[22] S. Rolewicz, *Remarks on linear metric Montel spaces*, ibidem III. 7 (1959), p. 195-197.

[23] — *On isomorphic representation of spaces of holomorphic functions by matrix spaces* $M(a_{m,n})$, Reports of the Conference on Functional Analysis, Warszawa 1960.

[24] С. Ролевич, *Об изоморфизме и аппроксимативной размерности пространств голоморфных функций*, ДАН 133 (1960), p. 31-33.

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Bases, lacunary sequences and complemented subspaces in the spaces L_p

by

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In this paper we investigate the isomorphic structure (invariants of linear homeomorphisms) of subspaces of the space L_p $(1 \leqslant p < +\infty)$. We consider especially the properties of basic sequences (bases in subspaces), as well as the properties of subspaces complemented in L_p. These properties are connected with classical problems concerning lacunary series. We explain them in a more detailed way.

Let $p > 2$ and let (φ_n) be an orthonormal system. Then

$$\Big(\int_0^1 \Big|\sum_{i=1}^n t_i\varphi_i(t)\Big|^p dt\Big)^{1/p} \geqslant \Big(\int_0^1 \Big|\sum_{i=1}^n t_i\varphi_i(t)\Big|^2 dt\Big)^{1/2} = \Big(\sum_{i=1}^n |t_i|^2\Big)^{1/2}$$

for any scalars $t_1, t_2, \ldots, t_n$ $(n = 1, 2, \ldots)$.

An orthonormal system is said to be *p-lacunary* iff (1) the converse inequality

$$\Big(\int_0^1 \Big|\sum_{i=1}^n t_i\varphi_i(t)\Big|^p dt\Big)^{1/p} \leqslant C\Big(\sum_{i=1}^n |t_i|^2\Big)^{1/2}$$

holds for some C depending only on (φ_n) and for any $t_1, t_2, \ldots, t_n$ $(n = 1, 2, \ldots)$.

In the language of the functional analysis this means that there is an isomorphism (linear homeomorphism) of Hilbert space l_2 onto the closed linear manifold in L_p spanned on the functions φ_n. Under this isomorphism the unit vectors in l_2 correspond the functions φ_n, i. e. the basic sequence (φ_n) is equivalent to the unit vector basis in l_2 (see the definition in section 1). Moreover, the operator $T: x \to \big(\int_0^1 x(t)\varphi_n(t)\,dt\big)$ is a projection of L_p onto this manifold.

(1) We write "iff" instead of "if and only if".

We prove the converse implication. Namely, if E is a subspace of L_p isomorphic to l_2, then E may be obtained as a closed linear manifold spanned on some p-lacunary system (Theorem 3).

The classical problem considered by Banach [2] whether any orthonormal system contains a p-lacunary subsystem may be generalized to the following one:

Given a sequence (x_n) in L_p $(p>2)$, give a necessary and sufficient condition in order that (x_n) contain a basic sequence (x_{n_k}) equivalent to the unit vector basis in l_2.

This problem is solved in Corollary 5. Moreover, we shall show that if $p>2$, then every basic sequence contains a subsequence equivalent to one of two typical basic sequences. They are: the unit vector basis in l_2, e. g. any p-lacunary system, and the unit vector basis of l_p (p is fixed), e. g. the sequence of characteristic functions of mutually disjoint sets.

Using this fact we prove a few results concerning unconditional bases in L_p $(1<p<+\infty)$ generalizing earlier results of Gapoškin [7], [8].

On the basis of our Theorem 2 we show that if X is an infinite-dimensional subspace complemented in L_p $(1<p<+\infty)$, then either X is isomorphic to l_2, or X contains a complemented subspace isomorphic to l_p. This result completes a similar one obtained for other spaces in the paper [17].

In the last part of this paper we give a characterization of a non-reflexive subspace of the space L_1.

Our paper is closely connected with the earlier one [14] of the first of the authors, in which the classes M_ε^p are introduced. Our Theorem 2 is only a slight modification of Theorem 1 in [14]. The equivalence of conditions 3a, 3c, 3d is also proved here.

For simplicity we restrict our attention to the case of the space L_p. However, all our results may be extended to the case of the spaces $L_p(S,\Sigma,\mu)$ defined in [6], p. 241.

1. Terminology and notation. We shall employ the notation and terminology adopted in [6]. We write "space" instead of "B-space". The term "subspace of a space X" denotes a closed manifold in X. The smallest subspace spanned on the sequence (x_n) is denoted by $[x_n]$. The symbol $[x_n]_p$ is reserved for the smallest linear manifold spanned on a sequence (x_n) of real-valued and measurable functions on $[0,1]$, closed in L_p, i. e. closed under the norm $\|x\|_p=(\int_0^1|x(t)|^p dt)^{1/p}$. The symbol X^* denotes the conjugate space to the space X. The Cartesian product of spaces X and Y is denoted by $X\times Y$.

The subspace E of a space X is said to be *complemented* in X iff there is a *projection*, i. e. a linear idempotent mapping, from X onto E. A space X is said to be *isomorphic* to a space Y iff there is a linear homeomorphism from X onto Y. The sequence (x_n) is said to be a *basis* in a space X iff any element x in X has the unique expansion $x=\sum_{n=1}^{\infty}t_n x_n$. The basis (x_n) is *unconditional* iff this series converges unconditionally, for any x in X (see [5], p. 67-77). If (x_n) is an (unconditional) basis of a subspace of a space X, then (x_n) is said to be an (*unconditional*) *basic sequence* in X. The basic sequences (x_n) and (y_n) are said to be equivalent iff, for any sequence of scalars (t_i) the convergence of the series $\sum_{i=1}^{\infty}t_i x_i$ implies the convergence of the series $\sum_{i=1}^{\infty}t_i y_i$ and conversely. We recall that if the basic sequences (x_n) and (y_n) are equivalent, then the spaces $[x_n]$ and $[y_n]$ are isomorphic. The sequence (x_n^*) in X^* is said to be *biorthogonal* sequence to the sequence (x_n) iff $x_m^*(x_n)=\delta_n^m$ $(n,m=1,2,\ldots)$. The *unit vector basis* in l_p is the unconditional basis consisting of vectors $e_i=(\delta_n^i)$ for $i=1,2,\ldots$

2. Definition 1 [14]. Suppose that $p\geqslant 1$ and $\varepsilon>0$. We set

$$M_\varepsilon^p=\{x\in L_p:\ \mathrm{mess}\{t:\ |x(t)|\geqslant\varepsilon\|x\|_p\}\geqslant\varepsilon\}\ (^2).$$

THEOREM 1. *The classes M_ε^p have the following properties:*

1a. *if $\varepsilon_1<\varepsilon_2$, then $M_{\varepsilon_1}^p\supset M_{\varepsilon_2}^p$,*

1b. $\bigcup_{\varepsilon>0}M_\varepsilon^p=L_p$,

1c. *if $x\neq 0$ does not belong to M_ε^p, then there is a set A such that $\mathrm{mess}\,A<\varepsilon$ and $\int_A\left|\frac{x(t)}{\|x\|}\right|^p dt>1-\varepsilon$,*

1d. *if $p\geqslant 2$, $\varepsilon>0$, then $\|x\|_p\geqslant\|x\|_2\geqslant\varepsilon^{3/2}\|x\|_p$, for every x in M_ε^p,*

1e. *if $p>2$, $0<c\leqslant 1$ and $\|x\|_p\geqslant\|x\|_2\geqslant C\|x\|_p$, for some x, then x belongs to $M_{\varepsilon_0}^p$, where $\varepsilon_0=(c/2)^{2p/(p-2)}$,*

1f. *if $p\geqslant 2$, $\varepsilon>0$ and (x_n) is a sequence in M_ε^p such that the series $\sum_{n=1}^{\infty}x_n$ is unconditionally convergent in L_p, then $\sum_{n=1}^{\infty}\|x_n\|_p^2<+\infty$.*

Proof. The properties 1a, 1b and 1c are obvious.

1d. The inequality $\|x\|_p\geqslant\|x\|_2$ for $p>2$ is well known. To prove that $\|x\|_2\geqslant\varepsilon^{3/2}\|x\|_p$ write $S_\varepsilon^p(x)=\{t:\ |x(t)|\geqslant\varepsilon\|x\|_p\}$. Since x is in M_ε^p, $\mathrm{mess}\,S_\varepsilon^p(x)\geqslant\varepsilon$ and

$$\|x\|_2=\Big(\int_0^1|x(t)|^2dt\Big)^{1/2}\geqslant\Big(\int_{S_\varepsilon^p(x)}|x(t)|^2dt\Big)^{1/2}\geqslant\big(\varepsilon^2\|x\|_p^2\,\mathrm{mess}\,S_\varepsilon^p(x)\big)^{1/2}\geqslant\varepsilon^{3/2}\|x\|_p.$$

(²) By mess A we denote the Lebesgue measure of a set A.

1e. Suppose that x does not belong to M_ε^p $(\varepsilon < 1)$. Hence $\operatorname{mess} S_\varepsilon^p(x) < \varepsilon$. Using the elementary inequality

$$\Big(\int_E |x(t)|^2 dt\Big)^{1/2} \leqslant (\operatorname{mess} E)^{(p-2)/2p} \Big(\int_E |x(t)|^p dt\Big)^{1/p}$$

we obtain

$$\|x\|_2 = \Big(\int_0^1 |x(t)|^2 dt\Big)^{1/2} = \Big(\int_{S_\varepsilon^p(x)} |x(t)|^2 dt + \int_{[0,1]-S_\varepsilon^p(x)} |x(t)|^2 dt\Big)^{1/2}$$

$$\leqslant \Big(\int_{S_\varepsilon^p(x)} |x(t)|^2 dt\Big)^{1/2} + \Big(\int_{[0,1]-S_\varepsilon^p(x)} |x(t)|^2 dt\Big)^{1/2}$$

$$\leqslant \big(\operatorname{mess} S_\varepsilon^p(x)\big)^{(p-2)/2p} \|x\|_p + \varepsilon \|x\|_p < 2\varepsilon^{(p-2)/2p}\|x\|_p.$$

Thus, if $\|x\|_2 \geqslant C\|x\|_p$, then $C < 2\varepsilon^{(p-2)/2p}$, i. e. $\varepsilon > (c/2)^{2p/(p-2)}$.

1f. Since the identical embedding $u(x) = x$ of L_p into L_2 is continuous for $p > 2$, every unconditionally convergent series in L_p is unconditionally convergent in L_2 again. Hence, according to a result of Orlicz [16], it follows that $\sum_{n=1}^{\infty} \|x_n\|_2^2 < +\infty$. Thus, x_n belonging to M_ε^p $(n = 1, 2, \ldots)$, we obtain $\sum_{n=1}^{\infty} \|x_n\|_p^2 \leqslant \varepsilon^{3/2} \sum_{n=1}^{\infty} \|x_n\|_2^2 < +\infty$ by 1d.

THEOREM 2. *Let (x_n) be a sequence in L_p $(p \geqslant 1)$ such that for every $\varepsilon > 0$ there is an index n_ε such that x_{n_ε} does not belong to M_ε^p. Then there exists a sequence (x_n'), where $x_n' = x_{k_n}$ $(k_1 < k_2 < \ldots)$, such that:*

2a. the sequence $(x_n'/\|x_n'\|)_p$ is a basic sequence equivalent to the unit vector basis in l_p,

2b. the space $[x_n']_p$ has a complement in L_p.

LEMMA 1. *Let (A_n') be a sequence of mutually disjoint sets of positive measure and let (y_n) be a sequence in L_p such that $\|y_n\|_p = 1$ and the support of the function y_n is contained in A_n' $(n = 1, 2, \ldots)$. Then (y_n) is a basic sequence satisfying the conditions 2a and 2b.*

Proof. Since $\|\sum_{i=1}^{n} t_i y_i\|_p^p = \sum_{i=1}^{n} |t_i|^p \int_{A_i'} |y_i(s)|^p ds = \sum_{i=1}^{n} |t_i|^p$ for any scalars $t_1, t_2, \ldots, t_k$ $(n = 1, 2, \ldots)$, 2a is satisfied. To establish 2b we put

$$Px = \sum_{n=1}^{\infty} \int_{A_n'} y_n^*(s) \cdot x(s)\, ds \cdot y_n \quad \text{for any } x \text{ in } L_p,$$

where y_n^* is a function in L_q $(q^{-1} + p^{-1} = 1)$ such that $\|y_n\|_q = \int_{A_n'} y_n^*(s) y_n(s) ds = 1$ $(n = 1, 2, \ldots)$. It is easily seen that P is the required projection of L_p onto $[y_n]_p$ with the norm $\|P\| = 1$.

Proof of Theorem 2. According to [4], Theorems 2 and 3, it is sufficient to choose a sequence $(x_n'\|x_n'\|_p^{-1})$ "a little translated" with respect to some sequence (y_n) satisfying the assumptions of Lemma 1.

If x is in L_p, then the set function $\Phi(A) = \int_A |x(t)|^p \cdot dt$ is absolutely continuous. Hence, by the assumptions and by 1a, 1b and 1c we may define by an induction process a subsequence (x_n') of the sequence (x_n) and a sequence of sets (A_n) so that

$$\int_{A_n} \left|\frac{x_n'(t)}{\|x_n'\|_p}\right|^p dt > 1 - 4^{-(n+1)p} \quad (n = 1, 2, \ldots), \tag{1}$$

$$\int_{A_{n+1}} \sum_{i=1}^{n} \left|\frac{x_i'(t)}{\|x_i'\|_p}\right|^p \cdot dt < 4^{-(n+1)p} \quad (n = 1, 2, \ldots). \tag{2}$$

Let us write

$$A_n' = A_n - \bigcup_{i=n+1}^{\infty} A_i, \tag{3}$$

$$z_n(t) = \begin{cases} \dfrac{x_n'(t)}{\|x_n'\|_p} & \text{for } t \in A_n', \\ 0 & \text{for } t \notin A_n', \end{cases} \tag{4}$$

$$y_n = \frac{z_n}{\|z_n\|_p} \quad (n = 1, 2, \ldots). \tag{5}$$

Obviously, if $n \neq m$ then $A_n' \cap A_m' = \emptyset$. By (1)-(5), we have (for each n)

$$\left\|\frac{x_n'}{\|x_n'\|_p} - z_n\right\|_p^p \leqslant \int_{[0,1]-A_n'} \left|\frac{x_n'(t)}{\|x_n'\|_p}\right|^p dt \leqslant \int_{[0,1]-A_n} \left|\frac{x_n'(t)}{\|x_n'\|_p}\right|^p dt + \tag{6}$$

$$+ \int_{A_n - A_n'} \left|\frac{x_n'(t)}{\|x_n'\|_p}\right|^p dt < 4^{-(n+1)p} + \sum_{i=n+1}^{\infty} \int_{A_i} \left|\frac{x_n'(t)}{\|x_n'\|_p}\right|^p dt$$

$$< 4^{-(n+1)p} + \sum_{i=n+1}^{\infty} 4^{-ip} < 4^{-np},$$

$$1 \geqslant \|z_n\|_p^p = \int_{A_n'} \left|\frac{x_n'(t)}{\|x_n'\|_p}\right|^p dt \geqslant \int_{A_n} \left|\frac{x_n'(t)}{\|x_n'\|_p}\right|^p dt - \sum_{\nu=n+1}^{\infty} \int_{A_\nu} \left|\frac{x_n'(t)}{\|x_n'\|^p}\right|^p dt \tag{7}$$

$$\geqslant 1 - 4^{-(n+1)p} - \sum_{\nu=n+1}^{\infty} 4^{-(\nu+1)p} \geqslant 1 - 4^{-np}.$$

If follows by (5)-(7) that

$$\left\|\frac{x_n'}{\|x_n'\|_p}-y_n\right\|_p \leqslant \left\|\frac{x_n'}{\|x_n'\|_p}-z_n\right\|_p+\|z_n-y_n\|_p \leqslant 4^{-n}+\|y_n\|_p(1-\|z_n\|_p)<2\cdot 4^{-n}.$$

Thus

$$\|P\|\sum_{n=1}^{\infty}\|y_n^*\|_q\left\|\frac{x_n'}{\|x_n'\|_p}-y_n\right\|_p<1.$$

Hence the sequence $(x_n'/\|x_n'\|_p)$ fulfils the assumptions of Theorems 2 and 3 of [4].

THEOREM 3. *Let $p>2$ and let E be an infinite dimensional subspace of L_p. Then the following conditions are equivalent:*

3a. *E is isomorphic to the space l_2,*

3b. *no subspace of E is isomorphic to l_p,*

3c. *no subspace of E complemented in L_p is isomorphic to l_p,*

3d. *$E\subset M_\varepsilon^p$ for some $\varepsilon>0$,*

3e. *the norms $\|\ \|_p$ and $\|\ \|_2$ are equivalent on E, i. e. there is a constant $C_E>0$ such that $\|x\|_p\geqslant\|x\|_2\geqslant C_E\|x\|_p$, for any x in E,*

3f. *there is a p-lacunary orthonormal system (φ_n) such that $E=[\varphi_n]_p$,*

3g. *there are $\varepsilon>0$ and an unconditional basis (e_n) in E such that $e_n\in M_\varepsilon^p$ for $n=1,2,\ldots$*

Proof. The implications 3a $\Rightarrow$ 3b $\Rightarrow$ 3c are well known ([1], chap. XII).

3c $\Rightarrow$ 3d is an immediate consequence of Theorem 2.

3d $\Rightarrow$ 3e is an immediate consequence of 1d.

3e $\Rightarrow$ 3f. Using the Schmidt orthogonalization process we choose an orthonormal system (φ_n) in E such that $[\varphi_n]_p=[\varphi_n]_2=E$ (it is possible because E is simultaneously closed in L_p and L_2, by 3e). By 3e, we have

$$\left\|\sum_{i=1}^{n}t_i\varphi_i\right\|_p\leqslant C_E^{-1}\left\|\sum_{i=1}^{n}t_i\varphi_i\right\|_2=C_E^{-1}\Big(\sum_{i=1}^{n}t_i^2\Big)^{1/2}$$

for each of the scalars $t_1,t_2,\ldots,t_n$ $(n=1,2,\ldots)$.

Hence (φ_n) is p-lacunary and $[\varphi_n]_p=E$.

3f $\Rightarrow$ 3a. Let (φ_n) be an orthonormal p-lacunary system and let $E=[\varphi_n]_p$. Hence, it follows that there is a constant C_E such that the inequality $\|x\|_p\geqslant\|x\|_2=\left\|\sum_{n=1}^{\infty}t_ne_n\right\|_2=\big(\sum_{n=1}^{\infty}t_n^2\big)^{1/2}\geqslant C_E\|x\|_p$ holds for every x in E, where $t_n=\int_0^1 x(t)\varphi_n(t)\,dt$ $(n=1,2,\ldots)$. Thus the mapping $x\leftrightarrow\big(\int_0^1 x(t)\varphi_n(t)\,dt\big)$ is an isomorphism between E and l_2.

The condition 3g follows immediately from 3a and 3d.

Now assume 3g. Without loss of generality we may assume that $\|e_n\|_p=1$ $(n=1,2,\ldots)$. We shall show that the series $\sum_{n=1}^{\infty}t_ne_n$ converges iff $\sum_{n=1}^{\infty}t_n^2<+\infty$, i. e. that the basis (e_n) is equivalent to the unit vector basis in l_2.

Suppose that the series $\sum_{n=1}^{\infty}t_ne_n$ converges. Since (e_n) is an unconditional basis in E, the series $\sum_{n=1}^{\infty}t_ne_n$ converges unconditionally and, by a result of [16], $\sum_{n=1}^{\infty}\|t_ne_n\|_p^2=\sum_{n=1}^{\infty}|t_n|^2<+\infty$.

Conversely, supposet hat $\sum_{n=1}^{\infty}t_n^2<+\infty$. Then, by a result of [12] one may choose a sequence (ε_n), $\varepsilon_n=\pm1$ $(n=1,2,\ldots)$, so that

$$(8)\qquad \left\|\sum_{n=1}^{N}\varepsilon_nt_ne_n\right\|_p\leqslant B_p\Big(\sum_{n=1}^{N}t_n^2\Big)^{1/2}\quad(N=1,2,\ldots),$$

where B_p is a constant depending only on p (3).

Since E is reflexive, the basis (e_n) is boundedly complete (see [10] or [5], p. 71), by (8), it follows that the series $\sum_{n=1}^{\infty}\varepsilon_nt_ne_n$ converges. Hence, (e_n) being an unconditional basis, the series $\sum_{n=1}^{\infty}t_ne_n$ is also convergent.

3. COROLLARY 1. *If E is a subspace of L_p $(p>2)$ isomorphic to l_2, then E is complemented in L_p.*

Proof. By 3f there exists a p-lacunary orthonormal system (e_n) such that $[e_n]_p=E$. Put

$$(9)\qquad Px=\sum_{n=1}^{\infty}\Big(\int_0^1 e_n(t)x(t)\,dt\Big)e_n\quad\text{for any } x \text{ in } L_p.$$

(3) This is a consequence of the following result.
Let (r_n) be a sequence in L_p $(p\geqslant 2)$ such that

$$\int_0^1\Big|\sum_{i=1}^{k}r_i(t)\Big|\cdot r_{k+1}(t)\operatorname{sign}\Big(\sum_{i=1}^{k}r_i(t)\Big)\,dt\leqslant 0\quad\text{for}\quad k=1,2,\ldots,N-1.$$

Then $\left\|\sum_{k=1}^{N}r_k\right\|_p\geqslant B_p\big(\sum_{k=1}^{N}\|r_k\|_p^2\big)^{1/2}$ $(N=1,2,\ldots)$, where B_p depends only on p ([12], proof of Theorem 1).

In view of Theorem 3, formula (9) well defines a linear mapping from L_p into E. Since $Pe_n = e_n$ $(n = 1, 2, \ldots)$ and $[e_n]_p = E$, P is the desired projection.

Remark 1. We do not know whether Corollary 1 can be extended to the case where $1 < p < 2$.

No subspace of L_1 isomorphic to l_2 has a complement in L_1 (see e. g. [17], p. 216). The smallest closed manifold spanned in L_1 on Rademacher functions is an example of a non complemented subspace of L_1 isomorphic to l_2.

Corollary 2 ([14], Corollary 3). *Let $p > 2$ and let E be an infinite-dimensional subspace of L_p. Then, either E is isomorphic to l_2, or E contains a subspace isomorphic to l_p and complemented in L_p.*

This immediately follows from Theorems 2 and 3.

Remark 2. Let $1 \leqslant p \neq q < 2$. Then there is a subspace X_q of L_p which is isomorphic to l_q [13]. In view of [1], chap. XII, no subspace of X_q is isomorphic to l_p. This example shows that Theorem 3 and Corollary 2 cannot be extended to the case where $1 \leqslant p < 2$.

Corollary 3. *Let $1 < p < \infty$ and let X be an infinite-dimensional subspace of L_p complemented in L_p. Then, either X is isomorphic to l_2, or X contains a complemented subspace isomorphic to l_p.*

Proof. The case $p = 2$ is trivial. In the case where $p > 2$ we apply Corollary 2. The case where $1 < p < 2$, can be reduced to the preceding one according to a well-known result showing that if X is reflexive, then Y is a complemented subspace of X iff Y^* is a complemented subspace of X^*.

Corollary 4. *Let $p > 2$ and let (x_n) be an unconditional basic sequence in L_p with $0 < \inf_n \|x_n\|_p \leqslant \sup_n \|x_n\|_p < +\infty$. Then (x_n) is equivalent to the unit vector basis in l_2 iff there is an $\varepsilon > 0$ such that x_n is in M_ε^p for $n = 1, 2, \ldots$*

This immediately follows from the analysis of the proof of Theorem 3.

Remark 3. In particular from Corollary 4 we obtain the result of Bari [3] and Gelfand [9], which states that all unconditional bases (e_n) in L_2 with $0 < \inf_n \|e_n\|_2 < \sup_n \|e_n\|_2 < +\infty$ are equivalent.

To prove that consider a subspace of L_p isomorphic to L_2 and use Corollary 4 and condition 3d.

Corollary 5. *Let $p > 2$ and let (x_n) be a sequence in L_p satisfying the following conditions:*

(i) *(x_n) weakly converges to 0,*

(ii) $\limsup_n \|x_n\|_p > 0$.

Then there is a subsequence (x_{n_k}) which is equivalent either (α) to the unit vector basis in l_p, or (β) to the unit vector basis in l_2. Moreover, (β) holds iff

(iii) *there is $\varepsilon > 0$ such that x_n is in M_ε^p for infinite many n.*

Proof. If (iii) is satisfied, then without loss of generality we may assume that all x_n are in M_ε^p for some $\varepsilon > 0$ (ε does not depend on n). Since the space L_p $(p > 1)$ has an unconditional basis [15], by (i), (ii) and a result in [4], p. 56, C1, we may choose an unconditional basic sequence (x_{n_k}) with $0 < \inf_k \|x_{n_k}\|_p \leqslant \sup_k \|x_{n_k}\|_p < +\infty$. Now (β) follows from Corollary 4.

If (iii) does not hold, then we apply Theorem 2.

Corollary 6. *Let $p > 2$ and let (x_n) be a basic sequence in L_p with $0 < \inf_n \|x_n\|_p \leqslant \sup_n \|x_n\|_p < +\infty$. Then there is a basic sequence (x_{n_k}) which is equivalent either (α) to the unit vector basis in l_p, or (β) to the unit vector basis in l_2. Moreover, (β) holds iff condition (iii) is satisfied.*

Proof. Since every bounded basic sequence in any reflexive space weakly converges to 0 [4], this Corollary follows immediately from Corollary 5.

Remark 4. The example considered in Remark 2 shows that Corollaries 5 and 6 cannot be extended to the case where $1 \leqslant p < 2$.

However, in the space l_p $(1 < p < +\infty)$ every basic sequence contains a subsequence equivalent to the unit vector basis in l_p (It may be deduced in the same way as in [4], p. 157, C. 5).

Corollary 7 ([11], p. 246). *Let $p > 2$ and let (x_n) be an orthonormal system such that*

(iv) $\sup_n \|x_n\|_p = C < +\infty$.

Then there exists a p-lacunary subsequence (x_{n_k}).

Proof. The p-lacunarity of a sequence (x_{n_k}) means that (x_{n_k}) is an orthonormal system which is a basic sequence equivalent (under the norm $\|\cdot\|_p$) to the unit vector basis in l_2. Hence, in view of Corollary 5, to complete the proof it is sufficient to show that (i), (ii) and (iii) are satisfied. Since (x_n) is an orthonormal system, $\lim_n \int_0^1 x_n(t) y(t)\, dt = 0$ for every y in L_2. Thus (i) holds because (x_n) is a bounded sequence

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[4] Indeed, let (x_n) with $\sup_n \|x_n\| < +\infty$ be a basis in a reflexive space X and let (x_n^*) be the biorthogonal sequence to (x_n). Since (x_n^*) is a total set of functionals and $\lim_n x_m^*(x_n) = 0$, by the reflexivity of X and the boundedness of (x_n) it follows that (x_n) weakly converges to 0.

in a reflexive space which tends to zero for a dense set of functionals (the class of all square-integrable functions is dense in L_q for $1 \leqslant q \leqslant 2$). Condition (ii) follows from the inequality $\|x_n\|_p \geqslant \|x_n\|_2 = 1$ ($n = 1, 2, \ldots$). Since $\|x_n\|_p \geqslant \|x_n\|_2 = 1 \geqslant C^{-1} \|x_n\|_p$ ($n = 1, 2, \ldots$), we obtain (iii) by 1e.

THEOREM 4. *Let (x_n) be an unconditional basis in L_p $(1 < p < +\infty)$ such that $0 < \inf_n \|x_n\|_p \leqslant \sup_n \|x_n\|_p < +\infty$.*

Then

4a. *every subsequence of (x_n) contains a subsequence which is equivalent either to the unit vector basis in l_p, or to the unit vector basis in l_2,*

4b. *there exists a subsequence (x_{n_k}) which is equivalent to the unit vector basis in l_p.*

Proof. For $p = 2$ it follows from the result of Bari and Gelfand mentioned in Remark 3.

Let $p > 2$. Then 4a follows from Corollary 4. To prove 4b, suppose a contrario that no subsequence of (x_n) is equivalent to the unit vector basis in l_p. Hence, by Theorem 2, there is an $\varepsilon > 0$ such that all x_n are in M_ε^p. Thus, by Theorem 3, implication 3f $\Rightarrow$ 3a, we infer that the space $[x_n]_p = L_p$ is isomorphic to l_2. But it leads to a contradiction with ([1], chap. XII).

The proof in the case where $1 < p < 2$ can be reduced to the preceding one since the space L_q is conjugate to L_p for $q = p/(p-1) > 2$ and in view of the following lemma:

LEMMA 2. *Let (x_n) and (e_n) be unconditional bases in spaces X and E and let (x_n^*) and (e_n^*) be the corresponding biorthogonal sequences in the conjugate spaces X^* and E^* respectively. Then, (n_k) being an increasing sequence of indices, the basic sequences (x_{n_k}) and (e_k) are equivalent iff the basic sequences $(x_{n_k}^*)$ and (e_k^*) are equivalent.*

Proof. Denote by $\hat{x}^*$ the restriction of a functional x^* on X to the space $X_0 = [x_{n_k}]$ and set $\|\hat{x}^*\|_{X_0} = \sup_{0 \neq x \in X_0} |x^*(x)| \, \|x\|^{-1}$. If the basic sequences $(\hat{x}_{n_k})$ and (e_k) are equivalent, then the basic sequences $(\hat{x}_{n_k}^*)$ and (e_k^*) are also equivalent. Since (x_n) is an unconditional basis, $P = \sum_{k=1}^{\infty} x_{n_k}^*(\cdot) x_{n_k}$ is a well-defined projection operator from X onto X_0. Thus, for arbitrary scalars $t_1, t_2, \ldots, t_k$ $(k = 1, 2, \ldots)$, we have

$$\Big\| \sum_{i=1}^{k} t_i x_{n_i}^* \Big\| = \sup_{\|x\| \leqslant 1} \Big| \sum_{i=1}^{\infty} t_i x_{n_i}^*(x) \Big| = \sup_{\|x\| \leqslant 1} \Big| \sum_{i=1}^{k} t_i x_{n_i}^*(Px) \Big| \leqslant \|P\| \, \Big\| \sum_{i=1}^{k} t_i \hat{x}_{n_i}^* \Big\|_{X_0}.$$

On the other hand, $\|x^*\| \geqslant \|\hat{x}^*\|_{X_0}$ for every x^* in X^*. Thus, the basic sequences $(x_{n_k}^*)$ and $(\hat{x}_{n_k}^*)$ are equivalent, and the basic sequences $(x_{n_k}^*)$ and (e_k^*) are also equivalent.

The proof of the converse implication is analogous.

Remark 5. The assumption of Lemma 2 that (x_n) is an unconditional basis is essential. Let $X = E = c$. Consider the basis (x_n) where

$$x_n = \{\xi_i^{(n)}\}, \qquad \xi_i^{(n)} = \begin{cases} 0 & \text{for } i < n, \\ 1 & \text{for } i \geqslant n. \end{cases}$$

The biorthogonal sequence in $c^* = l$ is

$$(x_n^*) = (\{\eta_i^{(n)}\}), \qquad \text{where} \qquad \eta_i^{(1)} = \begin{cases} 1 & \text{for } i = 1, \\ 0 & \text{for } i > 1, \end{cases}$$

and

$$\eta_i^{(n)} = \begin{cases} -1 & \text{for } i = n-1, \\ 1 & \text{for } i = n, \\ 0 & \text{for other } i, \end{cases} \qquad (n = 2, 3, \ldots).$$

Let $n_k = 2k$ $(k = 1, 2, \ldots)$. Then the basic sequence (x_{2k}) is equivalent to the basis (x_n), but the basic sequence (x_{2k}^*) is equivalent to the unit vector basis in l, and thus it is not equivalent to the basis (x_n^*).

COROLLARY 8. *Let $1 < p \neq 2 < +\infty$. Then there is no unconditional basis in the space L_p such that the basic sequence (x_{k_n}) is equivalent to the basis (x_n) for every increasing sequence of indices (k_n).*

Indeed, if it were not so, then according to Theorem 4 the space L_p would be isomorphic to l_p, contrary to [1], chap. XII.

COROLLARY 9. *Let (x_n) with $\|x_n\|_p = 1$ $(n = 1, 2, \ldots)$ be an unconditional basis in L_p. Then $\liminf_n \|x_n\|_2 = 0$, for $p > 2$ ($\limsup_n \|x_n\|_2 = +\infty$, for $1 < p < 2$). In particular ([7], Theorem 2), if an orthonormal system is an unconditional basis in L_p, then $\limsup_n \|x_n\|_p = \infty$ for $p > 2$ ($\liminf_n \|x_n\|_p = 0$ for $1 < p < 2$).*

Proof. In the case where $p > 2$ this follows from 1e and Corollary 4. The case where $1 < p < 2$ reduces to the preceding one by the consideration concerning conjugate space.

Remark 6. The trigonometrical orthogonal system is a basis in L_p for $1 < p < +\infty$ ([20], p. 182). This basis satisfies 4a, by Corollary 7, but does not satisfy 4b for $p \neq 2$. This example shows that in Theorem 4 and Corollary 9 the assumption that the basis is unconditional is essential.

If $p \geqslant 2$ we may prove Corollary 8 without the assumption that the basis is unconditional. Probably this assumption is superflous also for $1 < p < 2$.

Remark 7. Let (χ_n) be the Haar orthonormal system[5]. It is well known that $(\chi_n\|\chi_n\|_p^{-1})$ is an unconditional basis in L_p $(1 < p < +\infty)$ [15]. This basis has the following property: there is only a finite number of functions χ_n belonging to M_ε^p for any $\varepsilon > 0$. Hence no subsequence of (χ_n) is equivalent to the unit vector basis in l_2 for $p \neq 2$. On the other hand, in L_p, for $1 < p \neq 2 < +\infty$, there is an unconditional basis $(\Psi_n^{(p)})$ with $\|\Psi_n^{(p)}\|_p = 1$ containing a subsequence $(\Psi_{n_k}^{(p)})$ equivalent to the unit vector basis l_2 ([17], Theorem 7). Obviously if $1 < p \neq 2 < +\infty$, then no permutation of the basis $(\Psi_n^{(p)})$ is equivalent to the basis $(\chi_n\|\chi_n\|_p^{-1})$.

4. Definition 2. An unconditional basis (x_n) in a B-space X is said to be *permutatively homogeneous* iff it is equivalent to the basis $(x_{p(n)})$ for any permutation $p(\cdot)$ of indices.

Remark 8. The unit vector bases in the spaces l_p $(1 \leqslant p < +\infty)$, c_0 and in Orlicz sequence spaces l_N are permutatively homogeneous. If (x_n) is a permutatively homogeneous basis in a B-space X, then (x_n^*) — the biorthogonal sequence to (x_n), is permutatively homogeneous basis in $[x_n^*] \subset X^*$.

THEOREM 5. *Let (x_n) be a permutatively homogeneous basis in a space X. Then*

5a. $0 < \inf_n \|x_n\| \leqslant \sup_n \|x_n\| < +\infty$,

5b. *the basis (x_n) is equivalent to the basic sequence (x_{k_n}) for any increasing sequence of indices (k_n).*

Proof. 5a. Suppose that $\liminf_n \|x_n\| = 0$ and choose two increasing sequences (k'_n) and (k''_n) such that $k'_i \neq k''_j$ $(i, j = 1, 2, \ldots)$ and $\sum_{n=1}^{\infty} \|x_{k'_n}\|\,\|x_{k''_n}\|^{-1} < +\infty$. Consider a permutation $p(\cdot)$ such that $p(k'_n) = k''_n$ and set

$$t_i = \begin{cases} \|x_{k''_{2n}}\| & \text{for } i = k'_{2n}, \\ 0 & \text{for other } i. \end{cases}$$

Then the bases (x_n) and $(x_{p(n)})$ are not equivalent, because the series $\sum_{n=1}^{\infty} t_n x_n$ converges but the series $\sum_{n=1}^{\infty} t_n x_{p(n)}$ diverges.

The proof that $\limsup_n \|x_n\| < +\infty$ is analogous.

5b. Let (k_n) be an increasing sequence of indices and let (t_n) be a sequence of scalars such that $\sum_{n=1}^{\infty} t_n x_n$ converges. According to 5a,

[5] For the definition and basic properties of the Haar orthogonal system see [11], p. 44.

$\lim t_n = 0$. Hence we may choose an increasing sequence of indices (r_n) so that $|t_{r_m}| < 1/2^m$ $(m = 1, 2, \ldots)$. Let us consider a permutation $p(\cdot)$ such that $p(n) = k_n$ for $n \neq r_m$. Since the basis (x_n) is permutatively homogeneous, the series $\sum_{n=1}^{\infty} t_n x_{p(n)}$ unconditionally converges. Thus the series $\sum_{n \neq r_m} t_n x_{p(n)} = \sum_{n \neq r_m} t_n x_{k_n}$ is unconditionally convergent. Finally since

$$\sum_{n=1}^{\infty} \|t_{r_n} x_{k_{r_n}}\| \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_n \|x_n\| < +\infty$$

by 5a, the series $\sum_{n=1}^{\infty} t_n x_{k_n}$ converges.

The proof that if the series $\sum_{n=1}^{\infty} t_n x_{k_n}$ converges then the series $\sum_{n=1}^{\infty} t_n x_n$ converges is analogous.

Remark 9. Singer [18] has introduced the notion of symmetric basis. Recently [19] he has proved that every symmetric basis is perfectly homogeneous. He has also proved our Theorm 5 and showed that if an unconditional basis satisfies 5b, then it is symmetric. However, the non-unconditional basis considered in Remark 5 satisfies 5a and 5b.

COROLLARY 10. *If a space X has a permutatively homogeneous basis (x_n), then it is isomorphic to its Cartesian square.*

This follows from 5b and the fact that $X = [x_{2n-1}] \times [x_{2n}]$.

COROLLARY 11. *If $p \neq 2$, then in L_p there is no permutatively homogeneous basis.*

This follows from 5b and Corollary 8.

Remark 10. A particular case of Corollary 11 is a result of Gapoškin ([8], Theorem 1) showing that if (χ_n) is the Haar orthonormal system, then there is a permutation $p(\cdot)$ such that the bases $(\chi_n\|\chi_n\|_p^{-1})$ and $(\chi_{p(n)}\cdot\|\chi_{p(n)}\|_p^{-1})$ are not equivalent for $1 < p \neq 2 < +\infty$.

COROLLARY 12. *If the space L_p is isomorphic to an Orlicz sequence space l_N, then $p = 2$.*

This follows from Corollary 11 and the fact that in each Orlicz sequence space the sequence of the unit vectors is a permutatively homogeneous basis.

This Corollary may be generalized to the following

COROLLARY 13. *Let X be a complemented subspace of L_p $(1 < p < +\infty)$ and let (x_p) be a permutatively homogeneous basis in X. Then X is isomorphic either to l_p or to l_2. Moreover, the basis (x_n) is equivalent either to the unit vector basis in l_p or to the unit vector basis in l_2.*

Proof. We shall write $Y_1 \sim Y_2$ iff the spaces Y_1 and Y_2 are isomorphic.

Since X is complemented in L_p, there exists a space Y such that $X \times Y \sim L_p$. Hence, by Corollary 9, we have $L_p \sim X \times Y \sim (X \times X) \times \times Y \sim X \times (X \times Y) \sim X \times L_p$.

Let (y_n) with $\|y_n\|_p = 1$ $(n = 1, 2, \ldots)$ be an unconditional basis in L_p.

Let

$$z_n = \begin{cases} \{x_k, 0\} & \text{for } n = 2k-1 \quad (k = 1, 2, \ldots), \\ \{0, y_k\} & \text{for } n = 2k \quad (k = 1, 2, \ldots). \end{cases}$$

It is easily seen that (z_n) is an unconditional basis in $X \times L_p$. Since $L_p \sim X \times L_p$, by 4a there is a subsequence $(x_{n_i}) = (z_{2n_i-1})$ equivalent either to the unit vector basis in l_p, or to the unit vector basis in l_2. To complete the proof we apply 5b.

5. THEOREM 6. *Let X be a non-reflexive subspace of the space L_1. Then X contains a subspace complemented in L_1 and isomorphic to l_1.*

Proof. Let K be a subset in L_1 and let $0 < \mu \leqslant 1$. We put

$$(10) \qquad \eta(x, \mu) = \sup_{\operatorname{mess} E = \mu} \int_E |x(t)|\,dt \quad \text{for any } x \text{ in } L_1,$$

$$(11) \qquad \eta(K, \mu) = \sup_{x \in K} \eta(x, \mu),$$

$$(12) \qquad (K, +0) = \lim_{\mu \to 0} \eta(K, \mu).$$

It is well known ([1], p. 136) that a set K is weakly compact in L_1 iff $\eta(K, +0) = 0$. Hence, in view of the Eberlein-Smulian theorem ([6], p. 430), if K is the unit ball of a non-reflexive subspace X of L_1, then $\eta(K, +0) = \eta^* > 0$. Thus, by (10)-(12), we may choose positive numbers μ_n, subsets E_n of $[0, 1]$ and x_n in L_1 so that

$$(13) \qquad \eta(x_n, \mu_n) = \eta^*, \qquad \lim_{n \to \infty} \mu_n = 0,$$

$$(14) \qquad \operatorname{mess} E_n = \mu_n, \qquad \int_{E_n} |x_n(t)|\,dt = \mu^*.$$

Let us write

$$(15) \qquad \hat{x}_n(t) = \begin{cases} x_n(t) & \text{for } t \in E_n, \\ 0 & \text{for } t \notin E_n, \end{cases} \qquad (n = 1, 2, \ldots).$$

By (13)-(15) the sequence (x_n) satisfies the assumptions of Theorem 2 and $\|\hat{x}_n\| = \eta^* > 0$ $(n = 1, 2, \ldots)$. Hence we may choose an increasing

sequence of indices (n_i) so that $(\hat{x}_{n_i})$ is a basic sequence equivalent to the unit vector basis in l and the space $[\hat{x}_{n_i}]$ is complemented in L_1.

Let us write

$$(16) \qquad \bar{x}_n = x_n - \hat{x}_n \quad (n = 1, 2, \ldots).$$

By (13)-(16) we have

$$(17) \qquad \eta(\bar{x}_n, \mu) \leqslant \eta(x_n, \mu_n + \mu) - \eta(\hat{x}_n, \mu_n) = \eta(x_n, \mu_n + \mu) - \eta(x_n, \mu_n)$$

for $0 < \mu \leqslant 1$ and $n = 1, 2, \ldots$ Hence

$$(18) \qquad \limsup_n \eta(\bar{x}_n, \mu) \leqslant \eta(K, \mu) - \eta(K, +0) \quad (0 < \mu \leqslant 1).$$

Thus $\eta((\bar{x}_n), +0) = 0$, i.e. the set consisting of elements of the sequence $(\bar{x}_n)$ is weakly compact in L_1. Hence we may assume that the sequence (n_i) is chosen so that the sequence $(\bar{x}_{n_{2i}} - \bar{x}_{n_{2i+1}})$ weakly converges to 0.

By Mazur's theorem ([6], p. 422) there exist linear convex combinations

$$(19) \qquad z_\nu = \sum_{i=k_\nu}^{k_{\nu+1}-1} a_i^{(\nu)} (x_{n_{2i}} - x_{n_{2i+1}}),$$

$$a_i^{(\nu)} \geqslant 0; \quad \sum_{i=k_\nu}^{k_{\nu+1}-1} a_i^{(\nu)} = 1; \quad k_1 < k_2 < \ldots \quad (\nu = 1, 2, \ldots)$$

such that

$$(20) \qquad \lim_\nu \|z_\nu - \hat{z}_\nu\| = \lim_\nu \|\bar{z}_\nu\| = 0,$$

where $\hat{z}_\nu = \sum_{i=k_\nu}^{k_{\nu+1}-1} a_i^{(\nu)} (\hat{x}_{n_{2i}} - \hat{x}_{n_{2i+1}})$ and $\bar{z}_\nu = z_\nu - \hat{z}_\nu$.

By the elementary properties of the unit vector basis in l ([17], Lemma 1) the space $[\hat{z}_\nu] \subset [\hat{x}_{n_i}]$ is isomorphic to l and has a complement in $[\hat{x}_{n_i}]$. Thus, since $[\hat{x}_{n_i}]$ is complemented in L_1, the space $[\hat{z}_\nu]$ is also complemented in L_1. Finally, by [4], Theorems 2 and 3, if we choose (z_ν) so that $\bar{z}_\nu = z_\nu - \hat{z}_\nu$ tends to zero "sufficiently quickly", then the subspace $[z_\nu] \subset X$, as a "translated subspace" with respect to $[\hat{z}_\nu]$, will have the desired properties.

Remark 11. We shall give an alternative proof of a slightly weaker result as Theorem 6.

Lex X be a non-reflexive subspace of L_1. Then the embedding operator $T: X \to L_1$ is not weakly compact. Hence, by a theorem of Gantmacher ([6], p. 485), the conjugate operator $T^*: M \to X^*$ is also not weakly compact. Thus, by [17], Theorem 5, X^* contains a subspace isomorphic to c_0. Finally, by [4], Theorem 4, we conclude that

If X is a non-reflexive subspace of L_1, then X contains a subspace isomorphic to l and complemented in X.

References

[1] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.

[2] — *Sur les séries lacunaires*, Bull. Acad. Polonaise (1933), p. 149-154.

[3] N. K. Bari, *Biorthogonal systems and bases in Hilbert space*, Moskov, Gos. Univ. Uč. Zap. 148, Matematika 4 (1951), p. 69-107 (Russian).

[4] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), p. 151-164.

[5] M. M. Day, *Normed linear spaces*, Berlin 1958.

[6] N. Dunford and J. T. Schwartz, *Linear Operators I*, London 1958.

[7] W. F. Gapoškin, *On unconditional bases in the spaces L_p ($p > 1$)*, Uspehi Mat. Nauk. (N. S.) 13 (1958), p. 179-184 (Russian).

[8] — *On certain property of unconditional bases in the spaces L_p ($p > 1$)*, ibidem 14 (1959), p. 143-148 (Russian).

[9] I. M. Gelfand, *Remark on the work of N. K. Bari "Biorthogonal system and bases in Hilbert spaces"*, Moskov. Gos. Univ. Uč. Zap. 148, Matematika 4 (1951), p. 224-225 (Russian).

[10] R. C. James, *Bases and reflexivity in Banach spaces*, Annals of Math. 52 (1950), p. 518-527.

[11] S. Kaczmarz und H. Steinhaus, *Theorie der Orthogonalreihen*, Warszawa 1935.

[12] M. I. Kadec, *On conditionally convergent series in the space L_p*, Uspehi Mat. Nauk (N. S.) 11 (1954), p. 107-109 (Russian).

[13] — *On linear dimension of the spaces L_p*, Uspehi (N. S.) 13 (1958), p. 95-98 (Russian).

[14] — *On linear dimension of the space L_p ($p > 2$)*, Nauč. Dokl. Vyš. Školy 2 (1958), p. 104-107 (Russian).

[15] J. Marcinkiewicz, *Quelques théorèmes sur les séries orthogonales*, Ann. Soc. Polon. Math. 7 (1938), p. 51-56.

[16] W. Orlicz, *Über unbedingte Konvergenz in Funktionenräumen I*, Studia Math. 4 (1933), p. 33-37.

[17] A. Pełczyński, *Projections in certain Banach spaces*, ibidem 19 (1960), p. 209-228.

[18] I. Singer, *On Banach spaces with symmetric basis*, Revue de Mathématiques Pures et Appliquées 7 (1961), p. 159-166 (in Russian).

[19] — *Some characterizations of symmetric bases*, Bull. Acad. Pol Sci., Série math., astr. et phys., 10 (1962), in print.

[20] A. Zygmund, *Trigonometrical Series, I*, Cambridge 1959.

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Mercerian theorems and inverse transformations

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1. A sequence-to-sequence summability method defined by a matrix A is called a U-method for bounded sequences if the A-transform of every non-zero bounded sequence is non-zero ([6], p. 132). Let A be the matrix of a conservative (i. e. convergence-preserving) sequence-to-sequence method which is a U-method for bounded sequences. It will be shown that A sums no bounded divergent sequence if and only if there exists a conservative matrix B which is a left reciprocal of A, or equivalently, if and only if there exists a matrix $C = (c_{n,k})$ which is a left reciprocal of A and which satisfies

$$\sup_n \sum_{k=1}^{\infty} |c_{n,k}| < \infty.$$

The hypothesis that the method is a U-method for bounded sequences may be omitted if the matrices B, C mentioned above satisfy $BA = I+P$, $CA = I+P$ instead of $BA = I$, $CA = I$, where

$$I = (\delta_{n,k}), \quad \delta_{n,n} = 1, \quad \delta_{n,k} = 0 \quad (k \neq n),$$

and P is a "trivial" conservative matrix $(p_{n,k})$ such that

$$p_{n,k} = 0 \quad (k \geqslant k_0,\ n = 1, 2, \ldots).$$

Parallel results are proved for certain classes of sequence-to-function methods, where the matrix C which occurs in the results stated above is replaced by a sequence $\{g_n\}$ of functions of bounded variation, with

$$\sup_n \operatorname{var} g_n < \infty.$$

These results depend upon a theorem on the existence of extensions of certain linear operators on subspaces of separable Banach spaces. Theorem 1 is the extension theorem, in a form more general than is required for the applications made here, as it may be of independent interest. A special case of the theorem was suggested by a remark of Zeller [11].