_2)\|^2 \leqslant k\|(s_1, s_2)^*(s_1, s_2)+(t_1, t_2)^*(t_1, t_2)\|$ for all (s_1, s_2) and (t_1, t_2) in $H\times H$, where k is a constant. Then there exists a compacvt space $\mathfrak{M}_H$ with an involutory homeomorphism σ such that H is *-isomorphic to the halfring $C^+(\mathfrak{M}_H)$ of all continuous complex-valued functions on $\mathfrak{M}_H$, for which $f(\sigma(M^+)) = \overline{f(M^+)}$.*

Proof. We embed H in $\mathfrak{R} = H\times H$, then $\mathfrak{R}$ is a real Banach *-algebra with the involution $\nu(s_1, s_2) \to \nu(s_1^*, s_2^*)$. Now

$$k\|\nu^*(s_1, s_2)\nu(s_1, s_2)+\nu^*(t_1, t_2)\nu(t_1, t_2)\|$$
$$= k\|\nu(s_1^*s_1+s_2^*s_2+t_1^*t_1+t_2^*t_2,\ s_1^*s_2+s_2^*s_1+t_1^*t_2+t_1t_2^*)\|$$
$$= k \inf_{\substack{(u_1,u_2)\in\nu^{-1}\nu(s_1,s_2)\\(v_1,v_2)\in\nu^{-1}\nu(t_1,t_2)}} \|u_1^*u_1+u_2^*u_2+v_1^*v_1+v_2^*v_2\|+\|u_1^*u_2+u_2^*u_1+v_1^*v_2+v_2^*v_1\|$$
$$\geqslant k \inf_{\substack{(u_1,u_2)\in\nu^{-1}\nu(s_1,s_2)\\(v_1,v_2)\in\nu^{-1}\nu(t_1,t_2)}} \|u_1^*u_1+u_2^*u_2+u_1^*u_2+u_2^*u_1+v_1^*v_1+v_2^*v_2+v_1^*v_2+v_2^*v_1\|$$
$$= k \inf_{\substack{(u_1,u_2)\in\nu^{-1}\nu(s_1,s_2)\\(v_1,v_2)\in\nu^{-1}\nu(t_1,t_2)}} \|(u_1, u_2)^*(u_1, u_2)+(v_1, v_2^*)(v_1, v_2)\|.$$

Since $\|(u_1, u_2)\|^2 \leqslant k\|(u_1, u_2)^*(u_1, u_2)+(v_1, v_2)^*(v_1, v_2)\|$ for all (u_1, u_2) and (v_1, v_2) in $H\times H$, we have

$$k\|\nu^*(s_1, s_2)\nu(s_1, s_2)+\nu^*(t_1, t_2)\nu(t_1, t_2)\| \geqslant \inf_{(u_1,u_2)\in\nu^{-1}\nu(s_1,s_2)} \|u_1, u_2)\|^2$$
$$= \Big(\inf_{(u_1,u_2)\in\nu^{-1}\nu(s_1,s_2)} \|(u_1, u_2)\|\Big)^2 = \|\nu(s_1, s_2)\|^2.$$

Hence, this condition which is satisfied in $\mathfrak{R}$ is precisely the one which Arens and Kaplansky [2] assumed and makes $\mathfrak{R}$ *-isomorphic to the ring of all those continuous complex-valued functions on $\mathfrak{M}_{\mathfrak{R}}$ which satisfy $f_s[\sigma(M)] = \overline{f_s(M)}$. Since for $s\in H$, $f_s(M^+) = f_s(M)$, we examine the map of H in the ring of all those continuous complex-valued functions on $\mathfrak{M}_H$.

Now $f_{(s_1-s_2)^*}(M) = \overline{f_{s_1-s_2}(M)} = \overline{f_{s_1}(M^+)} - \overline{f_{s_2}(M^+)}$. Since $(s_1-s_2)^* = s_1^*-s_2^*$ in $\mathfrak{R}$, $f_{(s_1-s_2)^*}(M) = f_{s_1^*}(M^+)-f_{s_2^*}(M^+)$. Therefore we have $f_{s_1^*}(M^+)-f_{s_2^*}(M^+) = \overline{f_{s_1}(M^+)} - \overline{f_{s_2}(M^+)}$ for all $s\in H$. In particular, if $s_2 = 0$ we obtain that $f_{s_1^*}(M^+) = \overline{f_{s_1}(M^+)}$. This states that the mapping $s \to f_s(M^+)$ is a *-isomorphism of H into $C(\mathfrak{M}_H)$.

The involutory homeomorphism $\sigma(M^+)$ of the compact space $\{\mathfrak{M}_H\}$ onto itself is defined by $f_s(\sigma(M^+)) = \overline{f_s(M^+)}$. Since $f_{s^*}(M^+) = \overline{f_s(M^+)}$ the map of H in $C^+(\mathfrak{M}_H)$ is contained in the set of those functions which satisfy $f_s(\sigma(M^+)) = \overline{f_s(M^+)}$.

Let $C^+(\mathfrak{M}_H, \sigma)$ denote the halfring of all those continuous functions on $\{\mathfrak{M}_H\}$ for which $f_s(\sigma(M^+)) = \overline{f_s(M^+)}$ for all $M^+\in\mathfrak{M}_H$. For any $f\in C^+(\mathfrak{M}_H, \sigma)$ then $f(\sigma(M^+)) = \overline{f(M^+)}$. Since f is also in $C^+(\mathfrak{M}_H)$, then $f(M^+) = f_s(M^+)+if_t(M^+)$, $s, t\in H$. Thus, $f(\sigma(M^+)) = f_s(\sigma(M^+)) + if_t(\sigma(M^+)) = \overline{f_s(M^+)} - \overline{if_t(M^+)}$. Since $f_s(\sigma(M^+)) = \overline{f_s(M^+)}$ and $f_t(\sigma(M^+)) = \overline{f_t(M^+)}$, then $\overline{f_s(M^+)+if_t(M^+)} = \overline{f_s(M^+)} - \overline{if_t(M^+)}$. Therefore $f_t(M^+) = 0$ and $f(M^+) = f_s(M^+)$, for all $M^+\in\mathfrak{M}_H$, and f is contained in the map of H in $C^+(\mathfrak{M}_H)$. Hence, the map of H in $C^+(\mathfrak{M}_H)$ is precisely $C^+(\mathfrak{M}_H, \sigma)$. In the case $k=1$, the *-isomorphism is complete.

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Asymptotische Auffassung der Operatorenrechnung

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W. A. Ditkin hat in seiner Arbeit [5] gezeigt, daß der Körper der Operatoren von J. Mikusiński [7] einem mit Hilfe von Restklassen und direkten Summen aus analytischen Funktionen gebildeten Körper isomorph ist. Identifiziert man den Operator s von Mikusiński mit einer komplexen Veränderlichen, so wird erreicht, daß man beim Aufbau der Analysis für Operatorfunktionen die klassische Funktionentheorie verwenden kann. Insbesondere bleiben die Laplace-Transformation und die komplexe Umkehrformel für die Rücktransformation als wertvolle Hilfsmittel in der Operatorenrechnung. In dem Vortrag [1] wurde bereits darüber berichtet, daß man die oben erwähnte algebraische Auffassung von Ditkin unter Beachtung von [12] auch durch eine asymptotische Auffassung ersetzen kann. Hierzu sollen jetzt nähere Einzelheiten entwickelt werden, wobei zugleich einige Vereinfachungen und Verbesserungen gegenüber von [1] vorgenommen werden, die man zum Teil auch in dem Buch [2] wiederfinden wird. Verschiedene Anregungen und Hinweise verdankt der Verfasser Herrn J. Mikusiński.

1. Die Grundlagen. Wie in [1] wollen wir hier nicht das Faltungsintegral, sondern in Anlehnung an M. Rajewski [10] (vgl. auch [5a]) das Duhamel-Integral

$$fg = \frac{d}{dt}\int_0^t f(t-\tau)g(\tau)\,d\tau \tag{1}$$

als Grundlage der Operatorenrechnung wählen. Die Menge der für $t \geqslant 0$ einmal stetig differenzierbaren Funktionen bildet dann mit der gewöhnlichen Addition und der Funktionenmultiplikation (1) einen kommutativen Ring $\mathfrak{F}$. Dabei schreiben wir die Funktionenmultiplikation (1) zur Unterscheidung von der gewöhnlichen Wertemultiplikation $f(t)g(t)$ entweder ohne Argument oder mit Malpunkt

$$fg = f(t)g = fg(t) = f(t)\cdot g(t)$$

und bezeichnen Funktionenpotenzen mit $f^n = f^{\cdot n}(t)$. Jeder Funktion