and thus (5.1) holds with $n = m+1$ if we set $t_{m+1} = 0$. Hence there exists a sequence $\{s_n\}$ such that (5.1) is satisfied for $n = 1, 2, \ldots$, and $t_n = 0$ for $n \geqslant 4$.

Let H be the set of all normal conservative matrices which sum no bounded divergent sequence. Since $G \subset H$, and since, by Theorem 11, $(\bar{G}-G) \cap H$ is void, it follows that $(\bar{G}-G) \subset (\bar{H}-H)$. The reverse inclusion is false, for the matrix A in the example given above satisfies $A \in \bar{H}-H$, since $A+\lambda I$ sums no bounded divergent sequence when $|\lambda-1| < 1$. To see this, consider the matrix obtained from $A+\lambda I$ by deleting the first two rows; this is of the form $100(I+B)$, where $\|B\|^* < 1$ if $|\lambda-1| < 1$, and hence it has a two-sided conservative reciprocal, and belongs to H. This does not contradict the example given above, for $A+\lambda I$ has no conservative right reciprocal for small λ.

These considerations suggest the problem of determining whether the condition $A \in \bar{H}-H$, which is a sufficient condition for a normal conservative matrix A to sum a bounded divergent sequence, is also necessary.

Bibliography

[1] R. G. Cooke, *Infinite matrices and sequence spaces*, London 1950.

[2] J. Copping, *K-matrices which sum no bounded divergent sequence*, Journal London M. S. 30 (1955), p. 123-127.

[3] — *Conditions for a K-matrix to evaluate some bounded divergent sequences*, ibidem 32 (1957), p. 217-227.

[4] — *Inclusion theorems for conservative summation methods*, Konikl. Nederl Akademie van Wetenschappen A, 61 (1958), p. 485-499.

[5] S. Mazur et W. Orlicz, *Sur les espaces métriques linéaires (I)*, Studia Math 10 (1948), p. 184-208.

[6] — *On linear methods of summability*, ibidem 14 (1954), p. 129-160.

[7] A. Wilansky and K. Zeller, *The inverse matrix in summability: reversible methods*, Journal London M. S. 32 (1957), p. 397-408.

[8] — *Banach algebra and summability*, Illinois Journal of Math. 2 (1958), p. 378-385.

[9] L. Włodarski, *Sur les méthodes continues de limitation (I)*, Studia Math. 14 (1954), p. 161-187.

[10] K. Zeller, *Faktorfolgen bei Limitierungsverfahren*, Math. Zeitschrift 56 (1952), p. 134-151.

[11] — *Review of* [4], Math. Reviews 20 (1959), p. 989.

Reçu par la Rédaction le 17. 2. 1961

STUDIA MATHEMATICA, T. XXI. (1962)

On some spaces of functions and distributions (I)

Spaces $\mathscr{D}_M$ and $\mathscr{D}'_M$

by

J. MUSIELAK (Poznań)

1. Definitions. L. Schwartz introduced in [5] the spaces $\mathscr{D}_{L^p}$ of functions and $\mathscr{D}'_{L^p}$ of distributions. The purpose of this paper is to investigate some properties of spaces $\mathscr{D}_M$ and $\mathscr{D}'_M$, the spaces $\mathscr{L}^p$ being replaced by Orlicz spaces $\mathscr{L}^*_M$. We adopt here the notations of [5] with the only exception that multiple integrals will be denoted by a single sign of integral. Further, $M(u)$ will denote an even continuous function which vanishes only at 0 and is convex for positive u. Moreover, we assume for simplicity $u^{-1}M(u) \to 0$ as $u \to 0$ and $u^{-1}M(u) \to \infty$ as $u \to \infty$. $M_{-1}(u)$ will mean the inverse function of $M(u)$ for $u \geqslant 0$, $M_{-1}(u) = M_{-1}(-u)$ for $u < 0$; $N(u)$ will mean the function complementary to $M(u)$ in the sense of Young. Let $\varrho_M(\varphi) = \int M(\varphi(x))\,dx$, where the integral is taken over the whole n-dimensional space. Then

$$\mathscr{L}^*_M = \{\varphi \text{ measurable: } \varrho_M(k\varphi) < \infty \text{ for a certain } k > 0\},$$

with the norm $\|\varphi\|_M = \inf\{\varepsilon > 0 : \varrho_M(\varphi/\varepsilon) \leqslant 1\}$ is a B-space, called an Orlicz space (cf. e. g. [2]).

We write

$$\mathscr{D}_M = \{\varphi \in \mathscr{E} : D^p\varphi \in \mathscr{L}^*_M \text{ for every } p\}.$$

The system of sets

$$U(m, \varepsilon) = \{\varphi \in \mathscr{D}_M : \|D^p\varphi\|_M \leqslant \varepsilon \quad \text{for} \quad |p| \leqslant m\},$$

$m = 0, 1, 2, \ldots$, $\varepsilon > 0$, being assumed to be a fundamental system of neighbourhoods of zero, $\mathscr{D}_M$ becomes a locally convex linear topological space and the topology is equivalent to that induced by the F-norm

$$\|\varphi\|^M = \sum_{m=0}^{\infty} \frac{m!}{2^{m+1}n^m} \sum_{|p|=m} \frac{1}{p!} \frac{\|D^p\varphi\|_M}{1+\|D^p\varphi\|_M},$$

where n is the number of variables. So $\mathscr{D}_M$ is a B_0-space (cf. [1], [3]); the completeness of $\mathscr{D}_M$ follows from 2.1.

If $\mathscr{X}$ and $\mathscr{Y}$ are two linear topological spaces, we shall write $\mathscr{X} \subset \mathscr{Y}$ to denote that $\mathscr{X}$ is a part of $\mathscr{Y}$; $\mathscr{X} \dot{\subset} \mathscr{Y}$ will mean that $\mathscr{X}$ as a linear space is isomorphic with a linear subspace $\mathscr{X}_1$ of $\mathscr{Y}$ and the topology induced in $\mathscr{X}_1$ by $\mathscr{X}$ is stronger than the topology induced in $\mathscr{X}_1$ by $\mathscr{Y}$, i. e. each neighbourhood of zero in $\mathscr{Y}$ contains a neighbourhood of zero in $\mathscr{X}_1$ in the topology induced by $\mathscr{X}$. Moreover, we shall denote by $\mathscr{D}'_M$ the strong dual of $\mathscr{D}_N$.

2. Theorems. The following theorems hold:

2.1. (a) *If* $\varphi \in \mathscr{D}_M$, *then* $D^p \varphi(x) \to 0$ *as* $|x| \to \infty$ *for every* p.

(b) *If* $\varphi_k \to 0$ *in* $\mathscr{D}_M$, *then* $\varphi_k(x)$ *are uniformly bounded and* $\varphi_k(x) \to 0$ *as* $|x| \to \infty$ *uniformly in* k.

2.2. (a) *If* $M_2(u) = O(M_1(u))$ *as* $u \to 0$, *then* $\mathscr{D}_{M_1} \dot{\subset} \mathscr{D}_{M_2}$; $\mathscr{D}_{L^1} \dot{\subset} \mathscr{D}_M \dot{\subset} \mathscr{B}$ *and* $\mathscr{D}_M \dot{\subset} \mathscr{L}^*_M$ *for every* $M(u)$.

(b) *If* $M(u)$ *satisfies the condition* (Δ_2) *for small* u, *then* $\mathscr{D}_M = \{\varphi \in \mathscr{E}: \varrho_M(D^p(k\varphi)) < \infty$ *for every* p *and every* $k > 0\}$.

(c) $\mathscr{D} \dot{\subset} \mathscr{D}_M$; *moreover, denoting for* $a > 0$, $q = 1, 2, \ldots$,

$$a_{a,q}(x) = \begin{cases} \exp[-|x|^{2q}(a^{2q}-|x|^{2q})^{-1}] & \text{for } |x| < a, \\ 0 & \text{for } |x| \geqslant a, \end{cases}$$

$$\varphi_i(x) = a_{s_i,i}(x)\varphi(x) \quad \text{for} \quad \varphi \in \mathscr{D}_M \text{ and suitable } s_i,$$

we have $a_{a,q} \in \mathscr{D}$, $\varphi_i \in \mathscr{D}$, *and assuming that* $M(u)$ *satisfies* (Δ_2) *for small* u, $\varphi_i \to \varphi$ *in* $\mathscr{D}_M$, *hence* $\mathscr{D}$ *is dense in* $\mathscr{D}_M$. (*The relation* $\varphi_i \to \varphi$ *holds also in* $\mathscr{B}$.)

2.3. (a) *If* $N_2(u) = O(N_1(u))$ *as* $u \to 0$, *then* $\mathscr{D}'_{M_1} \dot{\subset} \mathscr{D}'_{M_2}$.

(b) $\mathscr{L}^*_M \dot{\subset} \mathscr{D}'_M \dot{\subset} \mathscr{D}'$, $\mathscr{D}'_M \dot{\subset} \mathscr{B}'$.

(c) *If* $N(u)$ *satisfies* (Δ_2) *for all* u *and* $T \in \mathscr{D}'_M$, *then* $T * a \in \mathscr{L}^*_M$ *for every* $a \in \mathscr{D}$.

(d) *Under the same assumptions as in* (c), *distributions* $T \in \mathscr{D}'_M$ *are exactly finite sums of derivatives of functions from the space* $\mathscr{L}^*_M$, *and* $\mathscr{D}'_M = \{T \in \mathscr{D}': T * a \in \mathscr{L}^*_M$ *for every* $a \in \mathscr{D}\}$.

3. Proofs. First, we prove the following lemma:

3.11. *Denote by* P *the set of all systems* $p = (p_1, p_2, \ldots, p_n)$, *where* $p_i = 0, 1$ *and not all* p_i *are equal to* 0. *Let* $p_{a_1} = \ldots = p_{a_k} = 1$, $p_i = 0$ *for* $i \neq a_j$, $j = 1, 2, \ldots, k$, *and write for an* $x^0 = (x^0_1, \ldots, x^0_n)$,

$$D^p\varphi(x^0, u) = D^p\varphi(x^0_1, \ldots, x^0_{a_1-1}, u_{a_1}, x^0_{a_1+1}, \ldots, x^0_{a_k-1}, u_{a_k}, x^0_{a_k+1}, \ldots, x^0_n).$$

Assume $\varphi \in \mathscr{E}$ *and*

$$(*) \qquad \int M[D^p\varphi(x^0, u)]\, du_{a_1} \ldots du_{a_k} \leqslant I \quad \text{for all } p \in P,$$

where the integral is taken over a k*-dimensional space. Let* $x = (x_1, \ldots, x_n)$ *and* $x' = (x'_1, \ldots, x'_n)$ *be two arbitrary points and let* $\max_i |x_i - x'_i| \leqslant b$, $\max_i(|x_i - x^0_i|, |x'_i - x^0_i|, 1) \leqslant R$. *Then*

$$\varphi(x) - \varphi(x') \leqslant I \sum_{i=1}^{n} \binom{n}{i} 2^{n-i} \frac{M_{-1}(IR^{-n+i}b^{-i})}{IR^{-n+i}b^{-i}}.$$

In fact, we have

$$(\overset{*}{*}) \qquad \varphi(x) - \varphi(x') = \sum_{p \in P} \int_{x'_{a_1}}^{x_{a_1}} \ldots \int_{x'_{a_k}}^{x_{a_k}} D^p\varphi(x', u)\, du_{a_1} \ldots du_{a_k}.$$

For a fixed $p = (p_1, \ldots, p_n)$, we apply identity $(\overset{*}{*})$ to the function $\int_{x'_{a_1}}^{x_{a_1}} \ldots \int_{x'_{a_k}}^{x_{a_k}} D^p\varphi(x', u)\, du_{a_1} \ldots du_{a_k}$ and the points x', x^0 in place of $\varphi(x)$ and the points x, x'. We easily obtain

$$\int_{x'_{a_1}}^{x_{a_1}} \ldots \int_{x'_{a_k}}^{x_{a_k}} D^p\varphi(x', u)\, du_{a_1} \ldots du_{a_k} = \sum_{q \in Q} \int_{x^0_{\gamma_1}}^{x'_{\gamma_1}} \ldots \int_{x^0_{\gamma_l}}^{x'_{\gamma_l}} D^p\varphi(x^0, u)\, du_{\gamma_1} \ldots du_{\gamma_l},$$

where Q is the set of all systems $q \in P$, $q_{\gamma_j} = 1$ for $j = 1, 2, \ldots, l$, $q_i = 0$ for $i \neq \gamma_j$, such that $p_i = 1$ implies $q_i = 1$. Applying Jessen's inequality

$$\int_A f(x)\, dx \leqslant \mu(A) M_{-1}\left[\frac{1}{\mu(A)} \int_A M(f(x))\, dx\right],$$

$\mu(A)$ being the measure of A, to the integrals on the right-hand side of the last identity, substituted into $(\overset{*}{*})$, and taking into account the fact that $u^{-1}M_{-1}(u)$ is decreasing, we obtain the required inequality.

3.1. Now we give the proof of 2.1(a). Obviously, it is sufficient to prove $\varphi(x) \to 0$ as $|x| \to \infty$. Suppose it is not true, i. e. there is an $\varepsilon > 0$ and a sequence x^k such that $|x^k|$ increases to infinity and $\varphi(x^k) > \varepsilon$. Evidently, we may assume $I_0 = \max_{|p| \leqslant n} \varrho_M(D^p\varphi) < \infty$. Let V_n be the volume of the unit sphere in the n-dimensional space. The following lemma will be of importance:

For every $I > I_0$ *and for every point* x *there exists a point* x^0 *in the sphere* K *with centre in* x *and with radius* $R_0 = 2^{2n}V_n^{-1}$ *such that inequality* $(*)$ *holds for all* $p \in P$.

If $p = (1, \ldots, 1)$, the lemma is obvious. Assuming $p \neq (1, \ldots, 1)$

we prove it indirectly. Suppose that for a certain x every point $x^0 \in K$ satisfies the converse inequality

$$(**) \qquad \int M[D^p \varphi(x^0, u)]\, du_{a_1} \ldots du_{a_k} > I$$

for a $p \in P$, $p \neq (1, \ldots, 1)$. Given $p \in P$, write $A_p = \{x^0 \in K: (**) \text{ is satisfied}\}$. Obviously A_p are measurable and the measure $\mu(A_p)$ of at least one of these sets, say A_{p^0}, is greater than $V_n R_0^n 2^{-n}$. Let $p^0 = (p_1, \ldots, p_n)$, $p_{a_i} = 1$ for $i = 1, 2, \ldots, k$ and let $\beta_1, \ldots, \beta_{n-k}$ denote all indices among $1, 2, \ldots, n$ which are $\neq a_i$ for $i = 1, 2, \ldots, k$. Let A'_{p^0} be the projection of the set A_{p^0} on the $(n-k)$-dimensional space of points $(x_{\beta_1}, \ldots, x_{\beta_{n-k}})$. It is easily seen that $\mu(A'_{p^0}) > (2R_0)^{-k} \mu(A_{p^0})$, whence $\mu(A'_{p^0}) > 2^{-2n} V_n R_0^{n-k}$. But

$$I_0 \geqslant \int M[D^{p^0}\varphi(u)]\, du \geqslant \int\limits_{A'_{p^0}} \Big\{\int M[D^{p^0}\varphi(u)]\, du_{a_1} \ldots du_{a_k}\Big\} du_{\beta_1} \ldots du_{\beta_{n-k}}$$
$$\geqslant I\mu(A'_{p^0}) > I \cdot 2^{-2n} V_n R_0^{n-k}.$$

On the other hand, $R_0 = 2^{2n} V_n^{-1} > (2^{2n} I_0 I^{-1} V_n^{-1})^{1/(n-k)}$, whence $I_0 < I \cdot 2^{-2n} V_n R_0^{n-k}$ — a contradiction. Thus the lemma is proved.

Now we may apply 3.11 to any point x and x'. Let K^k be the greatest sphere in the n-dimensional space with centre in x^k and with radius r_k such that $\varphi(x) \geqslant \frac{1}{2}\varepsilon$ for $x \in K^k$. Obviously, we may suppose the spheres K^k to be disjoint, for

$$I_0 \geqslant \varrho_M(\varphi) \geqslant \int\limits_{\bigcup_1^\infty K^k} M(\varphi(x))\, dx \geqslant M(\tfrac{1}{2}\varepsilon)\, \mu\Big(\bigcup_1^\infty K^k\Big);$$

hence $V_n \sum_1^\infty r_k^n = \mu\big(\bigcup_1^\infty K^k\big) \leqslant I_0 M^{-1}(\frac{1}{2}\varepsilon) < \infty$, i. e. $r_k \to 0$.

It is clear that there is a point x'^k on the boundary of K^k such that $\varphi(x'^k) = \frac{1}{2}\varepsilon$. Further, let x^{0k} be a point corresponding to x^k such that 3.11 is satisfied and let $|x_i^k - x_i^{0k}| \leqslant R_0$, $i = 1, 2, \ldots, n$. Then $|x_i'^k - x_i^{0k}| \leqslant 2R_0$ for sufficiently large k and writing $R = \max(R_0, 1)$, we may apply 3.11 to x^k, x'^k, x^{0k} and $b = r_k$. We obtain

$$\tfrac{1}{2}\varepsilon \leqslant \varphi(x^k) - \varphi(x'^k) \leqslant I \sum_{i=1}^n \binom{n}{i} 2^{n-i} \frac{M_{-1}(IR^{-n+i} r_k^{-i})}{IR^{-n+i} r_k^{-i}}.$$

Since $u^{-1} M_{-1}(u) \to 0$ as $u \to \infty$, we get $\varepsilon \leqslant 0$ — a contradiction.

The proof of 2.1(b) will also be performed by reductio ad absurdum. Suppose there is a sequence x^k, $|x^k|$ increasing to infinity, and $\varphi_k(x^k) \geqslant \varepsilon > 0$. We adopt the notation of 3.1 with the exception that K^k will denote here the greatest sphere with centre at x^k such that $\varphi_k(x) \geqslant$

$\geqslant \frac{1}{2}\varepsilon$; let $I_0^{(k)} = \max_{|p| \leqslant n} \varrho_M(D^p \varphi_k)$. Then we have $r_k < \varepsilon$ and by 3.11, assuming $I_1 = I_0 = I_0^{(k)}$, we have

$$\tfrac{1}{2}\varepsilon \leqslant I_0^{(k)} \sum_{i=1}^n \binom{n}{i} 2^{n-i} \frac{M_{-1}(I_0^{(k)} R^{-n+i} \varepsilon^{-i})}{I_0^{(k)} R^{-n+i} \varepsilon^{-i}}$$
$$\leqslant 2^{2n-1} R^n \varepsilon M_{-1}(I_0^{(k)} R^{-n} \varepsilon^{-1}),$$

whence $I_0^{(k)} \geqslant R^n \varepsilon M(2^{-2n} R^{-n})$, in contradiction to the fact that $I_0^{(k)} \to 0$ as $k \to \infty$. Hence $\varphi_k(x) \to 0$ as $|x| \to \infty$ uniformly in k.

The proof that $\varphi_k(x)$ are uniformly bounded is similar.

3.2. Proof of 2.2 (a). By the assumption, there exist K, $u_0 > 0$ such that $M_2(u) \leqslant K M_1(u)$ for $0 \leqslant u \leqslant u_0$. Let $\varphi \in \mathscr{D}_{M_1}$; then by 2.1 (a), p being fixed, for any $\varepsilon > 0$ an $r > 0$ may be chosen so that $|D^p \varphi(x)| < \varepsilon$ for $|x| > r$. Write $A = \{x: |D^p \varphi(x)| > u_0 \varkappa_p^{-1}\}$, where $\varkappa_p > 0$ is chosen so that $\varrho_{M_1}(\varkappa_p D^p \varphi) < \infty$. By 2.1(a), A is bounded; we have

$$\varrho_{M_2}(\varkappa_p D^p \varphi) \leqslant K \varrho_{M_1}(\varkappa_p D^p \varphi) + \mu(A) \sup_{x \in A} M_2[\varkappa_p D^p \varphi(x)] < \infty,$$

whence $\varphi \in \mathscr{D}_{M_2}$. Thus we have proved that $\mathscr{D}_{M_1} \subset \mathscr{D}_{M_2}$. Now assume $\varphi_k \to 0$ in $\mathscr{D}_{M_1}$. Given any p and $\varkappa > 0$, put $A_k = \{x: |D^p \varphi_k(x)| > u_0 \varkappa^{-1}\}$. Then

$$\varrho_{M_2}(\varkappa D^p \varphi_k) \leqslant K \varrho_{M_1}(\varkappa D^p \varphi_k) + \mu(A_k) \sup_{x \in A_k} M_2[\varkappa D^p \varphi_k(x)].$$

By the assumption, $\varrho_{M_1}(\varkappa D^p \varphi_k) \to 0$ as $k \to \infty$ for any $\varkappa > 0$; by 2.1(b), the sets A_k are all contained in a compact and $\mu(A_k) \to 0$ as $k \to \infty$, since convergence in mean of $M_1[\varkappa D^p \varphi_k(x)]$ implies convergence in measure. Hence $\varrho_{M_2}(\varkappa D^p \varphi_k) \to 0$ as $k \to \infty$ for every $\varkappa > 0$, i. e. $\varphi_k \to 0$ in $\mathscr{D}_{M_2}$. Thus $\mathscr{D}_{M_1} \subset \mathscr{D}_{M_2}$.

The proofs of other parts of 2.2(a) and of 2.2(b) will be omitted here, being trivial. We proceed at once to the proof of the second part of 2.2(c). We shall base ourselves on the following lemma:

3.21. *The functions $a_{a,q}(x)$ being defined as in 2.2 (c), we have $a_{a,q} \in \mathscr{D}$. Moreover, given any $\eta > 0$, $R > 0$ and p, where $1 \leqslant |p| < 2q$, there exists a number a_0 such that $|a_{a,q}(x) - 1| \leqslant \eta$ and $|D^p a_{a,q}(x)| \leqslant \eta$ for $|x| \leqslant R$ and $a > a_0$. Here the constant $a_0 = \max[R(1+\eta^{-1})^{1/2q}$, $8R|p| a_{|p|} \eta^{-1} (5q^2)^{|p|} b_{|p|}, 1]$, where $a_1 = b_1 = 1$, $a_m = \prod_{\nu=2}^m [(2\nu-3)^2 n + \nu]$, $b_m = \prod_{\nu=2}^m [\frac{3}{2}(\nu+1)^2 + 1]$ for $m > 1$.*

This lemma follows essentially from the formula

$$(**) \qquad D^p a_{a,q}(x) = \frac{1}{a^{|p|}} \sum_{\nu=1}^{|p|} W_\nu\Big(\frac{x_1}{|x|}, \ldots, \frac{x_n}{|x|}\Big) \frac{a_{1,q}^{(\nu)}(|x| a^{-1})}{(|x| a^{-1})^{|p|-\nu}}$$

for $|x| \leqslant a$, where $W_\nu(\xi_1, \ldots, \xi_n)$ are polynomials independent of a (for the details of the proof cf. [4]).

Now take $\varphi \in \mathscr{D}_M$ and any $k > 0$ and write $\varphi_i(x) = \alpha_{s_i,i}(x)\varphi(x)$. By 2.2 (b), $\varrho_M(2k\varphi) < \infty$, whence for any $\varepsilon > 0$ there is an $A_0 > 0$ such that

$$\int_{|x|>A_0} M\big(2k\varphi(x)\big)\,dx < \tfrac{1}{2}\varepsilon.$$

Choose $\eta > 0$ so that $\varrho_M(k\eta\varphi) < \frac{1}{2}\varepsilon$. Applying 3.21 with $R = A_0$ we then have $|\alpha_{s_i,i}(x)-1| \leqslant \eta$ for $|x| \leqslant A_0$ and $i > a_0$, whence

$$\varrho_M\big(k(\varphi_i-\varphi)\big) \leqslant \varrho_M(k\eta\varphi) + \int_{|x|>A_0} M\big(2k\varphi(x)\big)\,dx < \varepsilon,$$

i. e. $\varrho_M\big(k(\varphi_i-\varphi)\big) \to 0$.

Now take an arbitrary p. We shall show that $D^p\left(\varphi\dfrac{\partial\alpha_{s_i,i}}{\partial x_j}\right) \to 0$ in $\mathscr{L}_M^*$ as $i\to\infty$ for every j. Since

$$D^p\left(\varphi\frac{\partial\alpha_{s_i,i}}{\partial x_j}\right) = \sum_{\nu_1=0}^{p_1}\cdots\sum_{\nu_n=0}^{p_n}\binom{p_1}{\nu_1}\cdots\binom{p_n}{\nu_n} D^{p(\nu)}(\varphi)\,D^{p'(\nu)}(\alpha_{s_i,i}),$$

where $p = (p_1, \ldots, p_n)$, $p(\nu) = (p_1-\nu_1, \ldots, p_n-\nu_n)$, $p'(\nu) = (\nu_1, \ldots, \nu_{j-1}, \nu_j+1, \nu_{j+1}, \ldots, \nu_n)$, it is sufficient to prove that the terms of this sum tend to 0 in $\mathscr{L}_M^*$, i. e. that $\psi D^p\alpha_{s_i,i} \to 0$ in $\mathscr{L}_M^*$ for an arbitrary $\psi \in \mathscr{D}_M$. Write $m_i = \max_x |D^p\alpha_{s_i,i}(x)|$. Applying (**) we can easily see that $m = \sup_i m_i < \infty$. Given $\varepsilon, k > 0$, let $A_1 > 0$ and $\eta > 0$ be such that $\int_{|x|>A_1} M\big(km\psi(x)\big)\,dx < \frac{1}{2}\varepsilon$ and $\varrho_M(k\eta\psi) < \frac{1}{2}\varepsilon$. Applying 3.21 with $R = A_1$, we get $|D^p\alpha_{s_i,i}(x)| \leqslant \eta$ for $|x| \leqslant A_1$ and $s_i \geqslant \max(a_0, \frac{1}{2}|p|)$, whence

$$\varrho_M\big(kD^p(\alpha_{s_i,i})\big) \leqslant \varrho_M(k\eta\psi) + \int_{|x|>A_1} M\big(km\psi(x)\big)\,dx < \varepsilon,$$

i. e. $\psi D^p(\alpha_{s_i,i}) \to 0$ as $i\to\infty$. Thus for any p and any index j, $D^p\left(\varphi\dfrac{\partial\alpha_{s_i,i}}{\partial x_j}\right) \to 0$ in $\mathscr{L}_M^*$ as $i\to\infty$.

Now let m be an arbitrary positive integer and assume $D^p(\psi\alpha_{s_i,i}) \to D^p(\psi)$ in $\mathscr{L}_M^*$ for an arbitrary $|p| < m$ and for any $\psi \in \mathscr{D}_M$. Then we prove $D^p(\varphi\alpha_{s_i,i}) \to D^p(\varphi)$ in $\mathscr{L}_M^*$ as $i\to\infty$ for $|p| = m$ and for an arbitrary $\varphi \in \mathscr{D}_M$. If we take $\psi = \partial\varphi/\partial x_j$, $p' = (p_1, \ldots, p_{j-1}, p_j-1, p_{j+1}, \ldots, p_n)$, we have

$$D^p(\varphi\alpha_{s_i,i}) - D^p(\varphi) = D^{p'}(\psi\alpha_{s_i,i}) - D^{p'}(\psi) + D^{p'}\left(\varphi\frac{\partial\alpha_{s_i,i}}{\partial x_j}\right) \to 0$$

and, consequently, $D^p(\varphi\alpha_{s_i,i}) \to D^p(\varphi)$. This proves that $\varphi\alpha_{s_i,i} \to \varphi$ in $\mathscr{D}_M$.

3.3. Since 2.3(a) follows from 2.2(a), we proceed to the proof of 2.3(b), $\mathscr{L}_M^* \subset \mathscr{D}_M'$. Assuming that $N(u)$ satisfies (Δ_2) for all u, $\mathscr{L}_M^*$ is the dual of $\mathscr{L}_N^*$, and the above inclusion follows from $\mathscr{D}_N \subset \mathscr{L}_N^*$. If (Δ_2) is not assumed, we perform the proof as follows. Let $\varphi \in \mathscr{L}_M^*$, i. e. $\varrho_M(k\varphi) < \infty$ for a $k > 0$. Then $\varphi(\psi) = \int\varphi(x)\psi(x)\,dx$ is finite for every $\psi \in \mathscr{L}_N^*$, for

$$|\varphi(\psi)| \leqslant \frac{1}{kk'}\,[\varrho_M(k\varphi) + \varrho_N(k'\psi)] < \infty,$$

if k' is sufficiently small. Assuming $\psi_i \to 0$ in $\mathscr{D}_N$, we have $\psi_i \to 0$ in $\mathscr{L}_N^*$, whence $\varphi(\psi_i) \to 0$ and so $\varphi(\psi)$ is a linear functional over $\mathscr{L}_N^*$. Hence $\mathscr{L}_M^* \subset \mathscr{D}_M'$. We now prove the topology in $\mathscr{L}_M^*$ to be stronger than in $\mathscr{D}_M'$. We fix a bounded set B in $\mathscr{D}_N$, i. e. for every p, $\|D^p\psi\|_N$ is bounded in B. In particular, $\|\psi\|_N \leqslant \lambda$ for a $\lambda > 0$ and all $\psi \in B$. Choosing $\varphi \in \mathscr{L}_M^*$, $\|\varphi\|_M \leqslant \lambda^{-1}$, we then have $\varphi(\psi) \leqslant 1$, i. e. $\varphi \in B^0$, where B^0 is the polar set defined by B. Hence each polar set in $\mathscr{D}_M'$ contains a sphere in $\mathscr{L}_M^*$, i. e. $\mathscr{L}_M^* \subset \mathscr{D}_M'$.

We come now to the proof of 2.3(c); the idea will follow essentially the lines of Schwartz's proof. $\mathscr{D}$ being dense in $\mathscr{L}_N^*$, the set $B = \{\varphi \in \mathscr{D}: \|\varphi\|_N \leqslant 1\}$ is dense in $\{\varphi \in \mathscr{L}_N^*: \|\varphi\|_N \leqslant 1\}$ in $\mathscr{L}_N^*$. For a fixed $\alpha \in \mathscr{D}$ and a variable $\varphi \in B$, the functions $\check{\alpha}*\varphi \in \mathscr{D}$ and the set of these functions is bounded in $\mathscr{D}_N$. In fact, $D^p(\check{\alpha}*\varphi) = D^p\check{\alpha}*\varphi$ and denoting by $S(\alpha)$ the support of $\alpha(x)$, $\mu =$ the measure of $S(\alpha)$, $\sup_x |D^p\alpha(x)| = K$, we have by Jessen's inequality

$$N\big[\big(D^p(\check{\alpha}*\varphi)(x)\big)\eta^{-1}\big] \leqslant \frac{1}{\mu}\int_{S(\alpha)} N[\mu K\varphi(x-t)\eta^{-1}]\,dt,$$

whence by Fubini's theorem

$$\varrho_N\big[\big(D^p(\check{\alpha}*\varphi)(x)\big)\eta^{-1}\big] \leqslant \int N\left(\frac{\varphi(x)}{\eta\mu^{-1}K^{-1}}\right)dx \leqslant 1$$

for $\eta > \mu K$. Hence $\|D^p(\check{\alpha}*\varphi)\|_N$ are bounded for $\varphi \in B$.

Now, for any $T \in \mathscr{D}'$, the numbers $(T*\alpha)(\varphi) = T(\check{\alpha}*\varphi)$ are bounded for $\varphi \in B$; indeed, $B^0 = \{S \in \mathscr{D}_M': S(\check{\alpha}*\varphi) \leqslant 1,\ \varphi \in B\}$ being polar in $\mathscr{D}_M'$, we must have $\pm\lambda^{-1}T \in B$ for a $\lambda > 0$, i. e. $|(T*\alpha)(\varphi)| = |\int(T*\alpha)(x)\varphi(x)\,dx| \leqslant \lambda$ for $\varphi \in B$. Hence $(T*\alpha)(\varphi)$ is a linear functional over $\mathscr{L}_N^*$, i. e. $T*\alpha \in \mathscr{L}_M^*$.

The proof of 2.3(d) follows the lines of Schwartz's proof, with the use of 2.3(c). The proof that distributions $T \in \mathscr{D}_M'$ are finite sums of derivatives of functions belonging to $\mathscr{L}_M^*$ may also be obtained by applying [3], II. Indeed, by [3], 2.21 and 2.23, every $T \in \mathscr{D}'_M$ may be represented in the form $T = \sum_1^k T_j$, where T_j is linear with respect to the pseudonorm $\|D^{p^j}\varphi\|_N$. Hence $T_j(\varphi) = \int f_j(x) D^{p^j}\varphi(x)\,dx = f_j(D^{p^j}\varphi)$ $= (-1)^{|p^j|} D^{p^j} f_j(\varphi)$ with $f_j \in \mathscr{L}_M^*$; thus $T = \sum_1^k (-1)^{|p^j|} D^{p^j} f_j$.

References

[1] J. Dieudonné et L. Schwartz, *La dualité dans les espaces (F) et (LF)*, Annales de l'Institut Fourier 1 (1950), p. 61-101.

[2] М. А. Красносельский, Я. Б. Рутицкий, *Выпуклые функции и пространства Орлича*, Москва 1958.

[3] S. Mazur et W. Orlicz, *Sur les espaces métriques linéaires (I)*, Studia Math. 10 (1948), p. 184-208; *(II)*, ibidem 13 (1953), p. 137-179.

[4] J. Musielak, *O pewnym twierdzeniu aproksymacyjnym dla funkcji nieskończenie różniczkowalnych*, Prace Matematyczne 7, in print.

[5] L. Schwartz, *Théorie des distributions I, II*, Paris 1950, 1951.

Reçu par la Rédaction le 25. 3. 1961

STUDIA MATHEMATICA, T. XXI. (1962)

On the radicals of p-normed algebras

by

W. ŻELAZKO (Warszawa)

A *p-normed algebra* is a complete metric algebra in which the metric is introduced by means of a p-homogeneous submultiplicative norm, i. e. such a norm $\|x\|$ that

(1) $$\|xy\| \leqslant \|x\|\,\|y\|,$$

(2) $$\|ax\| = |a|^p \|x\|,$$

where x, y are elements of the algebra in question, a is a real or complex scalar and p is a fixed real number satisfying $0 < p \leqslant 1$. For every locally bounded complete metric algebra there exists an equivalent metric introduced by a norm satisfying (1) and (2). The theory of commutative complete locally bounded algebras is developed in paper [2]. The present paper is a continuation of [2]. We give here a solution of the following problem 1 of [2]: "Is the radical of a commutative p-normed algebra R characterized by the relation

$$\operatorname{rad} R = \{x \in R\colon \|x\|_s = 0\}?"$$

Here

(3) $$\|x\|_s = \lim \sqrt[n]{\|x^n\|}$$

denotes the spectral norm in R (see [2], definition 1 and theorem 4). We shall show that the answer is in the affirmative. It is based upon the following

THEOREM 1. *Let R be a commutative complex p-normed algebra. Then the unit sphere of the spectral norm*

$$K = \{x \in R\colon \|x\|_s \leqslant 1\}$$

is a convex subset of R.

Proof. By theorem 4 of [2], property S7, K is a closed subset of R; consequently it is sufficient to prove that $\|x\|_s \leqslant 1$ and $\|y\|_s \leqslant 1$ imply $\|(x+y)/2\|_s \leqslant 1$. It may easily be seen that it is sufficient to prove