derivatives of functions belonging to $\mathscr{L}_M^*$ may also be obtained by applying [3], II. Indeed, by [3], 2.21 and 2.23, every $T\in\mathscr{D}'_M$ may be represented in the form $T=\sum_1^k T_j$, where T_j is linear with respect to the pseudonorm $\|D^{p^j}\varphi\|_N$. Hence $T_j(\varphi)=\int f_j(x)D^{p^j}\varphi(x)\,dx=f_j(D^{p^j}\varphi)$ $=(-1)^{|p^j|}D^{p^j}f_j(\varphi)$ with $f_j\in\mathscr{L}_M^*$; thus $T=\sum_1^k(-1)^{|p^j|}D^{p^j}f_j$.

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STUDIA MATHEMATICA, T. XXI. (1962)

On the radicals of p-normed algebras

by

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A *p-normed algebra* is a complete metric algebra in which the metric is introduced by means of a p-homogeneous submultiplicative norm, i. e. such a norm $\|x\|$ that

$$\|xy\|\leqslant\|x\|\,\|y\|, \tag{1}$$

$$\|ax\|=|a|^p\|x\|, \tag{2}$$

where x,y are elements of the algebra in question, a is a real or complex scalar and p is a fixed real number satisfying $0<p\leqslant 1$. For every locally bounded complete metric algebra there exists an equivalent metric introduced by a norm satisfying (1) and (2). The theory of commutative complete locally bounded algebras is developed in paper [2]. The present paper is a continuation of [2]. We give here a solution of the following problem 1 of [2]: "Is the radical of a commutative p-normed algebra R characterized by the relation

$$\operatorname{rad}R=\{x\in R\colon \|x\|_s=0\}?"$$

Here

$$\|x\|_s=\lim\sqrt[n]{\|x^n\|} \tag{3}$$

denotes the spectral norm in R (see [2], definition 1 and theorem 4). We shall show that the answer is in the affirmative. It is based upon the following

THEOREM 1. *Let R be a commutative complex p-normed algebra. Then the unit sphere of the spectral norm*

$$K=\{x\in R\colon \|x\|_s\leqslant 1\}$$

is a convex subset of R.

Proof. By theorem 4 of [2], property S7, K is a closed subset of R; consequently it is sufficient to prove that $\|x\|_s\leqslant 1$ and $\|y\|_s\leqslant 1$ imply $\|(x+y)/2\|_s\leqslant 1$. It may easily be seen that it is sufficient to prove that $\|x\|_s < 1$ and $\|y_s\| < 1$ imply $\|(x+y)/2\|_s \leqslant 1$; otherwise if $\|x\|_s \leqslant 1$, $\|y\|_s \leqslant 1$ and $\|(x+y)/2\|_s > 1$, and taking a suitable scalar a we should obtain $\|ax\|_s < 1$, $\|ay\|_s < 1$ and $\|(ax+ay)/2\|_s > 1$. Assume then that $\|x\|_s < 1$ and $\|y\|_s < 1$. By formulae (1) and (2) we have $\|(x+y)/2\|_s = (\frac{1}{2})^p \lim \sqrt[n]{\|(x+y)^n\|} \leqslant (\frac{1}{2})^p \lim \sqrt[n]{\sum_{k=0}^{n} \binom{n}{k}^p \|x^k y^{n-k}\|}$. Now by the same arguments as in [2] (proof of the subadditivity of $\|x\|_s$ in theorem 4) we may prove that for sufficiently large n, say $n > N$, we have

$$\|x^k y^{n-k}\| < 1 \quad \text{for} \quad k = 1, 2, \ldots, n.$$

Hence

$$\left\|\frac{x+y}{2}\right\|_s \leqslant (\tfrac{1}{2})^p \lim \sqrt[2n]{\sum_{k=0}^{n} \binom{2n}{k}^p} = (\tfrac{1}{2})^p \lim \sqrt[2n]{(2n+1)\binom{2n}{n}^p} = (\tfrac{1}{2})^p \left(\lim \sqrt[2n]{\binom{2n}{n}}\right)^p = 1.$$

Thus

$$\left\|\frac{x+y}{2}\right\|_s \leqslant 1, \quad \text{q. e. d.}$$

COROLLARY 1. *The p-homogeneous spectral norm $\|x\|_s$ of a p-normed algebra is equivalent to a homogeneous norm $\|x\|_s'$. The homogeneous spectral norm is given by*

$$\|x\|_s' = (\|x\|_s)^{1/p}.$$

COROLLARY 2. *Every p-normed complex field is the field of complex numbers.*

In fact, a p-normed field equipped with the homogeneous spectral norm is a normed field. This is another proof of theorem 6 of [2].

THEOREM 2. *Let R be a commutative p-normed algebra and let $\mathfrak{M}$ be the set of all multiplicative linear functionals defined on R; then*

$$\|x\|_s = \sup_{f \in \mathfrak{M}} |f(x)|^p, \tag{4}$$

or, which is equivalent,

$$\|x\|_s' = \sup_{f \in \mathfrak{M}} |f(x)|. \tag{4'}$$

Proof. By theorem 9 of [2] every multiplicative linear functional defined on R is continuous with respect to the norm $\|x\|_s'$. The set $I = \{x \in R: \|x\|_s' = 0\}$ is either R itself, in which case the proof is obvious, or is a closed ideal in R. Every functional $f \in \mathfrak{M}$ is constant on cosets of R/I. Hence for $X \in R/I$ we may define

$$\hat{f}(X) = f(x), \tag{5}$$

where $x \in X$ and $f \in \mathfrak{M}$. It is easy to see that $\hat{f}$ is a multiplicative linear functional defined on R/I. Moreover, $\hat{f}$ is continuous with respect to the norm $\|X\| = \inf_{x \in X} \|x\|_s'$, and every multiplicative linear functional defined on R/I is of form (5). It is also evident that

$$\sup_{\hat{f}} |\hat{f}(X)| = \sup_{f} |f(x)| \tag{6}$$

for every $X \in R/I$ and every $x \in X$. But R/I is a normed algebra with the norm $\|X\|$. Hence, by [1], Satz 8 (see also theorem 2′ below) we have

$$\sup_{f \in \mathfrak{M}} |f(X)| = \lim_n \sqrt[n]{\|X^n\|} \tag{7}$$

and the desired conclusion follows from (6), (7), and from the fact that $\|X^n\| = \|x^n\|_s' = (\|x\|_s')^n$ for every $X \in R/I$ and every $x \in X$, q. e. d.

COROLLARY 3. *For every commutative p-normed algebra R we have* $\operatorname{rad} R = \{x \in R: \|x\|_s = 0\}$.

Remark 1. For p-normed algebras we cannot prove all the theorems proved in [1] for the Banach algebras. The constructions in Gelfand's paper [1] are based upon the Riemann integral. The Riemann integral cannot be applied to spaces which are not locally convex. Consequently there arise some difficulties in introducing analytic functions in locally bounded algebras. So we pose the following

Problem. Let R be a commutative p-normed algebra. Let $a \in R$; we define the spectrum of a as $\sigma(a) = \{f(a): f \in \mathfrak{M}\}$ where $\mathfrak{M}$ is the set of all multiplicative linear functionals defined on R. Let $a \in R$ and F be a holomorphic function defined on an open set U containing $\sigma(a)$. Does there exist an element $x \in R$ such that $F(f(a)) = f(x)$ for every $f \in \mathfrak{M}$? (1)

Remark 2. Our proof of theorem 2 reduces to the case where R is a normed algebra. Gelfand's original proof in this case makes use of the notion of analytic functions with valuas from Banach algebras. Now we shall give an alternative proof of Gelfand's theorem without using these notions.

THEOREM 2′. *Let R be a commutative Banach algebra with the unit e. Then*

$$\max_{f \in \mathfrak{M}} |f(x)| = \lim \sqrt[n]{\|x^n\|},$$

where $\mathfrak{M}$ is the set of all multiplicative linear functionals defined on R.

(1) The answer is positive. The proof will appear in [3] *(added in proof)*.

Proof. We have

1. $|f(x)| \leqslant \|x\|$ for every $x \in R$, $f \in \mathfrak{M}$, whence $|f(x)| = \sqrt[n]{|f(x^n)|} \leqslant \sqrt[n]{\|x^n\|}$, and $\max |f(x)| \leqslant \underline{\lim} \sqrt[n]{\|x^n\|}$.

2. Let F be a linear functional defined on R; then $g_F(z) = F((e-zx)^{-1})$ is a holomorphic function defined for $|z| < 1/a$, where $a = \max_{f \in \mathfrak{M}} |f(x)|$, and x is a fixed element of R. It may easily be seen that $g_F(z) = \sum_{n=0}^{\infty} F(x^n) z^n = \sum_{n=0}^{\infty} F(x^n z^n)$. Consequently the sequence $x^n z^n$ is bounded, being weakly convergent to 0. We have $\|x^n z^n\| \leqslant M_z$, and $\sqrt[n]{\|x^n\|} \leqslant \sqrt[n]{M_z}/|z|$. Consequently $\overline{\lim} \sqrt[n]{\|x^n\|} \leqslant 1/|z|$ for every $|z| < 1/a$, and $\overline{\lim} \sqrt[n]{\|x^n\|} \leqslant a$.

By 1 and 2 we have $a = \lim \sqrt[n]{\|x^n\|}$, q. e. d.

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On Banach *-semialgebras

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1. Preliminaries

We shall use the term *halfring* in the sense given by the author in a previous paper [3]. We repeat that a *halfring* is a semiring which is embeddable in a ring. Since addition in our semiring S is commutative, a necessary and sufficient condition for S to be a halfring is that the additive semigroup of S be cancellative. Following [3], we construct the ring in which H is embedded. The product set $H \times H$ again forms a halfring according to the laws of addition and multiplication: $(s_1, s_2) + (t_1, t_2) = (s_1+t_1, s_2+t_2)$, $(s_1, s_2)(t_1, t_2) = (s_1t_1+s_2t_2,\ s_1t_2+s_2t_1)$. The diagonal $\Delta = \{(x, x) \mid \epsilon H\}$ is an ideal in $H \times H$. We say that $(s_1, s_2) \equiv (u_1, u_2)(\Delta)$ if and only if there exist elements (x, x) and (y, y) in Δ such that $(s_1, s_2) + (x, x) = (t_1, t_2) + (y, y)$. The quotient ring $\mathfrak{R} = H \times H/\Delta$ is called the *ring generated by H*. Let ν denote the natural homomorphism of $H \times H$ onto $\mathfrak{R}$, then the halfring H is embedded in the ring $\mathfrak{R}$, for the mapping $h \leftrightarrow \nu(h+a, a)$, for any a, is an isomorphism of H into $\mathfrak{R}$. We designate by $\nu(H)$ this isomorphic map of H in $\mathfrak{R}$ and by $\nu(s_1, s_2)$ the equivalence class of (s_1, s_2). A *division semiring* is a semiring, in which the elements $\neq 0$, form a multiplicative group. A semifield is a commutative division semiring. A halffield is a semifield which is embeddable in a field.

In a recent paper [4], we introduced the concept of a *normed semialgebra*. For the sake of completeness we repeat:

Definition 1. A semiring S is said to be a *semialgebra over a commutative semiring Σ with unit*, if a law of composition $(\sigma, s) = \sigma s$ of the product $\Sigma \times S$ is defined such that

(i) $(S, +)$ is a unital left Σ-semimodule relative to the composition $(\sigma, s) = \sigma s$,

(ii) for all $\sigma \in \Sigma$ and $s, t \in S$, $\sigma(s, t) = (\sigma s)t = s(\sigma t)$.

Definition 2. A *semivector space* is a semialgebra over a semifield.

Definition 3. A metric for a semilinear space S is said to be *invariant* if and only if $d(s+x, t+x) = d(s, t)$ for all $s, t, x \in S$.