Bases, lacunary sequences and complemented subspaces in the spaces $L_p$

by

M. I. KADLEC (Klarkov) and A. PEŁCZYŃSKI (Warszawa)

In this paper we investigate the isomorphic structure (invariants of linear homeomorphisms) of subspaces of the space $L_p$ ($1 \leq p < +\infty$). We consider especially the properties of basic sequences (bases in subspaces), as well as the properties of subspaces complemented in $L_p$. These properties are connected with classical problems concerning lacunary series. We explain them in a more detailed way.

Let $p > 2$ and let $(e_n)$ be an orthonormal system. Then

$$
\left( \int \frac{1}{s} \sum_{n=1}^{s} |t_n \varphi_n(t)|^p \, dt \right)^{1/p} \geq \left( \int \left( \sum_{n=1}^{s} |t_n \varphi_n(t)|^2 \right)^{1/2} \right)^{1/2} = \left( \sum_{n=1}^{s} |t_n|^2 \right)^{1/2}
$$

for any scalars $t_1, t_2, \ldots, t_n$ ($n = 1, 2, \ldots$).

An orthonormal system is said to be $p$-lacunary iff (1) the converse inequality

$$
\left( \int \frac{1}{s} \sum_{n=1}^{s} |t_n \varphi_n(t)|^p \, dt \right)^{1/p} \leq C \left( \sum_{n=1}^{s} |t_n|^2 \right)^{1/2}
$$

holds for some $C$ depending only on $(\varphi_n)$ and for any $t_1, t_2, \ldots, t_n$ ($n = 1, 2, \ldots$).

In the language of the functional analysis this means that there is an isomorphism (linear homeomorphism) of Hilbert space $l_2$ onto the closed linear manifold in $L_p$ spanned on the functions $\varphi_n$. Under this isomorphism the unit vectors in $l_2$ correspond to the functions $\varphi_n$, i.e. the basic sequence $(\varphi_n)$ is equivalent to the unit vector basis in $l_2$ (see the definition in section 1). Moreover, the operator $T: x \rightarrow \left( \frac{1}{s} \int \varphi(t) \varphi_n(t) \, dt \right)$ is a projection of $L_p$ onto this manifold.

(1) We write “iff” instead of “if and only if”.

Studia Mathematica XXI
We prove the converse implication. Namely, if \( E \) is a subspace of \( L_p \) isomorphic to \( l_1 \), then \( E \) may be obtained as a closed linear manifold spanned on some \( p \)-lacunary system (Theorem 3).

The classical problem considered by Banach [2] whether any orthonormal system contains a \( p \)-lacunary subsystem may be generalized to the following one:

Given a sequence \((x_n)\) in \( L_p \) (\( p > 2 \)), give a necessary and sufficient condition in order that \((x_n)\) contain a basic sequence \((x_n)\) equivalent to the unit vector basis in \( l_1 \).

This problem is solved in Corollary 5. Moreover, we shall show that if \( p > 2 \) then every basic sequence contains a subsequence equivalent to one of two typical basic sequences. Thus they: the unit vector basis in \( l_1 \), e.g. any \( p \)-lacunary system, and the unit vector basis of \( L_p \) (\( p \) is fixed), e.g. the sequence of characteristic functions of mutually disjoint sets.

Using this fact we prove a few results concerning unconditional bases in \( L_p \) (\( 1 < p < +\infty \)) generalizing earlier results of Gaping [7], [8].

On the basis of our Theorem 2 we show that if \( X \) is an infinite-dimensional subspace complemented in \( L_p \) (\( 1 < p < +\infty \)), then either \( X \) is isomorphic to \( l_1 \) or \( X \) contains a complemented subspace isomorphic to \( l_1 \). This result completes a similar one obtained for other spaces in the paper [17].

In the last part of this paper we give a characterization of a non-reflexive subspace of the space \( L_p \).

Our paper is closely connected with the earlier one [14] of the first of the authors, in which the classes \( M^p \) are introduced. Our Theorem 3 is only a slight modification of Theorem 1 in [14]. The equivalence of conditions 3a, 3c, 3d is also proved here.

For simplicity we restrict our attention to the case of the space \( L_p \).

However, all our results may be extended to the case of the spaces \( L_p(S, \Sigma, \mu) \) defined in [6], p. 241.

1. Terminology and notation. We shall employ the notation and terminology adopted in [6]. We write “space” instead of “\( L_p \) space.” The term “subspace of a space \( X \)” denotes a closed manifold in \( X \). The smallest subspace spanned on the sequence \((x_n)\) is denoted by \([x_n]\).

The symbol \([x_n]\) is reserved for the smallest linear manifold spanned on a sequence \((x_n)\) of real-valued and measurable functions on \([0, 1]\), closed in \( L_p \), i.e. closed under the norm \( |x_n| = (\int |x(t)|^p dt)^{1/p} \). The symbol \( X^* \) denotes the conjugate space to the space \( X \). The Cartesian product of spaces \( X \) and \( Y \) is denoted by \( X \times Y \).

The subspace \( E \) of a space \( X \) is said to be complemented in \( X \) iff there is a projection, i.e. a linear idempotent mapping, from \( X \) onto \( E \). A space \( X \) is said to be isomorphic to a space \( Y \) iff there is a linear homeomorphism from \( X \) onto \( Y \). The sequence \((x_n)\) is said to be a basis in a space \( X \) iff any element \( x \) in \( X \) has the unique expansion \( x = \sum a_n x_n \). The basis \((x_n)\) is unconditional iff this series converges unconditionally, for any \( x \) in \( X \). If \((x_n)\) is an (unconditional) basis of a subspace of a space \( X \), then \((x_n)\) is said to be a (unconditional) basic sequence in \( X \). The basic sequences \((y_n)\) and \((x_n)\) are said to be equivalent iff, for any sequence of scalars \((t_i)\) the convergence of the series \( \sum t_i x_i \) implies the convergence of the series \( \sum t_i y_i \), and conversely. We recall that if the basic sequences \((x_n)\) and \((y_n)\) are equivalent, then the spaces \([x_n]\) and \([y_n]\) are isomorphic. The sequence \((x_n)\) in \( X^* \) is said to be biorthogonal sequence to the sequence \((y_n)\) iff \( x_n^* (y_m) = \delta_{nm} \) (\( n, m = 1, 2, \ldots \)). The unit vector basis in \( l_p \) is the unconditional basis consisting of vectors \( e_i = (\delta_{ij}) \) for \( i = 1, 2, \ldots \).

2. Definition 1 [14]. Suppose that \( p > 1 \) and \( \varepsilon > 0 \). We set

\[
M^p = \{x \in L_p : \text{mes} \{t : |x(t)| \geq \varepsilon |x||1| \} > \varepsilon \}.
\]

Theorem 1. The classes \( M^p \) have the following properties:

1a. If \( \varepsilon_1 < \varepsilon_2 \), then \( M^p_{\varepsilon_1} \subset M^p_{\varepsilon_2} \).
1b. \( \bigcup_{\varepsilon > 0} M^p_\varepsilon = L_p \).
1c. If \( x \) does not belong to \( M^p_\varepsilon \), then there is a set \( A \) such that

\[
\text{mes} A < \varepsilon \quad \text{and} \quad \left( \int \frac{|x(t)|^p}{|t|} \right) > 1 - \varepsilon,
\]

1d. If \( p \geq 2 \), \( \varepsilon > 0 \), then \( |x||1| > \varepsilon |x||1| \) for every \( x \) in \( M^p_\varepsilon \),
1e. If \( p > 2 \), \( 0 < \varepsilon < 1 \) and \( |x||1| > \varepsilon |x||1| \), for some \( x \), then \( x \) belongs to \( M^p_{\varepsilon} \), where \( \varepsilon = \varepsilon (\varepsilon /2)^{2/p-2} \),
1f. If \( p > 2 \), \( \varepsilon > 0 \) and \( x \) is a sequence in \( M^p_\varepsilon \) such that the series \( \sum t_n x_n \) is unconditionally convergent in \( L_p \), then \( \sum |x_n|^2 < +\infty \).

The properties 1a, 1b and 1c are obvious.

1d. The inequality \( |x||1| > \varepsilon |x||1| \) for \( p > 2 \) is well known. To prove that \( |x||1| > \varepsilon |x||1| \) write \( S^p(x) = \{t : |x(t)| \geq \varepsilon |x||1| \} \). Since \( x \) is in \( M^p_\varepsilon \), \( \text{mes} S^p(x) > \varepsilon \) and

\[
\text{mes} S^p(x) = \frac{1}{p} \left( \int |x(t)|^p dt \right)^{1/p} \geq \left( \int |x||1| dt \right)^{1/p} \geq \left( \int |x||1| \text{mes} S^p(x) dt \right)^{1/p} \geq \varepsilon |x||1|.
\]

(*) By \( \text{mes} A \) we denote the Lebesgue measure of a set \( A \).
1e. Suppose that $x$ does not belong to $M_\alpha^p$ ($\alpha < 1$). Hence $\text{mess}_X^p(x) < \varepsilon$. Using the elementary inequality

$$\left( \int_{X} |x(t)|^q \, dt \right)^{1/q} \leq \left( \int_{\mathbb{R}^q} |x(t)|^q \, dt \right)^{1/q}$$

we obtain

$$\|x\|_a = \left( \int_{\mathbb{R}^q} |x(t)|^q \, dt \right)^{1/q} \leq \left( \int_{\mathbb{R}^q} |x(t)|^q \, dt \right)^{1/q} + \left( \int_{\mathbb{R}^q} |x(t)|^q \, dt \right)^{1/q}$$

$$\leq \left( \int_{\mathbb{R}^q} |x(t)|^q \, dt \right)^{1/q} + \left( \int_{\mathbb{R}^q} |x(t)|^q \, dt \right)^{1/q}$$

$$\leq \text{mess}_X^p(x) \|x\|_{L^p} + \varepsilon \|x\|_{L^p} < 2\alpha \|x\|_{L^p}.$$

Thus, if $\|x\|_{L^p} > C\|x\|_{L^p},$ then $C > 2\alpha \|x\|_{L^p},$ i.e., $x \in (C/2)^{\|x\|_{L^p} - 0}.$

Since the identical embedding $u(x) = x$ of $L^p$ into $L^p$ is continuous for $p > 2$, every unconditionally convergent series in $L^p$ is unconditionally convergent in $L^p$ again. Hence, according to a result of Orlicz [18], it follows that $\sum_{n=1}^{\infty} \|x_n\|_{L^p} < +\infty$. Thus, $x_n$ belonging to $M_\alpha^p$ ($n = 1, 2, \ldots$), we obtain $\sum_{n=1}^{\infty} \|x_n\|_{L^p} < \varepsilon\|x\|_{L^p}$, such that:

1a. the sequence $(x_n/\|x_n\|_{L^p})_{n=1}^{\infty}$ is a basic sequence equivalent to the unit vector basis in $L^p$.

2b. the space $[x_n/\|x_n\|_{L^p}]_{n=1}^{\infty}$ has a complement in $L^p$.

Proof of Theorem 2. According to [1], Theorems 2 and 3, it is sufficient to choose a sequence $(x_n/\|x_n\|_{L^p})_{n=1}^{\infty}$ a "little translated" with respect to some sequence $(y_n)$ satisfying the assumptions of Lemma 1.

If $x$ is in $L^p$, then the set function $\Phi(A) = \int_A |x(t)|^p \, dt$ is absolutely continuous. Hence, by the assumptions and by 1a, 1b and 1c we may define by an induction process a subsequence $(x_n)$ of the sequence $(x_n)$ and a sequence of sets $(A_n)$ so that

$$\int_{A_n} \left| \frac{x_n(t)}{\|x_n\|_{L^p}} \right|^p \, dt > 1 - 4^{-(n-1)p} \quad (n = 1, 2, \ldots),$$

$$\int_{A_{n+1}^c} \left| \frac{x_n(t)}{\|x_n\|_{L^p}} \right|^p \, dt < 4^{-(n+1)p} \quad (n = 1, 2, \ldots).$$

Let us write

$$A_n^c = A_n - \bigcup_{k=1}^{n-1} A_k,$$

$$y_n = \frac{x_n}{\|x_n\|_{L^p}} \quad (n = 1, 2, \ldots).$$

Obviously, if $n \neq m$ then $A_n^c \cap A_m^c = \emptyset$. By (1)-(5), we have (for each $n$)

$$\int_{A_n^c} \left| \frac{x_n(t)}{\|x_n\|_{L^p}} \right|^p \, dt < 4^{-(n+1)p},$$

$$\int_{A_{n+1}^c} \left| \frac{x_n(t)}{\|x_n\|_{L^p}} \right|^p \, dt < 4^{-(n+1)p} + \sum_{k=1}^{n} \int_{A_k} \left| \frac{x_k(t)}{\|x_k\|_{L^p}} \right|^p \, dt$$

$$< 4^{-(n+1)p} + \sum_{k=1}^{n} 4^{-p} < 4^{-np},$$

$$\int_{A_n} \left| \frac{x_n(t)}{\|x_n\|_{L^p}} \right|^p \, dt > 1 - 4^{-(n-1)p} - \sum_{k=1}^{n} \int_{A_k} \left| \frac{x_k(t)}{\|x_k\|_{L^p}} \right|^p \, dt$$

$$> 1 - 4^{-(n-1)p} - \sum_{k=1}^{n} 4^{-p} > 1 - 4^{-np}.$$
If follows by (5)-(7) that
\[ \left\| \frac{a_n}{\|a_n\|} - y \right\|_p \leq \left\| \frac{a_n}{\|a_n\|} - a_n \right\|_p + \|a_n - y_n\|_p \leq 4^{-n} + \|y_n\|_p (3 - \|y_n\|_p) < 2 \cdot 4^{-n}. \]

Thus
\[ \left\| P \right\| \sum_{n=1}^{N} \| y_n \|_p \left\| \frac{a_n}{\|a_n\|} - y_n \right\|_p < 1. \]

Hence the sequence \( \frac{a_n}{\|a_n\|} \) fulfills the assumptions of Theorems 2 and 3 of [4].

**Theorem 3.** Let \( p > 2 \) and let \( E \) be an infinite dimensional subspace of \( L_p \). Then the following conditions are equivalent:
3a. \( E \) is isomorphic to the space \( l_1 \).
3b. \( E \) is isomorphic to \( l_2 \).
3c. \( E \) is isomorphic to \( L_p \) for some \( p \).
3d. \( E \) is isomorphic to \( L_1 \).
3e. \( E \) is isomorphic to \( L_2 \).
3f. \( E \) is isomorphic to \( L_p \).

3g. there are \( \varepsilon > 0 \) and an unconditional basis \( (e_n) \) in \( E \) such that \( e_n \cdot M^*_\varepsilon \) for \( n = 1, 2, \ldots \).

Proof. The implications 3a \( \Rightarrow \) 3b \( \Rightarrow \) 3e are well known (1, chap. XII).
3o. \( \Rightarrow \) 3d is an immediate consequence of Theorem 2.
3d \( \Rightarrow \) 3e is an immediate consequence of 1d.
3e \( \Rightarrow \) 3f. Using the Schmidt orthonormalization process we choose an orthonormal system \( \{e_n\} \) in \( E \) such that \( \|e_n\|_p = \|e_n\|_1 = \|e_n\|_2 \) (it is possible because \( E \) is simultaneously closed in \( L_p \) and \( L_2 \), by 3o). By 3e, we have
\[ \left\| \sum_{n=1}^{N} t_n e_n \right\|_p \leq C_1 \left\| \sum_{n=1}^{N} t_n e_n \right\|_2 = C_1 \left( \sum_{n=1}^{N} t_n^2 \right)^{1/2} \]
for each of the scalars \( t_1, t_2, \ldots, t_N \) \( (n = 1, 2, \ldots) \).
Hence \( \{e_n\} \) is \( p \)-lacunary and \( \{e_n\}_n = E \).
3f \( \Rightarrow \) 3a. Let \( \{e_n\} \) be an orthonormal \( p \)-lacunary system and let \( E = \{e_n\}_n \). Hence, it follows that there is a constant \( C_E \) such that the inequality \( \|z\|_p \leq \|z\|_1 = \left\| \sum_{n=1}^{N} t_n e_n \right\|_2 \leq C_E \|z\|_p \) holds for every \( z \) in \( E \), where \( t_n = \int_0^1 a(t) \varphi_n(t) \, dt \) \( (n = 1, 2, \ldots) \). Thus the mapping \( \varphi (\int_0^1 a(t) \varphi_n(t) \, dt) \) is an isomorphism between \( E \) and \( l_1 \).

The condition 3g follows immediately from 3a and 3d.

Now assume 3g. Without loss of generality we may assume that \( \|e_n\|_p = 1 \) \( (n = 1, 2, \ldots) \). We shall show that the series \( \sum_{n=1}^{N} t_n e_n \) converges if \( \sum_{n=1}^{N} t_n^2 < +\infty \), i.e. that the basis \( (e_n) \) is equivalent to the unit vector basis in \( l_1 \).

Suppose that the series \( \sum_{n=1}^{N} t_n e_n \) converges. Since \( (e_n) \) is an unconditional basis in \( E \), the series \( \sum_{n=1}^{N} t_n e_n \) converges unconditionally and, by a result of [16], \( \sum_{n=1}^{N} \|e_n\|_p^2 = \sum_{n=1}^{N} \|e_n\|_1^2 < +\infty \).

Conversely, suppose that \( \sum_{n=1}^{N} t_n^2 < +\infty \). Then, by a result of [12], one may choose a sequence \( (e_n) \), \( e_n = \pm 1 \) \( (n = 1, 2, \ldots) \), so that
\[ \left\| \sum_{n=1}^{N} t_n e_n \right\|_p \leq B_p \left( \sum_{n=1}^{N} t_n^2 \right)^{1/2} \]
(\( N = 1, 2, \ldots \)),
where \( B_p \) is a constant depending only on \( p \).

Since \( E \) is reflexive, the basis \( (e_n) \) is boundedly complete (see [10] or [5], p. 71), by (5), it follows that the series \( \sum_{n=1}^{N} e_n e_n \) converges.

Hence, \( (e_n) \) being an unconditional basis, the series \( \sum_{n=1}^{N} t_n e_n \) is also convergent.

**Corollary 1.** If \( E \) is a subspace of \( L_p \) \( (p > 2) \) isomorphic to \( l_1 \), then \( E \) is complemented in \( L_p \).

Proof. By 3f there exists a \( p \)-lacunary orthonormal system \( (e_n) \) such that \( \{e_n\}_n = E \). Put
\[ f = \sum_{n=1}^{N} \left( \int_0^1 a(t) \varphi_n(t) \, dt \right) e_n \]
for any \( a \) in \( L_p \).

This is a consequence of the following result.
Let \( (e_n) \) be a sequence in \( L_p \) \( (p > 2) \) such that
\[ \int_0^1 \left[ \sum_{n=1}^{N} t_n e_n \right] \varphi_n(t) \, dt < 0 \quad \text{for} \quad k = 1, 2, \ldots, N - 1. \]

Then \( \sum_{n=1}^{N} t_n e_n \) \( \in B_p \left[ \sum_{n=1}^{N} \|e_n\|_p^2 \right]^{1/2} \)
(\( N = 1, 2, \ldots \)), where \( B_p \) depends only on \( p \) ([12], proof of Theorem 1).
In view of Theorem 3, formula (9) well defines a linear mapping from \( L_p \) into \( E \). Since \( P_e = e_n \ (n = 1, 2, \ldots) \) and \([e_n]_n = E\), \( P \) is the desired projection.

Remark 1. We do not know whether Corollary 1 can be extended to the case where \( 1 < p < 2 \).

No subspace of \( L_1 \) isomorphic to \( l_1 \) has a complement in \( L_1 \) (see e.g. [17], p. 216). The smallest closed manifold spanned in \( L_1 \) on Hahn-Banach functions is an example of a non complemented subspace of \( L_1 \) isomorphic to \( l_1 \).

**Corollary 2** ([14], Corollary 3). Let \( p > 2 \) and let \( E \) be an infinite-dimensional subspace of \( L_p \). Then, either \( E \) is isomorphic to \( l_1 \), or \( E \) contains a subspace isomorphic to \( l_1 \), and complemented in \( L_p \).

This immediately follows from Theorems 2 and 3.

Remark 2. Let \( 1 < p < q < 2 \). Then there is a subspace \( X \) of \( L_p \) which is isomorphic to \( l_q \) [13]. In view of [1], chap. XI, no subspace of \( X \) is isomorphic to \( l_1 \). This example shows that Theorem 3 and Corollary 2 cannot be extended to the case where \( 1 < p < 2 \).

**Corollary 3**. Let \( 1 < p < \infty \) and let \( X \) be an infinite-dimensional subspace of \( L_p \) complemented in \( L_p \). Then, either \( X \) is isomorphic to \( l_1 \), or \( X \) contains a complemented subspace isomorphic to \( l_1 \).

Proof. The case \( p = 2 \) is trivial. In the case where \( p > 2 \) we apply Corollary 2. The case where \( 1 < p < 2 \), can be reduced to the preceding one according to a well-known result showing that if \( X \) is reflexive, then \( X \) is a complemented subspace of \( L_p \).

**Corollary 4**. Let \( p > 2 \) and let \( (a_n) \) be a unconditional basic sequence in \( L_p \) with \( 0 < \inf \|a_n\|_p < \sup \|a_n\|_p < \infty \). Then \( (a_n) \) is equivalent to the unit vector basis of \( l_1 \) if and only if there is an \( \varepsilon > 0 \) such that \( a_n \) is in \( M^n_{\varepsilon} \) for all \( n = 1, 2, \ldots \).

This immediately follows from the analysis of Theorem 3.

Remark 3. In particular from Corollary 4 we obtain the result of Bari [3] and Gelfand [9], which states that all unconditional bases \( (e_n) \) in \( L_p \) with \( 0 < \inf \|e_n\|_p < \sup \|e_n\|_p < \infty \) are equivalent.

To prove that consider a subspace of \( L_p \) isomorphic to \( l_\infty \) and use Corollary 4 and condition 3d.

**Corollary 5**. Let \( p > 2 \) and let \( (a_n) \) be a sequence in \( L_p \) satisfying the following conditions:

(i) \( (a_n) \) weakly converges to 0,

(ii) \( \limsup_{n} \|a_n\|_p > 0 \).

Then there is a subsequence \( (a_{n_k}) \) which is equivalent either (a) to the unit vector basis in \( L_p \), or (b) to the unit vector basis in \( l_1 \). Moreover, (9) holds if

(iii) there is \( \varepsilon > 0 \) such that \( a_n \) is in \( M^n_{\varepsilon} \) for infinitely many \( n \).

Proof. (ii) is satisfied, then without loss of generality we may assume that all \( a_n \) are in \( M^n_{\varepsilon} \) for some \( \varepsilon > 0 \) (\( \varepsilon \) does not depend on \( n \)). Since the space \( L_p \) (\( p > 1 \)) has an unconditional basis [13], by (i), (ii) and a result in [4], p. 26, Ch. I, we may choose an unconditional basic sequence \( (a_n) \) with \( 0 < \inf \|a_n\|_p < \sup \|a_n\|_p < \infty \). Now (9) follows from Corollary 4.

If (iii) does not hold, then we apply Theorem 2.

**Corollary 6**. Let \( p > 2 \) and let \( (a_n) \) be a basic sequence in \( L_p \) with \( 0 < \inf \|a_n\|_p < \sup \|a_n\|_p < \infty \). Then there is a basic sequence \( (a_n) \) which is equivalent either (a) to the unit vector basis in \( L_p \), or (b) to the unit vector basis in \( l_1 \). Moreover, (9) holds if and only if (10) is satisfied.

Proof. Since every bounded basic sequence in any reflexive space weakly converges to 0 (4), this Corollary follows immediately from Corollary 5.

**Remark 4**. The example considered in Remark 2 shows that Corollaries 5 and 6 cannot be extended to the case where \( 1 \leq p < 2 \).

However, in the space \( L_p (1 < p < \infty) \) every basic sequence contains a subsequence equivalent to the unit vector basis in \( l_1 \), if it is defined in the same way as in [4], p. 167. (C. 6).

**Corollary 7** ([11], p. 246). Let \( p > 2 \) and let \( (a_n) \) be a normalized system such that

(i) \( \sup_n \|a_n\| = C < \infty \).

Then there exists a \( p \)-lacunary subsequence \( (a_{n_k}) \).

Proof. The \( p \)-lacunarity of the sequence \( (a_{n_k}) \) means that \( (a_{n_k}) \) is an orthonormal system which is a basic sequence equivalent (under the norm \( \|\cdot\| \)) to the unit vector basis in \( l_1 \). Hence, in view of Corollary 5, to complete the proof it is sufficient to show that (i), (ii) and (iii) hold if and only if (10) is satisfied. Since \( (a_{n_k}) \) is an orthonormal system, \( \lim \int a_{n_k}(t)g(t)dt = 0 \) for every \( g \) in \( L_1 \). Thus (i) holds because \( (a_{n_k}) \) is a bounded sequence
in a reflexive space which tends to zero for a dense set of functionals (the class of all square-integrable functions is dense in $L_p$ for $1 \leq q \leq 2$). Condition (ii) follows from the inequality $\|x_n\|_p \geq \|x_n\|_q = 1$ ($n = 1, 2, \ldots$). Since $\|x_n\|_p > \|x_n\|_q = 1 > 0^+$, we obtain (iii) by 1e.

**Theorem 4.** Let $(a_n)$ be an unconditional basis in $L_p$ ($1 < p < +\infty$) such that $0 < \inf \|a_n\|_p \leq \sup \|a_n\|_p < +\infty$.

Then $\forall n$, every subsequence of $(a_n)$ contains a subsequence which is equivalent to the unit vector basis in $L_p$, or to the unit vector basis in $L_1$.

**Theorem 5.** There exists a subsequence $(a_{n_k})$ which is equivalent to the unit vector basis in $L_p$.

**Proof.** For $p = 2$, it follows from the result of Bari and Gelfand mentioned in Remark 3.

Let $p > 2$. Then 4a follows from Corollary 4. To prove 4b, suppose there is a subsequence $(a_{n_k})$ is equivalent to the unit vector basis in $L_p$. Hence, by Theorem 2, there is a $c > 0$ such that $a_{n_k}$ is in $M_p^n$. Thus, by Theorem 3, implication 3f $\Rightarrow 3a$, we infer that the space $[a_{n_k}] = L_p$ is isomorphic to $A$. It leads to a contradiction with (11), chap. XII.

The proof in the case where $1 < p < 2$ can be reduced to the preceding one since the space $L_p$ for $1 < p < 2$ is isomorphic to $l_2$. We infer that the basic sequences $(a_{n_k})$ and $(a_n)$ are equivalent if and only if the basic sequences $(a_{n_k})$ and $(a_n)$ are equivalent.

**Proof.** Denote by $a^*$ the restriction of a functional $a$ on $X$ to the space $X_0 = [a_{n_k}]$ and set $\|\cdot\|_{X_0} = \sup a^*(a)\|a\|^{-1}$. If the basic sequences $(a_{n_k})$ and $(a_n)$ are equivalent, then the basic sequences $(a_{n_k})$ and $(a_n)$ are equivalent also. Since $(a_n)$ is an unconditional basis, $P = \sum \lambda^k a^k$ is a well-defined projection operator from $X$ onto $X_0$. Thus, for arbitrary scalars $t_1, t_2, \ldots, t_k$ ($k = 1, 2, \ldots$), we have

$$\left\| \sum_{k=1}^k t_k a_{n_k} \right\| = \sup_{\|a\|_P = 1} \left\| \sum_{k=1}^k t_k a^k(Pa) \right\| < \|P\| \left\| \sum_{k=1}^k t_k a_{n_k} \right\|_{X_0}.$$

On the other hand, $\|a^*\| = \|a_{n_k}\|_{X_0}$ for every $a^*$ in $X^*$. Thus, the basic sequences $(a_{n_k})$ and $(a_{n_k})$ are equivalent, and the basic sequences $(a_{n_k})$ and $(a_n)$ are also equivalent.

**Spaces $L_p$.**

The proof of the converse implication is analogous.

**Remark 6.** The assumption of Lemma 2 that $(a_n)$ is an unconditional basis is essential. Let $X = Z = c$. Consider the basis $(a_n)$ where

$$a_n = (a^{(n)}_i), \quad \xi_i^{(n)} = \begin{cases} 0 & \text{for } i < n, \\ 1 & \text{for } i \geq n. \end{cases}$$

The biorthogonal sequence in $c^\ast = l$ is

$$(a_n^*) = \left(\eta_i^{(n)}\right), \quad \eta_i^{(n)} = \begin{cases} 1 & \text{for } i = 1, \\ 0 & \text{for } i > 1, \\ 1 & \text{for } i = n-1, \\ 0 & \text{for } i = n, \quad (n = 2, 3, \ldots). \end{cases}$$

Let $n_k = 2k$ ($k = 1, 2, \ldots$). Then the basic sequence $(a_{n_k})$ is equivalent to the basis $(a_n)$, but the basic sequence $(a_{n_k})$ is equivalent to the unit vector basis in $l_1$ and thus is not equivalent to the basis $(a_n)$.

**Corollary 8.** Let $1 < p < 2 < +\infty$. Then there is no unconditional basis in the space $l_2$ such that the basic sequence $(a_{n_k})$ is equivalent to the basic sequence $(a_n)$ for every increasing sequence of indices $(a_n)$.

Indeed, if it were not so, then according to Theorem 4 the space $L_p$ would be isomorphic to $l_2$, contrary to (11), chap. XII.

**Corollary 9.** Let $(a_n)$ with $\|a_n\|_p = 1$ ($n = 1, 2, \ldots$) be an unconditional basis in $l_2$. Then $\lim \inf \|a_n\|_1 = 0$, for $p > 2$ ($\lim \sup \|a_n\|_1 = +\infty$, for $1 < p < 2$). In particular, if an orthonormal system is an unconditional basis in $l_2$, then $\lim \sup \|a_n\|_p = +\infty$ for $p > 2$ ($\lim \inf \|a_n\|_p = 0$ for $1 < p < 2$).

**Proof.** In the case where $p > 2$ this follows from 1f and Corollary 4. The case where $1 < p < 2$ reduces to the preceding one by the consideration concerning conjugate space.

**Remark 6.** The trigonometrical orthogonal system is a basis in $l_2$ for $1 < p < +\infty$ ([2], p. 182). This basis satisfies 4a, by Corollary 7, but does not satisfy 4b for $p = 2$. This example shows that in Theorem 4 and Corollary 9 the assumption that the basis is unconditional is essential.

If $p > 2$ we may prove Corollary 8 without the assumption that the basis is unconditional. Probably this assumption is superfluous also for $1 < p < 2$. 


Remark 7. Let \((x_{\alpha})\) be the Haar orthonormal system\(^{(*)}\). It is well known that \((x_{\alpha}||x_{\alpha}||^{-1})\) is an unconditional basis in \(L_p (1 < p < \infty)\) [15]. This basis has the following property: there is only a finite number of functions \(x_{\alpha}\) belonging to \(M_{p}'\) for any \(\varepsilon > 0\). Hence no subsequence of \((x_{\alpha})\) is equivalent to the unit vector basis in \(l_1\) for \(p \neq 2\). On the other hand, in \(L_p\) for \(1 < p < \infty\), there is an unconditional basis \((y_{\alpha}||y_{\alpha}||^{-1})\) with \(||y_{\alpha}||_{\infty} = 1\) containing a subsequence \((y_{\alpha'}||y_{\alpha'}||^{-1})\) equivalent to the unit vector basis \(l_1\) ([17], Theorem 2). Obviously if \(1 < p < \infty\), \(-\infty < \alpha < +\infty\), then no permutation of the basis \((y_{\alpha}||y_{\alpha}||^{-1})\) is equivalent to the basis \((x_{\alpha}||x_{\alpha}||^{-1})\).

4. Definition 2. An unconditional basis \((x_{\alpha})\) in a Banach space \(X\) is said to be permutatively homogeneous iff it is equivalent to the basis \((x_{\alpha}\|x_{\alpha}\|^{-1})\) for any permutation \(p(\cdot)\) of indices.

Remark 8. The unit vector bases in the spaces \(l_1 (1 \leq p < +\infty)\), \(c_0\), and in Orlicz sequence spaces \(l_p\) are permutatively homogeneous. If \((x_{\alpha})\) is a permutatively homogeneous basis in a Banach space \(X\), then \((x_{\alpha}') = \text{the biorthogonal sequence to} \ (x_{\alpha})\) is permutatively homogeneous basis in \([x_{\alpha}'] \subset X^*\).

Theorem 5. Let \((x_{\alpha})\) be a permutatively homogeneous basis in a Banach space \(X\). Then

5a. \(0 < \inf ||x_{\alpha}|| < \sup ||x_{\alpha}|| < +\infty\),

5b. the basis \((x_{\alpha})\) is equivalent to the basic sequence \((x_{\alpha})(\alpha)\) for any increasing sequence of indices \((\alpha)\).

Proof. 5a. Suppose that \(\liminf ||x_{\alpha}|| = 0\) and choose two increasing sequences \((\alpha_i)\) and \((\beta_i)\) such that \(\beta_i \geq \alpha_i (i = 1, 2, \ldots)\) and \(\sum ||x_{\alpha_i}|| ||x_{\beta_i}||^{-1} < +\infty\). Consider a permutation \(p(\cdot)\) such that \(p(\alpha_i) = \beta_i\) and set

\[ t_i = ||x_{\alpha_i}|| \quad \text{for} \quad i = \alpha_i, \]

\[ t_i = 0 \quad \text{for} \quad \text{other} \ i. \]

Then the bases \((x_{\alpha})\) and \((x_{\beta})\) are not equivalent, because the series \(\sum t_i x_{\alpha_i}\) converges but the series \(\sum t_i x_{\beta_i}\) diverges.

The proof that \(\limsup ||x_{\alpha}|| < +\infty\) is analogous.

5b. Let \((\alpha_i)\) be an increasing sequence of indices and let \((t_i)\) be a sequence of scalars such that \(\sum t_i x_{\alpha_i}\) converges. According to 5a,

\[ \lim t_i = 0. \]

Hence we may choose an increasing sequence of indices \((\alpha_i)\) so that \(t_{\alpha_i} < 1/2^m (m = 1, 2, \ldots)\). Let us consider a permutation \(p(\cdot)\) such that \(p(\alpha_i) = \beta_i\) for \(i \neq \alpha_i\). Since the basis \((x_{\beta})\) is permutatively homogeneous, the series \(\sum \sum t_{\alpha_i} x_{\alpha_i}\) is unconditionally convergent.

Thus the series \(\sum \sum t_{\alpha_i} x_{\alpha_i} = \sum t_i x_{\alpha_i}\) is unconditionally convergent.

Finally since

\[ \sum \sum ||x_{\beta_i}||^{-1} < \sum \sup ||x_{\beta_i}|| < +\infty \]

by 5a, the series \(\sum t_i x_{\beta_i}\) converges.

The proof that if the series \(\sum \sum t_{\alpha_i} x_{\alpha_i}\) converges then the series \(\sum \sum t_{\beta_i} x_{\beta_i}\) converges is analogous.

Remark 9. Singer [18] has introduced the notion of symmetric basis. Recently [19] he has proved that every symmetric basis is perfectly homogeneous. He has also proved our Theorem 5 and showed that if an unconditional basis satisfies 5b, then it is symmetric. However, the non-unconditional basis considered in Remark 5 satisfies 5a and 5b.

Corollary 10. If a space \(X\) has a permutatively homogeneous basis \((x_{\alpha})\), then it is isomorphic to its Cartesian square.

This follows from 5b and the fact that \(X = [x_{\alpha}] \times [x_{\beta}]\).

Corollary 11. If \(p \neq 2\), then in \(L_p\) there is no permutatively homogeneous basis.

This follows from 5b and Corollary 8.

Remark 10. A particular case of Corollary 11 is a result of Gapoškin ([8], Theorem 1) showing that if \((x_{\alpha})\) is the Haar orthonormal system, then there is a permutation \(q(\cdot)\) such that the bases \((x_{\alpha}||x_{\alpha}||^{-1})\) and \((x_{\alpha}||x_{\alpha}||^{-1})\) are not equivalent for \(1 < p \neq 2 < +\infty\).

Corollary 12. If the space \(L_p\) is isomorphic to an Orlicz sequence space \(l_{p^*}\), then \(p = 2\).

This follows from Corollary 11 and the fact that in each Orlicz sequence space the sequence of the unit vectors is a permutatively homogeneous basis.

This Corollary may be generalised to the following

Corollary 13. Let \(X\) be a complemented subspace of \(L_p\) \((1 < p < +\infty)\) and let \((x_{\alpha})\) be a permutatively homogeneous basis in \(X\). Then \(X\) is isomorphic either to \(l_p\) or to \(l_1\). Moreover, the basis \((x_{\alpha})\) is equivalent either to the unit vector basis in \(l_p\) or to the unit vector basis in \(l_1\).
Proof. We shall write \( Y_1 \sim Y_2 \) iff the spaces \( Y_1 \) and \( Y_2 \) are isomorphic.

Since \( X \) is complemented in \( L_p \), there exists a space \( Y \) such that \( X \times Y \sim L_p \). Hence, by Corollary 9, we have \( L_p \sim X \times Y \sim (X \times X) \times Y \sim X \times (X \times Y) \sim X \times L_p \).

Let \( y_n^* \) with \( \|y_n^*\| = 1 \) \((n = 1, 2, \ldots)\) be an unconditional basis in \( L_p \).

Let

\[
x_n^* = \begin{cases} (x_n, 0) & \text{for } n = 2k - 1 \quad (k = 1, 2, \ldots), \\ (0, y_k) & \text{for } n = 2k \quad (k = 1, 2, \ldots). \end{cases}
\]

It is easily seen that \( \{x_n^*\} \) is an unconditional basis in \( X \times L_p \). Since \( L_p \sim X \times L_p \), by \( 4a \) there is a subsequence \( \{x_n^*\} = \{x_m^*\} \) equivalent to the unit vector basis in \( L_p \), or to the unit vector basis in \( L_q \). To complete the proof we apply \( 5b \).

5. Theorem 6. Let \( X \) be a non-reflexive subspace of the space \( L_1 \). Then \( X \) contains a subspace complemented in \( L_1 \) and isomorphic to \( l_1 \).

Proof. Let \( K \) be a subset in \( L_1 \) and let \( 0 < \mu \leq 1 \). We put

\[
\eta(x, \mu) = \sup_{\|x\| = 1} \int \|x(t)\| \, dt \quad \text{for any } x \in L_1,
\]

\[
\eta(K, \mu) = \sup_{x \in K} \eta(x, \mu),
\]

\[
(K, +0) = \lim_{x \to +0} \eta(K, \mu).
\]

It is well known \((1), p. 336)\) that a set \( K \) is weakly compact in \( L_1 \) iff \( \eta(K, +0) = 0 \). Hence, in view of the Eberlein-Smulian theorem \((10), p. 430)\), if \( K \) is the unit ball of a non-reflexive subspace \( X \) of \( L_1 \), then \( \eta(K, +0) = \eta^* > 0 \). Thus, by \( 10 \)-\( 12 \), we may choose positive numbers \( \mu_n \) and subnets \( E_n \) of \([0, 1]\) and \( x_n \) in \( L_1 \) so that

\[
\eta(x_n, \mu_n) = \eta^*, \quad \lim_{m \to +0} \mu_n = 0,
\]

\[
\eta(x_n, \mu_n) = \mu_n, \quad \int \|x(t)\| \, dt = \mu^*.
\]

Let us write

\[
\hat{x}_n(t) = \begin{cases} x_n(t) & \text{for } t \in E_n, \\ 0 & \text{for } t \notin E_n, \end{cases} \quad (n = 1, 2, \ldots).
\]

By \( 13 \)-\( 15 \) the sequence \( \{x_n\} \) satisfies the assumptions of Theorem 2 and \( \|z_n\| = \eta^* > 0 \). \((n = 1, 2, \ldots)\). Hence we may choose an increasing sequence of indices \( \{n_i\} \) so that \( \{\hat{x}_{n_i}\} \) is a basic sequence equivalent to the unit vector basis in \( l_1 \) and the space \( [z_{n_i}] \) is complemented in \( L_1 \).

Let us write

\[
z_n = x_n - \hat{x}_n \quad (n = 1, 2, \ldots).
\]

By \( 13 \)-\( 16 \) we have

\[
\eta(x_n, \mu) = \eta(x_n, \mu_n + \mu) - \eta(\hat{x}_n + \mu_n)
\]

for \( 0 < \mu \leq 1 \) and \( n = 1, 2, \ldots \). Hence

\[
\limsup \eta(z_n, \mu) = \eta(K, \mu) - \eta(K, +0) \quad (0 < \mu \leq 1).
\]

Thus \( \eta(z_n, +0) = 0 \), i.e. the set consisting of elements of the sequence \( \{z_n\} \) is weakly compact in \( L_1 \). Hence we may assume that the sequence \( \{z_n\} \) is chosen so that the sequence \( \{z_{n_i}\} \) converges to \( 0 \).

By Mazur’s theorem \((9), p. 422)\) there exist linear convex combinations

\[
z_n = \sum_{i=1}^{k_{n+1} + 1} a_i^0(z_{n-i} - z_{n-i+1}),
\]

\[
a_i^0 \geq 0, \quad \sum_{i=1}^{k_{n+1} + 1} a_i^0 = 1, \quad k_1 < k_2 < \ldots \quad (r = 1, 2, \ldots)
\]

such that

\[
\lim_{n \to +0} \|z_n - z_i\| = \lim_{n \to +0} \|z_i\| = 0,
\]

where \( \hat{z}_i = \sum_{i=1}^{k_{n+1} + 1} a_i^0(z_{n-i} - z_{n-i+1}) \) and \( \hat{z}_i = z_n - \hat{z}_i \).

By the elementary properties of the unit vector basis \( 1 \) \((17), \text{Lemma 1})\) the space \( [z_i] \subset [z_n] \) is isomorphic to \( I \) and has a complement in \( [z_n] \). Thus, since \([z_n]\) is complemented in \( L_1 \), the space \( [z_i] \) is also complemented in \( L_1 \). Finally, by \( 4 \), Theorems 2 and 3, if we choose \( \{z_n\} \) so that \( \hat{z}_i = z_n - \hat{z}_i \) tends to zero “sufficiently quickly”, then the subspace \( [z_i] \subset X \), as a “translated subspace” with respect to \( [z_n] \), will have the desired properties.

Remark 11. We shall give an alternative proof of a slightly weaker result as Theorem 6.

Let \( X \) be a non-reflexive subspace of \( L_1 \). Then the embedding
operator \( T : X \rightarrow L_1 \) is not weakly compact. Hence, by a theorem of Gantmacher ([9], p. 485), the conjugate operator \( T^* : M \rightarrow X^* \) is also not weakly compact. Thus, by [17], Theorem 5, \( X^* \) contains a subspace isomorphic to \( a_\infty \). Finally, by [4], Theorem 1, we conclude that

If \( X \) is a non-reflexive subspace of \( L_1 \), then \( X \) contains a subspace isomorphic to \( l_1 \) and complemented in \( X \).

References


Mercerian theorems and inverse transformations

by

J. COPPING (Nottingham)

1. A sequence-to-sequence summability method defined by a matrix \( A \) is called a \( U \)-method for bounded sequences if the \( A \)-transform of every non-zero bounded sequence is non-zero ([8], p. 133). Let \( A \) be the matrix of a conservative (i.e., convergence-preserving) sequence-to-sequence method which is a \( U \)-method for bounded sequences. It will be shown that \( A \) sums no bounded divergent sequence if and only if there exists a conservative matrix \( B \) which is a left reciprocal of \( A \), or equivalently, if and only if there exists a matrix \( C = (c_{nk}) \) which is a left reciprocal of \( A \) and which satisfies

\[
\sup_n \sum_{k=1}^\infty |c_{nk}| < \infty.
\]

The hypothesis that the method is a \( U \)-method for bounded sequences may be omitted if the matrices \( B \), \( C \) mentioned above satisfy \( BA = I + P \), \( CA = I + P \) instead of \( BA = I \), \( CA = I \), where

\[
I = (\delta_{nk}), \quad \delta_{nn} = 1, \quad \delta_{nk} = 0 \quad (k \neq n),
\]

and \( P \) is a "trivial" conservative matrix \( (p_{nk}) \) such that

\[
p_{nk} = 0 \quad (k > n, \quad n = 1, 2, \ldots).
\]

Parallel results are proved for certain classes of sequence-to-function methods, where the matrix \( C \) which occurs in the results stated above is replaced by a sequence \( (y_n) \) of functions of bounded variation, with

\[
\sup_{n \geq k} |y_n| < \infty.
\]

These results depend upon a theorem on the existence of extensions of certain linear operators on subspaces of separable Banach spaces. Theorem 1 is the extension theorem, in a form more general than is required for the applications made here, as it may be of independent interest. A special case of the theorem was suggested by a remark of Zeller [11].

Studia Mathematica XXI