operator $T: X \to L_1$ is not weakly compact. Hence, by a theorem of Gantmacher ([6], p. 485), the conjugate operator $T^*: M \to X^*$ is also not weakly compact. Thus, by [17], Theorem 5, X^* contains a subspace isomorphic to c_0. Finally, by [4], Theorem 4, we conclude that

If X is a non-reflexive subspace of L_1, then X contains a subspace isomorphic to l and complemented in X.

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Mercerian theorems and inverse transformations

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1. A sequence-to-sequence summability method defined by a matrix A is called a U-method for bounded sequences if the A-transform of every non-zero bounded sequence is non-zero ([6], p. 132). Let A be the matrix of a conservative (i. e. convergence-preserving) sequence-to-sequence method which is a U-method for bounded sequences. It will be shown that A sums no bounded divergent sequence if and only if there exists a conservative matrix B which is a left reciprocal of A, or equivalently, if and only if there exists a matrix $C = (c_{n,k})$ which is a left reciprocal of A and which satisfies

$$\sup_n \sum_{k=1}^{\infty} |c_{n,k}| < \infty .$$

The hypothesis that the method is a U-method for bounded sequences may be omitted if the matrices B, C mentioned above satisfy $BA = I+P$, $CA = I+P$ instead of $BA = I$, $CA = I$, where

$$I = (\delta_{n,k}), \quad \delta_{n,n} = 1, \quad \delta_{n,k} = 0 \quad (k \neq n),$$

and P is a "trivial" conservative matrix $(p_{n,k})$ such that

$$p_{n,k} = 0 \quad (k \geqslant k_0,\ n = 1, 2, \ldots).$$

Parallel results are proved for certain classes of sequence-to-function methods, where the matrix C which occurs in the results stated above is replaced by a sequence $\{g_n\}$ of functions of bounded variation, with

$$\sup_n \operatorname{var} g_n < \infty .$$

These results depend upon a theorem on the existence of extensions of certain linear operators on subspaces of separable Banach spaces. Theorem 1 is the extension theorem, in a form more general than is required for the applications made here, as it may be of independent interest. A special case of the theorem was suggested by a remark of Zeller [11].

Theorem 2 is not essentially different from part of a theorem of Mazur and Orlicz ([6], 3.7.1), but the proof is given in full, since Mazur and Orlicz considered only matrix methods.

Theorems 4 to 9 give the conditions for a method to sum no bounded divergent sequence, first in the case in which the method is a U-method for bounded sequences, and then in the general case.

Finally, I discuss a conjecture of Wilansky and Zeller ([8], p. 384) concerning the maximal group of the Banach algebra of triangular conservative matrices.

2. When linear spaces are considered, it will be assumed that they are complex linear spaces, but all proofs are valid, with trivial modifications, for the real case.

Let X be a separable B_0-space ([5], p. 185), and E any closed linear subspace of X. Suppose that S is a Hausdorff space, and ξ a given point of S, such that there exists a countable fundamental set of neighbourhoods of ξ. Finally, let Y denote the linear space of all bounded complex-valued functions on S, continuous at ξ, with the norm

$$\|y\|_S = \sup_{s\in S} |y(s)| \qquad (y \in Y).$$

Clearly, Y is a B-space under this norm.

THEOREM 1. *With the notation of the preceeding paragraph, any continuous linear operator $T: E \to Y$ has a continuous linear extension $T_0: X \to Y$.*

COROLLARY. *If Y_1 is the subspace of Y consisting of all $y \in Y$ which vanish at ξ, then any continuous linear operator $T: E \to Y_1$ has a continuous linear extension $T_0: X \to Y_1$.*

Denote by $\| \|_n$ $(n = 1, 2, \ldots)$ the homogeneous semi-norms defined on X, with the property that for $x_m \in X$, $\lim_{m\to\infty} \|x_m\|_n = 0$ $(n = 1, 2, \ldots)$ implies and is implied by the convergence to zero of x_m under the F-norm defined on X ([5], p. 185).

For each $s \in S$, the functional $f(s)$ defined on E by $f(s, x) = y(s)$ $y = T(x)$, $x \in E$ is a continuous linear functional on E. By hypothesis, $f(s, x)$ is bounded for each fixed $x \in E$, as s varies in S, and it follows that there exist a positive integer p and a positive number P such that

$$|f(s, x)| \leqslant P \sup(\|x\|_1, \|x\|_2, \ldots, \|x\|_p) \qquad (s \in S, x \in E).$$

For in the contrary case, there would exist sequences $\{x_m\}$, $\{s_m\}$ $(m = 1, 2, \ldots)$, with $x_m \in E$ and $s_m \in S$, such that

$$|f(s_m, x_m)| \geqslant m \sup(\|x_m\|_1, \|x_m\|_2, \ldots, \|x_m\|_m),$$

which is false, by [6], 3.3, (β).

For each $s \in S$, there exists a linear extension $g(s)$ of $f(s)$, whose domain is X, and which satisfies

$$|g(s, x)| \leqslant P \sup(\|x\|_1, \|x\|_2, \ldots, \|x\|_p) \qquad (s \in S, x \in X). \tag{2.1}$$

Since X is separable, there exists a sequence $\{x_m\}$ $(x_m \in X$, $m = 1, 2, \ldots)$ such that

$$\|x_m\|_i \leqslant 1 \qquad (i = 1, 2, \ldots, p;\ m = 1, 2, \ldots),$$

and such that the set of all elements of the form tx_m, where t is any number, is dense in X. By (2.1),

$$|g(s, x_m)| \leqslant P \qquad (m = 1, 2, \ldots; s \in S). \tag{2.2}$$

We now construct the required extension T_0 of T, and we shall ensure that for each $x \in X$, $T_0(x)$ has the value $g(\xi, x)$ at ξ.

From the hypotheses concerning S, it follows that there exists a strictly decreasing sequence $\{N_i\}$ $(i = 1, 2, \ldots)$ of neighbourhoods of ξ, such that if $s \in S$, $s \neq \xi$, then $s \in S - N_i$ for some integer i. Denote by S^* the set obtained by deleting the point ξ from S, and by N_i^* $(i = 1, 2, \ldots)$ the set obtained by deleting the point ξ from N_i.

Select an increasing sequence of positive integers $\{n_q\}$ $(q = 1, 2, \ldots)$ as follows. Let q be fixed, and suppose that $1 = n_1 < n_2 < \ldots < n_{q-1}$ have been chosen. Let the set of all numbers z satisfying $|z| \leqslant P$ be expressed as a union of finitely many disjoint sets $Z_{p,q}$ $(p = 1, 2, \ldots, t_q)$, where $Z_{p,q}$ has diameter at most $1/q$. By (2.2), given any $s \in S^*$ and any positive integer m, there exists exactly one integer p such that $g(s, x_m) \in Z_{p,q}$.

Ordered collections $(Z_{p_1,q}, Z_{p_2,q}, \ldots, Z_{p_q,q})$ can be formed from q of the sets $Z_{p,q}$ in t_q^q different ways; denote these collections by $C_{j,q}$ $(j = 1, 2, \ldots, t_q^q)$. To each $s \in S^*$ corresponds exactly one $j = j(s)$ such that if

$$C_{j,q} = (Z_{p_1,q}, Z_{p_2,q}, \ldots, Z_{p_q,q}), \tag{2.3}$$

then

$$g(s, x_1) \in Z_{p_1,q},\ g(s, x_2) \in Z_{p_2,q}, \ldots, g(s, x_q) \in Z_{p_q,q}. \tag{2.4}$$

Let $S_{j,q}$ be the subset of S^* consisting of all $s \in S^*$ which correspond to j in this way; then

$$S^* = \bigcup_{j=1}^{t_q^q} S_{j,q}; \qquad S_{j,q} \cap S_{k,q} \text{ is void} \qquad (j, k = 1, 2, \ldots, t_q^q;\ j \neq k).$$

It may happen that for some values of j, $S_{j,q} \cap N_i^*$ is void for all large i, but we can choose a positive integer $n_q > n_{q-1}$ such that for each j, either $S_{j,q}\cap N^*_{n_q}$ is void or $S_{j,q}\cap N^*_i$ is non-void for $i=1,2,\ldots$ We have thus expressed $N^*_{n_q}$ as a union of finitely many disjoint non-void sets of the form $S_{j,q}\cap N^*_{n_q}$, and each of these sets has a non-void intersection with N^*_i for $i=1,2,\ldots$, so that each contains a sequence of elements tending to ξ.

But if $\{s_i\}$ $(i=1,2,\ldots)$ is any sequence such that $s_i\in S$, then, by passing to a subsequence and using (2.2), we may assume that $\lim_{i\to\infty} g(s_i, x_m)$ exists finitely for $m=1,2,\ldots$ By [6], 3.3.1, $\lim_{i\to\infty} g(s_i, x)$ exists for all $x\in X$, and defines a continuous linear functional on X. Moreover, if $s_i\to\xi$, then this functional is an extension of $f(\xi)$, since $g(s)$ is an extension of $f(s)$, and since $f(\xi, x)=\lim_{s\to\xi} f(s,x)$ $(x\in E)$.

Applying this argument to a sequence in each of the non-void sets $S_{j,q}\cap N^*_{n_q}$, we obtain a continuous linear extension $\Psi_{j,q}$ of $f(\xi)$, whose domain is X, and, by (2.1),

$$(2.5)\qquad |\Psi_{j,q}(x)|\leqslant P\sup(\|x\|_1, \|x\|_2, \ldots, \|x\|_p)\qquad (x\in X).$$

It is clear that, when $C_{j,q}$ is defined by (2.3), we obtain from (2.4),

$$\Psi_{j,q}(x_1)\in Z_{p_1,q},\ \Psi_{j,q}(x_2)\in Z_{p_2,q},\ \ldots,\ \Psi_{j,q}(x_q)\in Z_{p_q,q},$$

and hence

$$(2.6)\quad |\Psi_{j,q}(x_m)-g(s, x_m)|\leqslant 1/q\qquad (m=1,2,\ldots,q;\ s\in S_{j,q}\cap N^*_{n_q}).$$

Now define linear functionals $h(s)$ on X by

$$h(s)=g(s)\qquad (s\in S-N_{n_1}),$$

$$h(s)=g(s)+g(\xi)-\Psi_{j,q}\qquad (s\in S_{j,q}\cap(N^*_{n_q}-N^*_{n_{q+1}});\ q=1,2,\ldots),$$

$$h(\xi)=g(\xi).$$

Since $s\neq\xi$ implies that $s\in S-N_i$ for some i, it follows that $h(s)$ is defined for all $s\in S$. By (2.1) and (2.5), the operator T_0 defined by

$$h(s,x)=y(s),\qquad y=T_0(x)\qquad (x\in X)$$

satisfies

$$\|T_0(x)\|\leqslant 3P\sup(\|x\|_1, \|x\|_2, \ldots, \|x\|_p),$$

and hence is continuous. By (2.6)

$$|h(s, x_m)-g(\xi, x_m)|\leqslant 1/q\qquad (m=1,2,\ldots,q;\ s\in N^*_{n_q}),$$

and since elements tx_m are dense in X, it follows that

$$\lim_{s\to\xi} h(s,x)=g(\xi,x)\qquad (x\in X),$$

so that the range of T_0 is contained in Y. Finally, $g(\xi)$ and $\Psi_{j,q}$ are extensions of $f(\xi)$, and hence

$$h(s,x)=g(s,x)=f(s,x)\qquad (s\in S, x\in E),$$

and T_0 is an extension of T. Thus T_0 has all the required properties.

To prove the corollary, observe that $f(\xi)$ is now zero, and hence we may set $g(\xi)=0$ in the proof of the theorem.

We have shown incidentally that if X is a separable B-space, then we may construct T_0 so that $\|T_0\|\leqslant 3\|T\|$, and that, with the hypotheses of the corollary, we may ensure that $\|T_0\|\leqslant 2\|T\|$.

3. In this section, we shall suppose that ξ is either a fixed real number or $+\infty$, and that R_1 is a given subset of the real numbers, such that $\xi\notin R_1$ but ξ is a limit point of R_1. To avoid complicated notation, we shall write $\lim_{r\to\xi} f(r)$ when f is any complex-valued function defined on R_1, which tends to a limit as $r\to\xi$ through the values of r in R_1.

As usual, m, c, c_0 denote respectively the spaces of bounded sequences, convergent sequences, and null sequences (i. e., sequences converging to zero) of numbers, with the norm

$$\|u\|=\sup_k |t_k|\qquad (u=\{t_k\}).$$

Define

$$(3.1)\qquad A(r,u)=\sum_{k=1}^{\infty} a(r,k)t_k\qquad (r\in R_1, u=\{t_k\})$$

for any sequence u such that the series (3.1) converges for all $r\in R_1$.

If $\lim_{r\to\xi} A(r,u)=l(u)$ exists finitely, we say that the summability method A defined by (3.1) *sums* the sequence u to the value $l(u)$.

If $l(u)$ exists finitely for each $u\in c_0$ (each $u\in c$), we say that the method A defined by (3.1) is *conservative for null sequences* (*conservative*). If, moreover, $l(u)=\lim_{k\to\infty} t_k$ whenever $u=\{t_k\}\in c_0$ (whenever $u\in c$), then we say that A is *permanent for null sequences* (*permanent*). A method A is said to be a *U-method for bounded sequences* if the conditions $u\in m$ and $A(r,u)=0$ $(r\in R_1)$ imply that $u=0$.

It will be assumed henceforth that A satisfies the conditions

$$(3.2)\qquad \sup_{r\in R_1}\sum_{k=1}^{\infty}|a(r,k)|<\infty,$$

$$(3.3)\qquad \lambda_k(A)=\lim_{r\to\xi} a(r,k)\ \textit{exists finitely}\ (k=1,2,\ldots).$$

Conditions (3.2) and (3.3) are together sufficient for A to be conservative for null sequences. If A is a matrix method, then they are also necessary conditions, but (3.2) may be relaxed in the general case ([9], Theorem II). However, (3.2) is the necessary and sufficient condition for $A(r,u)$ to be bounded on R_1 for each $u \in m$, the proof of this in the case in which R_1 is an interval ([9], Theorem I) being clearly valid in the general case.

It will sometimes be supposed that A is conservative, and hence satisfies the extra condition

$$(3.4) \qquad \lambda(A) = \lim_{r\to\xi} \sum_{k=1}^{\infty} a(r,k) \ \textit{exists finitely.}$$

When (3.4) is satisfied, we define

$$(3.5) \qquad \chi(A) = \lambda(A) - \sum_{k=1}^{\infty} \lambda_k(A);$$

a conservative method A is said to be *coregular* if $\chi(A) \neq 0$ and *conull* if $\chi(A) = 0$.

Denote by $c_0(R_1, A)$, $c(R_1, A)$ respectively the spaces of functions $A(u)$ defined on R_1, with values $A(r,u)$ given by (3.1), such that $u \in c_0$, $u \in c$. These are subspaces of the space of all bounded complex-valued functions defined on R_1; the latter space, furnished with the norm

$$\|f\|_{R_1} = \sup_{r \in R_1} |f(r)|$$

is clearly a B-space, and will be denoted by $\tilde{M}(R_1)$. In general, the subspaces $c_0(R_1, A)$ and $c(R_1, A)$ are not closed under $\|\ \|_{R_1}$.

THEOREM 2. *With the notation given above, if A is a U-method for bounded sequences, and satisfies* (3.2) *and* (3.3), *but sums no bounded divergent sequence, then $c_0(R_1, A)$ and $c(R_1, A)$ are closed under $\|\ \|_{R_1}$.*

LEMMA 1. *Let $w_n = \{t_k^n\} \in m$ $(n = 1, 2, \ldots)$, with $\sup_n \|w_n\| < \infty$, and suppose that $\lim_{n\to\infty} t_k^n = t_k$ exists $(k = 1, 2, \ldots)$. If there exists an element $A_0 \in \tilde{M}(R_1)$ such that $\|A(w_n) - A_0\|_{R_1} \to 0$, then $A_0 = A(w)$, where $w = \{t_k\} \in m$.*

The proof of this lemma is obvious.

LEMMA 2. *Suppose that $v_n = \{t_k^n\} \in c_0$, with $\sup_n \|v_n\| < \infty$, $\|v_n\| \geqslant \geqslant a > 0$ $(n = 1, 2, \ldots)$, and $\lim_{n\to\infty} t_k^n = 0$ $(k = 1, 2, \ldots)$. Suppose also that $\|A(v_n)\|_{R_1} \to 0$. Then there exists an element $u \in m - c$ such that $\lim_{r\to\xi} A(r,u)$ exists.*

For by passing to a subsequence, we may assume that

$$(3.6) \qquad \begin{gathered} \|A(v_n)\|_{R_1} \leqslant 2^{-n}, \\ |t_k^n| \leqslant 4^{-n} a \quad (1 \leqslant k \leqslant q_{2n},\ k > q_{2n+1}), \end{gathered}$$

where $\{q_n\}$ is a suitably defined strictly increasing sequence of positive integers. Let $t_k = \sum_{n=1}^{\infty} t_k^n$; then $u = \{t_k\} \in m$, and

$$|t_k| \geqslant a - \sum_{i=1}^{\infty} 4^{-i} a = 2a/3$$

for some k between q_{2n} and q_{2n+1}; also

$$|t_k| \leqslant \sum_{i=1}^{\infty} 4^{-i} a = a/3 \qquad (q_{2n+1} < k \leqslant q_{2n+2}),$$

so that $u \in m - c$. By (3.6), there exists an element $A_0 \in \tilde{M}(R_1)$, such that $\|A(w_n) - A_0\|_{R_1} \to 0$, where

$$w_n = \sum_{i=1}^{n} v_i,$$

and hence by Lemma 1, $A_0 = A(u)$. By hypothesis, A is conservative for null sequences; also $w_n \in c_0$ $(n = 1, 2, \ldots)$, and $\|A(w_n) - A(u)\|_{R_1} \to 0$. Hence $\lim_{r\to\xi} A(r,u)$ exists, and Lemma 2 is proved.

Passing now to the proof of the theorem, we first show that $c_0(R_1, A)$ is closed, i. e. that if $u_n \in c_0 (n = 1, 2, \ldots)$, if $A_0 \in \tilde{M}(R_1)$, and if $\|A(u_n) - A_0\|_{R_1} \to 0$, then there exists an element $u_0 \in c_0$ such that $A_0 = A(u_0)$.

First suppose, if possible, that $\|u_n\| \to \infty$. Let $v_n = u_n / \|u_n\|$; then $\|A(v_n)\|_{R_1} \to 0$, and we may clearly assume that if $v_n = \{t_k^n\}$, then $\lim_{n\to\infty} t_k^n = t_k$ exists $(k = 1, 2, \ldots)$. Putting $v = \{t_k\}$, and applying Lemma 1, we obtain $A(v) = 0$, but A is a U-method for bounded sequences, and hence $v = 0$. By Lemma 2, A sums a bounded divergent sequence, contrary to hypothesis.

Therefore it may be assumed that $\sup_n \|u_n\| < \infty$, and so by Lemma 1, there exists an element $u_0 \in m$ such that $A_0 = A(u_0)$. Clearly $\lim_{r\to\xi} A(r, u_0)$ exists, but A sums no bounded divergent sequence, and hence $u_0 \in c$.

If $u_0 \in c - c_0$, then there exists a subsequence $\{u_n'\}$ of $\{u_n\}$ such that $v_n = u_{n+1}' - u_n'$ satisfies the conditions of Lemma 2, and hence A sums a bounded divergent sequence. Thus $u_0 \in c_0$ as required.

We now consider $c(R_1, A)$. If $u_n = \{t_k^n\} \in c$ $(n = 1, 2, \ldots)$, then $u_n = v_n + a_n e$, where $v_n \in c_0$, $a_n = \lim_{k\to\infty} t_k^n$, and $e = \{1\}$.

Suppose that $\|A(u_n)-A_0\|_{R_1}\to 0$. If $|a_n|\to\infty$, then

$$\|A(v_n/a_n)-A(-e)\|_{R_1}=\|A(u_n/a_n)\|_{R_1}\to 0.$$

By the result just proved for $c_0(R_1,A)$, there exists an element $v_0\in c_0$ such that $A(v_0)=A(-e)$, contrary to the hypothesis that A is a U-method for bounded sequences.

Thus we may assume that $a_n\to a$. Hence $\|A(v_n)-\{A_0-aA(e)\}\|_{R_1}\to\to 0$, and by the result proved for $c_0(R_1,A)$, there exists an element $v_0\in c_0$ such that $A_0-aA(e)=A(v_0)$; i. e., $A_0=A(ae+v_0)$, which completes the proof of the theorem.

Let R_2 be the set consisting of ξ and all the elements belonging to R_1, and denote by $M_\xi(R_2)$ the space of all bounded complex-valued functions f defined on R_2, such that $\lim\limits_{r\to\xi}f(r)=f(\xi)$. With the norm

$$\|f\|_{R_2}=\sup_{r\in R_2}|f(r)|,$$

this is a B-space. If $u\in m$, and if $A(r,u)$ is defined for $r\in R_1$ by (3.1), we may define

$$A(\xi,u)=\lim_{r\to\xi}A(r,u)$$

whenever this limit exists. The method A thus defines a bounded linear operator whose domain is a closed subspace of m, namely the set of all bounded A-summable sequences, and whose range is a subspace of $M_\xi(R_2)$. By (3.2) and (3.3), the domain contains c_0; moreover, if (3.4) is also satisfied, the domain contains c. Let U_0, U be the operators defined by restricting to c_0, c respectively the domain of the operator defined above by A. The ranges of U_0, U will be denoted by $c_0(R_2,A)$, $c(R_2,A)$. When considering U, we shall assume that (3.4) is satisfied so that $c(R_2,A)\subset M_\xi(R_2)$.

THEOREM 3. *With the notation of the preceeding paragraph, if A satisfies the hypotheses of Theorem 2, then $c_0(R_2,A)$ is closed under $\|\ \|_{R_2}$. If A satisfies also (3.4), then $c(R_2,A)$ is closed under $\|\ \|_{R_2}$.*

This follows immediately from Theorem 2.

4. Theorems 1 and 3 will now be applied to matrix methods, and to a class of sequence-to-function methods. It will be assumed that (3.4) is satisfied, and in applying Theorem 3 we use only the result for $c(R_2,A)$, but it will be seen that related results could be obtained by considering $c_0(R_2,A)$. When discussing matrix methods, we shall use A to denote the matrix, as well as the method defined by it.

THEOREM 4. *Let A be a conservative matrix method which is a U-method for bounded sequences. If A sums no bounded divergent sequence, then there exists a continuous linear operator $T_0\colon c\to c$ which is an extension of the inverse of the operator $U\colon c\to c$ defined by A. T_0 is defined by equations of the form*

$$t_n=\sum_{k=1}^{\infty}b_{n,k}\sigma_k\qquad (T_0(v)=u,\ v=\{\sigma_k\}\in c,\ u=\{t_n\}\in c),$$

where the matrix $B=(b_{n,k})$ is conservative, and $BA=I$.

The existence of T_0 follows from Theorem 3, the closed graph theorem, and Theorem 1. It is well-known that T_0 is defined by equations of the form

$$t_n=a_n\lim_{k\to\infty}\sigma_k+\sum_{k=1}^{\infty}b_{n,k}\sigma_k,$$

where $\sup\limits_n\big(|a_n|+\sum\limits_{k=1}^{\infty}|b_{n,k}|\big)<\infty$, $\lim\limits_{n\to\infty}b_{n,k}$ exists $(k=1,2,\ldots)$, and $\lim\limits_{n\to\infty}\big(a_n+\sum\limits_{k=1}^{\infty}b_{n,k}\big)$ exists. In particular,

$$\delta_{n,i}=a_n\lambda_i(A)+\sum_{k=1}^{\infty}b_{n,k}a_{k,i},$$

$$1=a_n\lambda(A)+\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}b_{n,k}a_{k,i},$$

and hence by (3.5), $a_n\chi(A)=0$. But if $\chi(A)=0$, then A sums a bounded divergent sequence ([10], 3.2), and thus $a_n=0$ $(n=1,2,\ldots)$, and B is conservative with $BA=I$.

If A is permanent, then the conservative matrix B of Theorem 4 may be replaced by a permanent matrix C, for by [1] (4.1, I), (δ),

$$0=\lim_{n\to\infty}\sum_{k=1}^{\infty}b_{n,k}a_{k,i}=\sum_{k=1}^{\infty}\lambda_k(B)a_{k,i}\qquad (i=1,2,\ldots);$$

hence if $c_{n,k}=b_{n,k}-\lambda_k(B)$, then $CA=I$ and C is conservative, with $\lambda_k(C)=0$ $(k=1,2,\ldots)$. From the equation $e=C(Ae)$, where $e=\{1\}$, it follows that C is permanent.

THEOREM 5. *Let A be a (conservative) sequence-to-function method satisfying (3.2), (3.3) and (3.4), where R_1 is the finite half-open interval $(r\colon b\leqslant r<\xi)$. Suppose that A is a U-method for bounded sequences, which sums no bounded divergent sequence, and that $a(r,k)$ $(k=1,2,\ldots)$ and $\sum\limits_{k=1}^{\infty}a(r,k)$ are continuous functions of r on R_1.*

Then A defines a continuous linear operator $U\colon c \to C(R_2)$, where $C(R_2)$ is the B-space of all functions continuous on the closed interval $R_2 = [b, \xi]$, with the norm $\| \, \|_{R_2}$. There exists a continuous linear operator $T_0\colon C(R_2) \to c$, which is an extension of the inverse of U. T_0 is defined by equations of the form

$$t_n = \int_b^\xi \sigma(r)\,dg_n(r) \qquad (T_0(v) = u,\ v = \sigma(r) \in C(R_2),\ u = \{t_n\} \in c),$$

where g_n is of bounded variation in $[b, \xi]$ $(n = 1, 2, \ldots)$, $\sup_n \operatorname{var} g_n < \infty$, and

$$\{g_n(\xi) - g_n(\xi-0)\}\lambda_k(A) + \int_b^{\xi-0} a(r, k)\,dg_n(r) = \delta_{n,k}, \tag{4.1}$$

$$\{g_n(\xi) - g_n(\xi-0)\}\lambda(A) + \int_b^{\xi-0} \sum_{k=1}^\infty a(r, k)\,dg_n(r) = 1. \tag{4.2}$$

If, moreover, A is coregular, then

$$g_n(\xi) = g_n(\xi-0) \qquad (n = 1, 2, \ldots).$$

The continuity of $a(r, k)$ and $\sum_{k=1}^\infty a(r, k)$ ensures that the range of U is contained in $C(R_2)$, when U is defined as in Section 3. The existence of T_0 then follows from Theorem 3, the closed graph theorem, and Theorem 1, and the expression for t_n is well-known. Define

$$a^*(r, k) = a(r, k)\ (b \leqslant r < \xi), \qquad a^*(\xi, k) = \lambda_k(A),$$

$$A^*(r) = \sum_{k=1}^\infty a(r, k)\ (b \leqslant r < \xi), \qquad A^*(\xi) = \lambda(A);$$

then, since T_0 is an extension of the inverse of U,

$$\delta_{n,k} = \int_b^\xi a^*(r, k)\,dg_n(r), \qquad 1 = \int_b^\xi A^*(r)\,dg_n(r), \tag{4.3}$$

and (4.1) and (4.2) are established.

By (4.3) and (3.2)

$$1 = \int_b^\xi \sum_{k=1}^\infty a^*(r, k)\,dg_n(r),$$

and hence

$$0 = \int_b^\xi \Big\{A^*(r) - \sum_{k=1}^\infty a^*(r, k)\Big\}\,dg_n(r) = \{g_n(\xi) - g_n(\xi-0)\}\chi(A),$$

by (3.5). Thus $g_n(\xi) = g_n(\xi-0)$ if A is coregular.

THEOREM 6. *Let A satisfy the conditions of Theorem 5, except that R_1 is now the infinite interval $(r\colon r \geqslant b)$, and $\xi = \infty$.*

Then A defines a continuous linear operator $U\colon c \to C(R_2)$, where $C(R_2)$ is the B-space of all functions continuous on R_2 (i. e. functions continuous for $r \geqslant b$, which tend to a finite limit as $r \to \infty$), with the norm $\| \, \|_{R_2}$. There exists a continuous linear operator $T_0\colon C(R_2) \to c$, which is an extension of the inverse of U. T_0 is defined by equations of the form

$$t_n = a_n \lim_{r\to\infty} \sigma(r) + \int_b^\infty \sigma(r)\,dg_n(r)$$

$$(T_0(v) = u,\ v = \sigma(r) \in C(R_2),\ u = \{t_n\} \in c),$$

where g_n is of bounded variation in the interval $(r\colon r \geqslant b)$ $(n = 1, 2, \ldots)$, $\sup_n \operatorname{var} g_n < \infty$, $\sup_n |a_n| < \infty$, and

$$a_n \lambda_k(A) + \int_b^\infty a(r, k)\,dg_n(r) = \delta_{n,k},$$

$$a_n \lambda(A) + \int_b^\infty \sum_{k=1}^\infty a(r, k)\,dg_n(r) = 1.$$

If, moreover, A is coregular, then $a_n = 0$ $(n = 1, 2, \ldots)$.

The proof is similar to that of Theorem 5.

In order to dispense with the condition that A be a U-method for bounded sequences, we require the following lemma of Mazur and Orlicz ([6], 3.7).

LEMMA 3. *If*

$$\sum_{k=1}^\infty |a(r, k)| < \infty \qquad (r \in R_3),$$

where R_3 is any infinite set of real numbers, and if the system of equations

$$\sum_{k=1}^\infty a(r, k)\,t_k = 0 \qquad (r \in R_3)$$

has no bounded divergent solution, then it has a finite number of linearly independent solutions in c_0.

This was proved by Mazur and Orlicz for the case in which $(a(r,k))$ is a matrix, but their proof holds without modification in the general case.

THEOREM 7. *For a conservative matrix method A to sum no bounded divergent sequence, it is necessary and sufficient that there exist conservative matrices B, P, with $p_{n,k} = 0$ $(k > k_0,\ n = 1, 2, \ldots)$, such that $BA = I+P$.*

An equivalent condition is that there exist matrices C, P, where P satisfies the same conditions as before, and

$$\sup_n \sum_{k=1}^{\infty} |c_{n,k}| < \infty,$$

such that $CA = I+P$.

The equivalence of the conditions involving B, C respectively follows from [4], Theorem 1.

Since P sums every sequence, it is obvious that the condition $BA = I+P$, where B is conservative, is sufficient for A to sum no bounded divergent sequence.

For the necessity, observe that by Lemma 3, if A sums no bounded divergent sequence, we may construct a matrix A_1, which defines a U-method for bounded sequences, by adjoining to the matrix A finitely many rows, each of which has finitely many non-zero elements ([6], 3.7.2). The method A_1 sums no bounded divergent sequence, and hence by Theorem 4, there exists a conservative matrix B_1 such that $B_1 A_1 = I$. If m is the number of rows adjoined to A, then the matrix B is obtained by deleting the first m columns of B_1.

THEOREM 8. *Let A be a (coregular) sequence-to-function method, satisfying* (3.2), (3.3), *and* (3.4), *with $\chi(A) \neq 0$, where $\chi(A)$ is defined by* (3.5). *Let R_1 be the finite halfopen interval $(r\colon\ b \leqslant r < \xi)$, and suppose that $a(r,k)$ $(k = 1, 2, \ldots)$ and $\sum_{k=1}^{\infty} a(r,k)$ are continuous functions of r on R_1.*

For A to sum no bounded divergent sequence, it is necessary that there exist a conservative matrix $P = (p_{n,k})$, with $p_{n,k} = 0$ $(k > k_0,$ $n = 1, 2, \ldots)$, and a sequence $\{g_n\}$ $(n = 1, 2, \ldots)$ of functions of bounded variation on $[b, \xi]$, with

$$\sup_n \operatorname{var} g_n < \infty, \tag{4.4}$$

$$g_n(\xi - 0) = g_n(\xi) \qquad (n = 1, 2, \ldots), \tag{4.5}$$

such that

$$\delta_{n,k} + p_{n,k} = \int_b^{\xi-0} a(r,k)\, dg_n(r). \tag{4.6}$$

If, moreover, $A(u)$ is continuous on R_1 whenever $u \in m$, then the conditions stated above are together sufficient for A to sum no bounded divergent sequence.

First suppose that A sums no bounded divergent sequence. Fix a number d such that $b < d < \xi$; the conditions of Lemma 3 are satisfied for the interval $R_3 = (r\colon\ d \leqslant r < \xi)$. Thus the system of equations

$$\sum_{k=1}^{\infty} a(r,k) t_k = 0 \qquad (d \leqslant r < \xi) \tag{4.7}$$

has finitely many linearly independent solutions in c_0, and hence finitely many in c. Define

$$a_1(r,k) = a(r,k) \qquad (d \leqslant r < \xi,\ k = 1, 2, \ldots),$$

$$a_1(r,k) = a(r,k) \qquad (b \leqslant r < d,\ k > k_0),$$

where k_0 is the number of linearly independent solutions of (4.7) in c. Let $r_1, r_2, \ldots, r_{k_0}$ be numbers such that $b < r_1 < r_2 < \ldots < r_{k_0} < d$, and define $a_1(r_i, k)$ $(i, k = 1, 2, \ldots, k_0)$ so that if (4.7) holds for some $\{t_k\} \in c$, then

$$\sum_{k=1}^{\infty} a_1(r_i, k) t_k \neq 0$$

for some integer i. Complete the definition of $a_1(r,k)$ for $k = 1, 2, \ldots, k_0$ by linearity in the remaining intervals of r, thus making the columns continuous functions of r. The method A_1 defined by $a_1(r,k)$ is coregular, a U-method for bounded sequences, and sums the same sequences as A. By Theorem 5, there exists a sequence $\{g_n\}$ $(n = 1, 2, \ldots)$ of functions of bounded variation on $[b, \xi]$, satisfying (4.4) and (4.5), such that

$$\delta_{n,k} = \int_b^{\xi-0} a_1(r,k)\, dg_n(r).$$

Also, $\{g_n\}$ defines an operator $T_0\colon C(R_2) \to c$, and hence $\{p_{n,k}\} \in c$ for each fixed k, where

$$p_{n,k} = \int_b^{\xi-0} \{a(r,k) - a_1(r,k)\}\, dg_n(r).$$

The conditions are therefore necessary.

To prove the converse, define

$$a^*(r,k) = a(r,k)\ (b \leqslant r < \xi), \quad a^*(\xi,k) = \lambda_k(A),$$

$$A^*(r) = \sum_{k=1}^{\infty} a(r,k)\ (b \leqslant r < \xi), \quad A^*(\xi) = \lambda(A).$$

Let $C(R_2)$ be the B-space of all functions continuous on the closed interval $R_2 = [b, \xi]$, with the norm $\| \ \|_{R_2}$, and let E be the closed linear subspace of $C(R_2)$ spanned by the functions $A^*(r)$ and $a^*(r,k)$ $(k = 1, 2, \ldots)$. By (3.2), (4.5) and (4.6),

$$(4.8) \qquad 1 + \sum_{k=1}^{k_0} p_{n,k} = \sum_{k=1}^{\infty} \int_b^{\xi} a^*(r,k)\, dg_n(r) = \int_b^{\xi} A^*(r)\, dg_n(r),$$

and hence by (4.4), (4.6), and (4.8), and since P is conservative, the sequence $\{g_n\}$ defines a continuous linear operator T: $E \to c$. By Theorem 1, T has a continuous linear extension T_0: $C(R_2) \to c$; moreover T_0 is defined by equations

$$t_n = \int_b^{\xi} \sigma(r)\, dh_n(r) \qquad (T_0(v) = u,\ v = \sigma(r) \in C(R_2),\ u = \{t_n\} \in c),$$

where h_n is of bounded variation on the interval $[b, \xi]$ $(n = 1, 2, \ldots)$, and $\sup_n \operatorname{var} h_n < \infty$. In particular,

$$(4.9) \qquad \delta_{n,k} + p_{n,k} = \int_b^{\xi} a^*(r,k)\, dh_n(r),$$

$$(4.10) \qquad 1 + \sum_{k=1}^{k_0} p_{n,k} = \int_b^{\xi} A^*(r)\, dh_n(r),$$

and hence by (3.2), (4.9), and (4.10),

$$0 = \int_b^{\xi} \Big\{A^*(r) - \sum_{k=1}^{\infty} a^*(r,k)\Big\}\, dh_n(r) = \{h_n(\xi) - h_n(\xi - 0)\}\chi(A),$$

so that

$$(4.11) \qquad h_n(\xi) = h_n(\xi - 0) \qquad (n = 1, 2, \ldots).$$

Now suppose that $u = \{t_k\} \in m$; then by (4.9),

$$(4.12) \quad \int_b^{\xi} \Big\{\sum_{k=1}^{\infty} a^*(r,k) t_k\Big\}\, dh_n(r) = \sum_{k=1}^{\infty} (\delta_{n,k} + p_{n,k}) t_k = t_n + \sum_{k=1}^{k_0} p_{n,k} t_k.$$

If u is also A-summable, then

$$A(\xi, u) = \lim_{r \to \xi} A(r, u) = \lim_{r \to \xi} \sum_{k=1}^{\infty} a(r,k) t_k$$

exists (finitely), and the function $A(u)$ with values $A(r,u)$ is continuous on the closed interval $[b, \xi]$, for by hypothesis it is continuous for $b \leqslant r < \xi$. By (3.2) and (4.11),

$$(4.13) \qquad \int_b^{\xi} A(r,u)\, dh_n(r) = \int_b^{\xi} \Big\{\sum_{k=1}^{\infty} a^*(r,k) t_k\Big\}\, dh_n(r),$$

but $\{h_n\}$ defines the operator T_0: $C(R_2) \to c$, and so by (4.12) and (4.13), $\{t_n + \sum_{k=1}^{k_0} p_{n,k} t_k\} \in c$. Since P is conservative, it follows that $\{t_n\} = u \in c$; thus A sums no bounded divergent sequence.

THEOREM 9. *Let A satisfy the conditions of Theorem 8, except that R_1 is now the interval $(r: r \geqslant b)$, and $\xi = \infty$.*

For A to sum no bounded divergent sequence, it is necessary that there exist a conservative matrix $P = (p_{n,k})$, with $p_{n,k} = 0$ $(k > k_0, n = 1, 2, \ldots)$, and a sequence $\{g_n\}$ of functions of bounded variation on the interval $(r: r \geqslant b)$, with $\sup_n \operatorname{var} g_n < \infty$, such that

$$\delta_{n,k} + p_{n,k} = \int_b^{\infty} a(r,k)\, dg_n(r).$$

If, moreover, $A(u)$ is continuous on R_1 whenever $u \in m$, then the conditions stated above are together sufficient for A to sum no bounded divergent sequence.

The proof is similar to that of Theorem 8.

THEOREM 10. *Suppose that A is a conservative matrix method, or a coregular sequence-to-function method of the type considered in Theorem 8 or Theorem 9, such that $a(r,k)$ $(k = 1, 2, \ldots)$ and $\sum_{k=1}^{\infty} a(r,k)$ are continuous functions of r. Suppose also that given any sequence of numbers $\{\theta_k\}$, such that $0 < \theta_k \leqslant \theta_{k+1} \to \infty$, there exists an unbounded sequence $\{t_k\}$ with $t_k = O(\theta_k)$, whose A-transform is bounded. Then A sums a bounded divergent sequence.*

For if A is a conservative matrix method which sums no bounded divergent sequence, then by Theorem 7, there exist conservative matrices B, P, with $p_{n,k} = 0$ $(k > k_0;\ n = 1, 2, \ldots)$ such that

$BA = I+P$. By [2], Lemma 2, there exists an unbounded non-decreasing sequence $\{\theta_k\}$ of positive numbers such that

$$\sum_{k=1}^{\infty} \theta_k \sum_{i=1}^{\infty} |b_{n,i}||a_{i,k}| < \infty \quad (n = 1, 2, \ldots).$$

Hence if $x = \{t_k\}$, $t_k = O(\theta_k)$, and if Ax is bounded, then $(I+P)x = (BA)x = B(Ax)$ is bounded, and thus x is bounded.

A similar proof holds for the sequence-to-function methods. For by a proof analogous to that of [2], Lemma 2, we have

$$\sum_{k=1}^{\infty} \theta_k \int_b^{\xi-0} |a(r, k)||dg_n(r)| < \infty \quad (n = 1, 2, \ldots)$$

for a suitable unbounded non-decreasing sequence $\{\theta_k\}$, when $\{g_n\}$ is the sequence of functions appearing in Theorem 8. The result then follows as before, since $\{g_n\}$ satisfies (4.4). The result for the case $\xi = \infty$ follows in the same way from Theorem 9.

For matrix methods, Theorem 10 was given in [3], Theorem 5(see also [7], Theorem 2).

5. In a recent paper [8], Wilansky and Zeller discuss the algebra of all triangular conservative matrices. A matrix $A = (a_{n,k})$ is triangular if $a_{n,k} = 0$ $(k > n)$. For any conservative matrix A, define

$$\|A\|^* = \sup_n \sum_{k=1}^{\infty} |a_{n,k}|;$$

then the algebra Δ of all triangular conservative matrices is a Banach algebra under the norm $\|\ \|^*$. The inverse of a matrix $A \in \Delta$ is the matrix $A^{-1} \in \Delta$ such that $AA^{-1} = A^{-1}A = I$. Clearly A has an inverse if and only if A is normal $(a_{nn} \neq 0)$, and sums no divergent sequence. Let G denote the maximal group of Δ, i. e., the set of elements of Δ with inverses, and let $\bar{G}$ denote the closure of G under $\|\ \|^*$. Wilansky and Zeller conjectured ([8], 7) that a normal conservative matrix A sums a bounded divergent sequence if and only if $A \in \bar{G}-G$.

THEOREM 11. *With the notation of the preceeding paragraph, if A is normal, and if $A \in \bar{G}-G$, then A sums a bounded divergent sequence.*

For suppose that $C \in G$, $C+F \in \Delta$, and $\|F\|^* \|C^{-1}\|^* < 1$, then $C+F \in G$, since

$$(C+F)^{-1} = C^{-1} \sum_{i=0}^{\infty} (-FC^{-1})^i \in \Delta.$$

It follows that if $A \in \bar{G}-G$, $A_n \in G$ $(n = 1, 2, \ldots)$, and $\|A_n - A\|^* \to 0$, then $\|A_n^{-1}\|^* \to \infty$.

Suppose, if possible, that A sums no bounded divergent sequence. By Theorem 4, there exists a conservative matrix B such that $BA = I$. Hence

$$B = B(A_n A_n^{-1}) = (BA_n)A_n^{-1} = (BA)A_n^{-1} + \{B(A_n - A)\}A_n^{-1}$$
$$= A_n^{-1} + Q_n A_n^{-1},$$

where $\|Q_n\|^* \leqslant \|B\|^* \|A_n - A\|^*$. Hence

$$\|B\|^* \geqslant \|A_n^{-1}\|^* \{1 - \|B\|^* \|A_n - A\|^*\},$$

which is false for large n. Thus A sums a bounded divergent sequence.

The converse, however, is false. Suppose, for example, that A is the matrix of the transformation $\{w_n\} \to \{z_n\}$ defined by

$$z_1 = -w_1, \quad z_2 = 99w_1 - w_2,$$
$$z_n = 100w_{n-2} + 99w_{n-1} - w_n \quad (n \geqslant 3).$$

Then A sums the sequence $\{(-1)^n\}$. Suppose that $B \in \Delta$, and that $\|B-A\|^* \leqslant 1/100$. We shall show that B sums an unbounded sequence $\{s_n\}$ satisfying

(5.1) $$|s_n| > 90|s_{n-1}| \quad (n = 1, 2, \ldots).$$

Define

$$t_n = \sum_{k=1}^{n} b_{n,k} s_k,$$

and suppose that (5.1) is satisfied for $n = 1, 2, \ldots, m$. Then

$$t_{m+1} = u_{m+1} - v_{m+1},$$

where

$$u_{m+1} = 100s_{m-1} + 99s_m + \beta_m,$$
$$\beta_m = \sum_{k=1}^{m} (b_{m+1,k} - a_{m+1,k}) s_k,$$
$$v_{m+1} = (1-\gamma_{m+1}) s_{m+1}, \quad \gamma_{m+1} = b_{m+1,m+1} - a_{m+1,m+1},$$
$$|\beta_m| \leqslant |s_m|/100, \quad |\gamma_{m+1}| \leqslant 1/100,$$

and hence by (5.1), with $n = m$,

$$97|s_m| < 99|s_m| - 100|s_{m-1}| - |\beta_m| \leqslant |u_{m+1}|,$$

but we have also

$$100|v_{m+1}| \leqslant 101|s_{m+1}|,$$

and thus (5.1) holds with $n = m+1$ if we set $t_{m+1} = 0$. Hence there exists a sequence $\{s_n\}$ such that (5.1) is satisfied for $n = 1, 2, \ldots$, and $t_n = 0$ for $n \geqslant 4$.

Let H be the set of all normal conservative matrices which sum no bounded divergent sequence. Since $G \subset H$, and since, by Theorem 11, $(\bar{G}-G) \cap H$ is void, it follows that $(\bar{G}-G) \subset (\bar{H}-H)$. The reverse inclusion is false, for the matrix A in the example given above satisfies $A \in \bar{H}-H$, since $A+\lambda I$ sums no bounded divergent sequence when $|\lambda-1| < 1$. To see this, consider the matrix obtained from $A+\lambda I$ by deleting the first two rows; this is of the form $100(I+B)$, where $\|B\|^* < 1$ if $|\lambda-1| < 1$, and hence it has a two-sided conservative reciprocal, and belongs to H. This does not contradict the example given above, for $A+\lambda I$ has no conservative right reciprocal for small λ.

These considerations suggest the problem of determining whether the condition $A \in \bar{H}-H$, which is a sufficient condition for a normal conservative matrix A to sum a bounded divergent sequence, is also necessary.

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On some spaces of functions and distributions (I)

Spaces $\mathscr{D}_M$ and $\mathscr{D}'_M$

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1. Definitions. L. Schwartz introduced in [5] the spaces $\mathscr{D}_{L^p}$ of functions and $\mathscr{D}'_{L^p}$ of distributions. The purpose of this paper is to investigate some properties of spaces $\mathscr{D}_M$ and $\mathscr{D}'_M$, the spaces $\mathscr{L}^p$ being replaced by Orlicz spaces $\mathscr{L}^*_M$. We adopt here the notations of [5] with the only exception that multiple integrals will be denoted by a single sign of integral. Further, $M(u)$ will denote an even continuous function which vanishes only at 0 and is convex for positive u. Moreover, we assume for simplicity $u^{-1}M(u) \to 0$ as $u \to 0$ and $u^{-1}M(u) \to \infty$ as $u \to \infty$. $M_{-1}(u)$ will mean the inverse function of $M(u)$ for $u \geqslant 0$, $M_{-1}(u) = M_{-1}(-u)$ for $u < 0$; $N(u)$ will mean the function complementary to $M(u)$ in the sense of Young. Let $\varrho_M(\varphi) = \int M(\varphi(x))\,dx$, where the integral is taken over the whole n-dimensional space. Then

$$\mathscr{L}^*_M = \{\varphi \text{ measurable: } \varrho_M(k\varphi) < \infty \text{ for a certain } k > 0\},$$

with the norm $\|\varphi\|_M = \inf\{\varepsilon > 0\colon \varrho_M(\varphi/\varepsilon) \leqslant 1\}$ is a B-space, called an Orlicz space (cf. e. g. [2]).

We write

$$\mathscr{D}_M = \{\varphi \in \mathscr{E}\colon D^p\varphi \in \mathscr{L}^*_M \text{ for every } p\}.$$

The system of sets

$$U(m, \varepsilon) = \{\varphi \in \mathscr{D}_M\colon \|D^p\varphi\|_M \leqslant \varepsilon \quad \text{for} \quad |p| \leqslant m\},$$

$m = 0, 1, 2, \ldots$, $\varepsilon > 0$, being assumed to be a fundamental system of neighbourhoods of zero, $\mathscr{D}_M$ becomes a locally convex linear topological space and the topology is equivalent to that induced by the F-norm

$$\|\varphi\|^M = \sum_{m=0}^{\infty} \frac{m!}{2^{m+1}n^m} \sum_{|p|=m} \frac{1}{p!} \frac{\|D^p\varphi\|_M}{1+\|D^p\varphi\|_M},$$

where n is the number of variables. So $\mathscr{D}_M$ is a B_0-space (cf. [1], [3]); the completeness of $\mathscr{D}_M$ follows from 2.1.