and thus (5.1) holds with \( n = m + 1 \) if we set \( t_n = 0 \). Hence there exists a sequence \( (t_n) \) such that (5.1) is satisfied for \( n = 1, 2, \ldots \), and \( t_n = 0 \) for \( n \geq 4 \).

Let \( H \) be the set of all normal conservative matrices which sum no bounded divergent sequence. Since \( G \subseteq H \), and since, by Theorem 11, \((G - G) \cdot R \) is void, it follows that \((G - G) \subseteq (H - H)\). The reverse inclusion is false, for the matrix \( A \) in the example given above satisfies \( A + H = H \), since \( A + \lambda I \) sums no bounded divergent sequence when \( |\lambda - 1| < 1 \). To see this, consider the matrix obtained from \( A + \lambda I \) by deleting the first two rows; this is of the form \( 100(I + \delta) \), where \( \|E\| < 1 \) if \( |\lambda - 1| < 1 \), and hence it has a two-sided conservative reciprocal, and belongs to \( H \). This does not contradict the example given above, for \( A + H \) has no conservative right reciprocal for small \( \lambda \).

These considerations suggest the problem of determining whether the condition \( A \cdot H = H \), which is a sufficient condition for a normal conservative matrix \( A \) to sum a bounded divergent sequence, is also necessary.

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### On some spaces of functions and distributions (I)

**Spaces \( S_M \) and \( D_M \)**

by

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1. **Definitions.** L. Schwartz introduced in [5] the spaces \( S_M \) of functions and \( D_M \) of distributions. The purpose of this paper is to investigate some properties of spaces \( S_M \) and \( D_M \), the spaces \( D^p \) being replaced by Orlicz spaces \( S^p \). We adopt here the notations of [8] with the only exception that multiple integrals will be denoted by a single sign of integral. Further, \( M(u) \) will denote an even continuous function which vanishes only at 0 and is convex for positive \( u \). Moreover, we assume for simplicity \( u^{-1} M(u) \to 0 \) as \( u \to 0 \) and \( u^{-1} M(u) \to \infty \) as \( u \to \infty \). \( M_{-1}(u) \) will mean the inverse function of \( M(u) \) for \( u \geq 0 \), \( M_{-1}(u) = M_{-1}(-u) \) for \( u < 0 \); \( X(u) \) will mean the function complementary to \( M(u) \) in the sense of Young. Let \( \varphi_M(x) = \int M[\varphi(x)] \, dx \), where the integral is taken over the whole \( n \)-dimensional space. Then

\[
\mathcal{L}_M = \{ \varphi \text{ measurable}; \varphi_M(\eta) < \infty \text{ for a certain } k > 0 \},
\]

with the norm \( \| \varphi \|_M = \inf \{ \varepsilon > 0; \varphi_M(\varepsilon) < 1 \} \) is a \( B \)-space, called an Orlicz space (cf. e.g. [2]).

We write

\[
S_M = \{ \varphi \in D^p; \varphi \in S_M \text{ for every } p \}.
\]

The system of sets

\[
U(a, \varepsilon) = \{ \varphi \in S_M; \| D^p \varphi \|_M < \varepsilon \text{ for } |p| \leq m \},
\]

\( m = 0, 1, 2, \ldots \), \( \varepsilon > 0 \), being assumed to be a fundamental system of neighbourhoods of zero, \( S_M \) becomes a locally convex linear topological space and the topology is equivalent to that induced by the \( P \)-norm

\[
\| \varphi \|_M = \sum_{k=0}^n \frac{1}{m!} \sum_{1 \leq j_1 \leq \ldots \leq j_k \leq m} \frac{1}{j_1! \ldots j_k!} \sum_{p=0}^m \frac{1}{p!} \| D^p \varphi \|_M^p,
\]

where \( m \) is the number of variables. So \( S_M \) is a \( B \)-space (cf. [1], [3]); the completeness of \( S_M \) follows from 2.1.
If $X$ and $Y$ are two linear topological spaces, we shall write $X \subset Y$ to denote that $X$ is a part of $Y$; $X \subset Y$ will mean that $X$ is a linear space isomorphic with a linear subspace $X$, of $Y$ and the topology induced in $X$, by $X$ is stronger than the topology induced in $X$, by $Y$, i.e. each neighbourhood of zero in $Y$ contains a neighbourhood of zero in $X$, in the topology induced by $X$. Moreover, we shall denote by $\mathcal{S}_M$ the strong dual of $\mathcal{N}$.

2. Theorems. The following theorems hold:

2.1. (a) If $\mathcal{S}_M$, then $D^k \varphi(x) \to 0$ as $|x| \to \infty$ for every $p$.

(b) If $\mathcal{S}_M$, then $\varphi_0(x)$ is uniformly bounded and $\varphi_0(x) \to 0$ as $|x| \to \infty$ uniformly in $k$.

2.2. (a) If $M(u) = O(M(u))$ as $u \to 0$, then $\mathcal{S}_M \subset \mathcal{S}_M$; $\mathcal{S}_M \subset \mathcal{S}_M$.

(b) If $M(u)$ satisfies the condition $(A_\beta)$ for small $u$, then $\mathcal{S}_M = \{f \in \mathcal{S}_M: D^p (D^k f) < \infty \text{ for every } p \text{ and every } k > 0\}$.

(c) $\mathcal{S}_M$; moreover, denoting for $a > 0$, $q = 1, 2, \ldots$,

\[ a_{q, k}(a) = \begin{cases} \exp(-|x|^q a^{q-1}) & \text{for } |x| < a, \\ 0 & \text{for } |x| \geq a, \end{cases} \]

we have $\varphi \in \mathcal{S}_M$, $\psi \in \mathcal{S}_M$, and assuming that $M(u)$ satisfies $(A_\beta)$ for small $u$$\gamma_1 \to \varphi$ in $\mathcal{S}_M$, hence $\mathcal{S}_M$ is dense in $\mathcal{S}_M$. (The relation $\varphi \to \varphi$ holds also in $\mathcal{S}_M$.)

2.3. (a) If $N_1(u) = O(N_2(u))$ as $u \to 0$, then $\mathcal{S}_M \subset \mathcal{S}_M$.

(b) $\mathcal{S}_M \subset \mathcal{S}_M \subset \mathcal{S}_M$.

(c) If $N(u)$ satisfies $(A_\beta)$ for all $u$ and $\mathcal{S}_M$, then $\mathcal{S}_M \subset \mathcal{S}_M$ for every $\mathcal{S}_M$.

(2) Under the same assumptions as in (c), distributions $\mathcal{S}_M \subset \mathcal{S}_M$ are exactly finite sums of derivatives of functions from the space $\mathcal{S}_M$, and $\mathcal{S}_M = \bigoplus \mathcal{S}_M$ for every $\mathcal{S}_M$.

3. Proofs. First, we prove the following lemma:

3.11. Denote by $P$ the set of all systems $p = (p_1, p_2, \ldots, p_n)$ such that $p_i = 0, 1$ and not all $p_i$ are equal to 0. Let $p_{n-1} = \cdots = p_{n-k} = 1$, $p_i = 0$ for $i \neq a_i$, $i = 1, 2, \ldots, k$, and write for an $x_0 = (x_{a_1}, \ldots, x_{a_k})$.

\[ D^p(x_0, u) = D^p(x_{a_1}, \ldots, x_{a_k}, u_{a_1}, \ldots, u_{a_k}, u_{a_1}, \ldots, u_{a_k}, \ldots), \]

Assume $\mathcal{S}_M$ and

\[ \int \mathcal{M}(D^p(x_0, u)) \, du_{a_1} \cdots du_{a_k} \leq I \quad \text{for all } p \in P, \]

where the integral is taken over a k-dimensional space. Let $x = (x_1, \ldots, x_n)$ and $x' = (x_1', \ldots, x_n')$ be two arbitrary points and let max $|x_i - x_i'| < b$, max $|x_i - x_i'|$, $|x_i - x_i'| < 1 \leq R$.

Then

\[ \varphi(x) - \varphi(x') \leq I \sum_{i=1}^{n} \left( \frac{|x_i - x_i'|^{2} + M_{-1}(b) M_{-1}(b)}{|x_i - x_i'|^{2} - b} \right). \]

In fact, we have

\[ \varphi(x) - \varphi(x') = \sum_{i=1}^{n} \int_{x_i}^{x_i'} D^p(x_0, u) du_{a_1} \cdots du_{a_k}. \]

For a fixed $p = (p_1, \ldots, p_n)$, we apply identity (2) to the function

\[ \int_{x_i}^{x_i'} D^p(x_0, u) du_{a_1} \cdots du_{a_k} \]

and the points $x, x'$. We easily obtain

\[ \sum_{i=1}^{n} \int_{x_i}^{x_i'} D^p(x_0, u) du_{a_1} \cdots du_{a_k} \leq \sum_{i=1}^{n} \int_{x_i}^{x_i'} \int_{x_i}^{x_i'} D^p(x_0, u) du_{a_1} \cdots du_{a_k}, \]

where $Q$ is the set of all systems $q \in P$, $q_i = 1$ for $i = 1, 2, \ldots, k$, $q_i = 0$ for $i \neq q_i$, such that $p_i = 1$ implies $q_i = 1$. Applying Jensen's inequality

\[ \int_{Q} I_{q} f(x) dx \leq \mu(A) M_{-1} \int_{\mu(A)} \mathcal{M}(f(x)) dx, \]

where $\mu(A)$ is the measure of $A$, to the integrals on the right-hand side of the last identity, substituted into (2), and taking into account the fact that $A^{-1}(u_1)\mathcal{N}(u)$ is decreasing, we obtain the required inequality.

3.1. Now we give the proof of 2.1(a). Obviously, it is sufficient to prove $\varphi(x) \to 0$ as $|x| \to \infty$. Suppose it is not true, i.e. there is an $x > 0$ and a sequence $x_0$ such that $|x_0| \to \infty$ and $\varphi(x_0) > c$. Evidently, we may assume $I_0 = \max_{x \in S} (D^p \varphi(x)) < \infty$. Let $V_0$ be the volume of the unit sphere in the n-dimensional space. The following lemma will be of importance:

For every $I > I_0$ and for every point $x$ there exists a point $x_0$ in the sphere $K$ with centre at $x$ and with radius $R_1 = 2^n V_n$ such that inequality (3) holds for all $x \in P$.

If $p = (1, \ldots, 1)$, the lemma is obvious. Assuming $p \neq (1, \ldots, 1)$
we prove it indirectly. Suppose that for a certain \( x \) every point \( x^* \in K \) satisfies the converse inequality

\[
(\ast) \quad \int M(D^n\varphi(x^*, u))du_{u_1} \cdots du_{u_n} > I
\]

for a \( p \in P \), \( p \neq (1, \ldots, 1) \). Given \( p \in P \), write \( A_p = \{ x^* \in K : (\ast) \) is satisfied \}. Obviously \( A_p \) are measurable and the measure \( \mu(A_p) \) of at least one of these sets, say \( A_p(x_0) \), is greater than \( \nu_{R^k}2^{-n} \). Let \( p = (p_1, \ldots, p_n) \), where \( p_i = 1 \) for \( i = 1, 2, \ldots, k \) and \( p_i = 0 \) for \( i = k + 1, \ldots, n \). Let \( A_p \) be the projection of the set \( A_p \) on the \((n-k)\)-dimensional space of points \( (\varepsilon_1, \ldots, \varepsilon_{n-k}) \). It is easily seen that \( \mu(A_p) > (2R_{k})^{-1}\mu(A_p) \), whence

\[
\mu(A_p(x_0)) > 2^{-n}\nu_{R^k}R_{k}^{-n}.
\]

But

\[
I = \int M(D^n\varphi(x^*, u))du_{u_1} \cdots du_{u_n} > \int M(D^n\varphi(x^*, u))du_{u_1} \cdots du_{u_n} > \int M(D^n\varphi(x^*, u))du_{u_1} \cdots du_{u_{n-k}}.
\]

On the other hand, \( R_{k} = 2^{n-k}\nu_{R^k} > (2^{n-k}\nu_{R^k}1^{-1})\nu_{R^k} \), whence \( I > 2^{-n}\nu_{R^k}R_{k}^{-n} \) — a contradiction. Thus the lemma is proved.

Now we may apply 3.11 to any point \( x \) and \( x^* \). Let \( R_{k} \) be the greatest sphere in the \( n \)-dimensional space with centre in \( x^* \) and with radius \( r_k \) such that \( \varphi(x) > \frac{1}{2} \) for \( x \in K \). Obviously, we may suppose the spheres \( R_{k} \) to be disjoint, for

\[
I \geq \varphi(x) \int M(D^n\varphi(x))du_{u_1} \cdots du_{u_n} > \int M(D^n\varphi(\varepsilon))du_{\varepsilon_1} \cdots du_{\varepsilon_{n-k}};
\]

hence

\[
\nu_{R^k} \frac{n}{2} R_{k}^{-n} \geq \int M_\varphi(\varepsilon) \mu(\varepsilon) R_{k}^{-n};
\]

it is clear that there is a point \( x^* \) on the boundary of \( K \) such that \( \varphi(x^*) = \frac{1}{2} \). Further, let \( x^* \) be a point corresponding to \( x^* \) such that \( 3.11 \) is satisfied and let \( |x - x^*| < R_{k} \). Then

\[
|\varepsilon - \varepsilon^*| \leq R_{k}, \quad i = 1, 2, \ldots, n.
\]

Then we may apply 3.11 to \( x^* \) and \( x^* \) and \( \varepsilon = R_k \), \( \mu(\varepsilon) \). We obtain

\[
\frac{1}{2} \geq \varphi(x^*) - \varphi(x^*) \int M(1 - R_k^{-n + 1})IR_k^{-n + 1}.
\]

Since \( R_k \rightarrow 0 \) as \( n \rightarrow \infty \), we get \( \varepsilon \rightarrow 0 \) — a contradiction.

The proof of 2.1(b) will also be performed by reductio ad absurdum. Suppose there is a sequence \( \{x^* \} \) increasing to infinity, and \( \varphi(x^*) \rightarrow 0 \) will denote here the greatest sphere with centre at \( x^* \) such that \( \varphi(x) \geq 0 \); let \( I = \max_{x_0 < x} I \).

Then we have \( r_k < \varepsilon \) and by 3.11,

\[
\int M_\varphi(\varepsilon) du \leq \int M_\varphi(\varepsilon) du_{u_1} \cdots du_{u_n} \int M_\varphi(\varepsilon) du_{u_1} \cdots du_{u_{n-k}}.
\]

Thus the lemma is proved.

2.2. From the definition of \( \varphi(x) \), one obtains

\[
\int M_\varphi(\varepsilon) du \geq \int M_\varphi(\varepsilon) du_{u_1} \cdots du_{u_n} \int M_\varphi(\varepsilon) du_{u_1} \cdots du_{u_{n-k}}.
\]

Thus the lemma is proved.

Also we have proved that \( \varphi(x) \rightarrow 0 \) in \( K \). Given any \( p > 0 \), \( \varphi(x) \rightarrow 0 \) in \( K \). Then we have

\[
\varphi(x) \rightarrow 0 \quad \text{as} \quad x \rightarrow 0
\]

and

\[
\varphi(x) \rightarrow 0 \quad \text{as} \quad x \rightarrow 0
\]

Therefore, we have proved that \( \varphi(x) \rightarrow 0 \) in \( K \). Now assume \( \varphi(x) \rightarrow 0 \) in \( K \). Given any \( p > 0 \), \( \varphi(x) \rightarrow 0 \) in \( K \). Then we have

\[
\varphi(x) \rightarrow 0 \quad \text{as} \quad x \rightarrow 0
\]

Thus the lemma is proved.

The proofs of other parts of 2.2(a) and 2.2(b) will be omitted here, being trivial. We proceed at once to the proof of the second part of 2.2(c). We shall base ourselves on the following lemmas:

2.3. The functions \( a_n(x) \) being defined as in 2.2(c), we have \( a_n \in \Psi \). Moreover, given any \( \eta > 0 \), \( R > 0 \) and \( p \), where \( 1 \leq |p| < 2 \), there exists a number \( a_n \) such that \( a_n \leq \eta \) and \( \varphi(x) \rightarrow 0 \) as \( x \rightarrow 0 \). For \( \eta < 0 \) and \( \varepsilon > a_n \). Here the constant \( a_n = \max(|R(1 + |p|)^n|) \).

2.4. The functions \( a_n(x) \) being defined as in 2.2(c), we have \( a_n \in \Psi \). Moreover, given any \( \eta > 0 \), \( R > 0 \) and \( p \), where \( 1 \leq |p| < 2 \), there exists a number \( a_n \) such that \( a_n \leq \eta \) and \( \varphi(x) \rightarrow 0 \) as \( x \rightarrow 0 \). For \( \eta < 0 \) and \( \varepsilon > a_n \).

This lemma follows essentially from the formula

\[
(*) \quad D^n a_n(x) = \frac{1}{a_{n+1}} \sum_{k=1}^{n+1} \varphi_{\varepsilon_1}^j \cdots \varphi_{\varepsilon_{n+1}}^{j-1} a_{n+1} \varphi_{\varepsilon_1}^{j-1} \cdots \varphi_{\varepsilon_{n+1}}^{j-1}
\]
for $|x| \leq a$, where $W_{i}(\xi_{1}, \ldots, \xi_{n})$ are polynomials independent of $a$
(for the details of the proof cf. [4]).

Now take $\varphi \in \mathcal{D}_{M}$ and any $k > 0$ and write $\varphi(x) = a_{k}(x)\varphi(x)$.
By 2.3(b), $g_{M}(2kp(x)) < \infty$, whence for any $\epsilon > 0$ there is an $\epsilon > 0$ such that
\[
\int_{|x|<\epsilon} M(2kp(x))dx < \frac{\epsilon}{k}.
\]
Choose $\epsilon > 0$ so that $g_{M}(k\epsilon) < \frac{\epsilon}{k}$. Applying 3.21 with $\epsilon = A_{k}$ we then have $|a_{k}(x)| < \epsilon$ for $|x| < A_{k}$ and $i > a_{k}$, whence
\[
ga_{k}\left(k\varphi(x)\right) \leq g_{M}(k\epsilon) + \int_{|x|\leq A_{k}} M(2k\epsilon)dx < \epsilon,
\]
\[\text{i.e. } g_{M}(k\varphi(x)) \rightarrow 0.
\]

Now take an arbitrary $p$. We shall show that $D^{p}\left(\varphi \frac{\partial a_{k}(x)}{\partial x_{j}}\right) \rightarrow 0$
in $\mathcal{D}_{M}$ as $i \rightarrow \infty$ for every $j$. Since
\[
D^{p}\left(\varphi \frac{\partial a_{k}(x)}{\partial x_{j}}\right) = \sum_{\eta_{1}} \cdots \sum_{\eta_{n}} \frac{\partial^{p}}{\partial x_{j}} \frac{\partial^{\eta_{n}}}{\partial x_{n}} \cdots \frac{\partial^{\eta_{1}}}{\partial x_{1}} \varphi^{(p)}(\eta_{1}, \ldots, \eta_{n}) a_{k}(x),
\]
where $p = (p_{1}, \ldots, p_{n})$, $\varphi^{(p)}(\eta_{1}, \ldots, \eta_{n})$, $\eta_{1}, \ldots, \eta_{n}$, $\eta_{1} \geq 1$, \ldots, $\eta_{n} \geq 1$, it is sufficient to prove that the terms of this sum tend to 0 in $\mathcal{D}_{M}$, i.e. that $D^{p}a_{k} \rightarrow 0$ in $\mathcal{D}_{M}$ for an arbitrary $\varphi \in \mathcal{D}_{M}$.
Write $m_{k} = \max|D^{p}a_{k}(x)|$. Applying (**) we can easily see that $m = m_{k} < \infty$. Given $x$, $k > 0$, let $A_{k} > 0$ and $\eta > 0$ be such that
\[
\int_{|x|\leq A_{k}} M(2k\eta)dx < \frac{\epsilon}{k} \text{ and } g_{M}(k\eta) < \frac{\epsilon}{k}.
\]
Applying 3.21 with $\epsilon = A_{k}$, we get $|D^{p}a_{k}(x)| \leq \eta$ for $|x| < A_{k}$ and $\eta \geq \max(a_{k}, 1, |p|)$, whence
\[
g_{M}(k\varphi(x)) \leq g_{M}(k\eta) + \int_{|x|\leq A_{k}} M(k\eta)dx < \epsilon,
\]
\[\text{i.e. } D^{p}a_{k} \rightarrow 0 \text{ as } i \rightarrow \infty.
\]

Now let $m$ be an arbitrary positive integer and assume $D^{p}(\varphi x_{j}) \rightarrow D^{p}(\varphi)$ in $\mathcal{D}_{M}$ for an arbitrary $|x| < m$ and for any arbitrary $\varphi \in \mathcal{D}_{M}$. Then we prove $D^{p}(\varphi x_{j}) \rightarrow D^{p}(\varphi)$ in $\mathcal{D}_{M}$ as $i \rightarrow \infty$ for $|x| = m$ and for an arbitrary $\varphi \in \mathcal{D}_{M}$.

The proof of 2.3(d) follows the lines of Schwartz's proof, with the use of 2.3(c). The proof that distributions $\mathcal{D}_{M}$ are finite sums of
derivatives of functions belonging to $S'_{M}$ may also be obtained by applying [3], II. Indeed, by [3], 2.21 and 2.23, every $T \in S'_{M}$ may be represented in the form $T = \sum T_{f}$, where $T_{f}$ is linear with respect to the pseudonorm $\|D^{\alpha}\phi\|$. Hence $T_{f}(\varphi) = \int f_{x}(x)D^{\alpha}\varphi(x)dx = f_{x}(D^{\alpha}\varphi) = (-1)^{\|\alpha\|}D^{\alpha}f_{x}(\varphi)$ with $f_{x} \in S'_{M}$; thus $T = \sum (-1)^{\|\alpha\|}D^{\alpha}f_{x}$.

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On the radicals of $p$-normed algebras

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A $p$-normed algebra is a complete metric algebra in which the metric is introduced by means of a $p$-homogeneous submultiplicative norm, i.e. such a norm $\|\cdot\|$ that

\[(1) \quad \|xy\| \leq \|x\||y|,\]
\[(2) \quad \|ax\| = |a|^p\|x|,\]

where $x, y$ are elements of the algebra in question, $a$ is a real or complex scalar and $p$ is a fixed real number satisfying $0 < p \leq 1$. For every locally bounded complete metric algebra there exists an equivalent metric introduced by a norm satisfying (1) and (2). The theory of commutative complete locally bounded algebras is developed in paper [2]. The present paper is a continuation of [2]. We give here a solution of the following problem 1 of [2]: “Is the radical of a commutative $p$-normed algebra $E$ characterized by the relation

\[\text{rad}E = \{x \in E : \|x\| = 0\}^{\circ}.\]

Here

\[(3) \quad \|x\| = \lim V^{\frac{1}{p}}|x|^p\]

denotes the spectral norm in $E$ (see [3], definition 1 and theorem 4). We shall show that the answer is in the affirmative. It is based upon the following

**THEOREM 1.** Let $E$ be a commutative complete $p$-normed algebra. Then the unit sphere of the spectral norm

\[K = \{x \in E : \|x\| \leq 1\}\]

is a convex subset of $R$.

**Proof.** By theorem 4 of [2], property $S7$, $K$ is a closed subset of $R$; consequently it is sufficient to prove that $\|x\| \leq 1$ and $\|y\| \leq 1$ imply $\|(|x|y)/2\| \leq 1$. It may easily be seen that it is sufficient to prove