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On spaces of holomorphic functions

by

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In this paper we consider locally convex metrisable spaces whose elements are analytic functions of several variables. We develop Kolmogorov's ideas [14], which concern the existence of continuous linear mappings at such spaces. In particular, we consider the following problem: For which domains D_1 and D_2 in Euclidean complex spaces of dimensions k_1 and k_2, respectively, the spaces $\mathscr{H}(D)$ and $\mathscr{H}(D_1)$ — of all holomorphic ([1]) functions defined on these domains — are isomorphic (linearly homeomorphic).

Our results (Corollaries 4.1, 4.2, and 4.3) contain as particular cases a result of Kolmogorov which states that if D_1 and D_2 are polycylinders and $\dim D_1 \neq \dim D_2$, then the spaces $\mathscr{H}(D_1)$ and $\mathscr{H}(D_2)$ are not isomorphic, and a result of Pełczyński [21] stating that the spaces of all entire functions of one variable and $\mathscr{H}(C_0)$ — of all holomorphic functions defined for $|z| < 1$ — are not isomorphic.

We also consider the spaces of all holomorphic functions having a given degree of growth.

Similarly to Kolmogorov [14] and Pełczyński [21], in order to establish that given spaces are not isomorphic we compute the so-called "approximative dimension", which is an invariant of linear homeomorphisms. Our method of computing approximative dimension is different from that of Kolmogorov-Tichomirov [15] and Erochin [9]. It is based on finding some matrix representations of a given space by basic expansions. A matrix representation determines the isomorphic structure of the space, in particular it determines its approximative dimension. Matrix representations of some spaces of holomorphic functions are found in § 3 and in § 5, section 2. Examples given in § 5 show that non-isomorphic spaces can have the same approximative dimension. However, under the assumption that spaces have matrix representations of a special form, the equality of approximative dimensions implies the isomorphism of the spaces (Theorem 5.3).

([1]) i. e. analytic and one-valued functions.

In many cases the matrix representation is obtained by replacing every function of a given space by its Taylor coefficients. This allows us to find estimations of Taylor coefficients of holomorphic functions for some functional classes. Examples of such estimations are given in § 6.

Most of the results of this paper were announced in [23] and [24]. Recently some results have been obtained independently by Aizenberg and Mitiagin [1], [2].

The results of § 1, section 3, and § 5 were obtained in collaboration with C. Bessaga and A. Pełczyński.

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§ 1. Preliminaries

1. B_0-spaces; bases. In the sequel we shall consider only B_0-spaces, i. e. locally convex complete metric linear spaces. For the basic properties of such spaces see Mazur and Orlicz [18] or Bourbaki's monograph [6], in which they are called F-spaces. Now we recall only that the topology in a B_0-space X can be given by a non-decreasing sequence of pseudonorms $(\|\cdot\|_\alpha)$, $\alpha = 1, 2, \ldots$

Linear topological spaces X and Y are said to be *isomorphic* (written $X \approx Y$) if and only if there exists a linear homeomorphic mapping from X onto Y. The symbol $X \not\approx Y$ will denote that the spaces X and Y are not isomorphic.

Let (X_j) be a sequence of B_0-spaces whose topologies are given by sequences of pseudonorms $(\|\cdot\|_{\alpha j})$, $j = 1, 2, \ldots$, respectively. The space $\mathfrak{X}$ whose elements are sequences (x_j), x_j being in X_j, with the topology determined by the pseudonorms $\|(x_j)\|_\alpha = \sup\{\|x_i\|_{\beta i}\colon i, \beta \leqslant \alpha\}$ will be called a *product in sense s* of spaces X_j and denoted by the symbol $(X_1 \times X_2 \times \ldots)_s$.

A series $\sum_{n=1}^{\infty} x_n$ of elements of a B_0-space X is said to be *absolutely convergent* if and only if $\sum_{n=1}^{\infty} \|x_n\| < \infty$ for every continuous pseudonorm $\|\cdot\|$ defined on X.

A sequence (e_n) with $e_n \in X$ is called a *basis* (an *absolute basis*) of the space X if and only if every vector x in X may be uniquely represented as the sum of a convergent (absolute convergent) series:

$$x = \sum_{n=1}^{\infty} t_n e_n \quad (t_n\text{—scalars}).$$

2. Köthe spaces. We shall consider a class of "Stufenräume" introduced by Köthe. Let $\mathcal{N}^k$ denote the set of all systems of k non-negative integers. Suppose we are given a $(k+1)$-dimensional matrix of numbers $a_{\alpha\boldsymbol{n}}$, where $\alpha = 1, 2, \ldots$, $\boldsymbol{n} \in \mathcal{N}^k$, such that

(11) $$a_{\alpha\boldsymbol{n}} \geqslant 0, \quad \sup_\alpha a_{\alpha\boldsymbol{n}} > 0, \quad a_{\alpha\boldsymbol{n}} \leqslant a_{\beta\boldsymbol{n}} \quad \text{for} \quad \alpha \leqslant \beta.$$

A *Köthe space* $\mathcal{M}(c_{\alpha\boldsymbol{n}})$ is the space of all (k-fold) sequences $\xi = (\xi_{\boldsymbol{n}})$ of complex numbers such that

$$\|\xi\|_\alpha = \sup_{\boldsymbol{n}} a_{\alpha\boldsymbol{n}} |\xi_{\boldsymbol{n}}| < \infty \quad (\alpha = 1, 2, \ldots).$$

Addition and multiplication by scalars in this space is defined as usual; the topology is given by pseudonorms $\|\cdot\|_\alpha$.

In the case of $k = 1$ we shall write n instead of $\boldsymbol{n}$; spaces $\mathcal{M}(a_{\alpha n})$ will be called *single Köthe spaces*.

If $X = \mathcal{M}(a_{\alpha\boldsymbol{n}})$, we shall say that X has a matrix representation $\mathcal{M}(a_{\alpha n})$.

THEOREM 1.1 [22]. *Let X be a B_0-space with topology determined by a non-decreasing sequence of pseudonorms $(\|.\|_\alpha)$ and with a basis (e_n). Let $a_{\alpha n} = \|e_n\|_\alpha$. If*

(12) *for every α there exists a β such that $\sum_{n=1}^{\infty} a_{\alpha n}/a_{\beta n} < \infty$ (here we understand $0/0 = 0$),*

then $x = \mathcal{M}(a_{\alpha n})$.

For the proof see [4].

A B_0-space X with a basis satisfying condition (12) is said to be *nuclear* (²).

Remark 1.1. It follows immediately from Theorem 1.1 that if a set of vectors $(e_{\boldsymbol{n}})$ can be reordered in such a way as to constitute a basis of X and if for every α there is a β such that $\sum_{\boldsymbol{n}\in\mathcal{N}^k} \|e_{\boldsymbol{n}}\|_\alpha/\|e_{\boldsymbol{n}}\|_\beta < \infty$, then $X \approx \mathcal{M}(\|e_{\boldsymbol{n}}\|_\alpha)$.

3. Approximative dimension. Let A, B be subsets of a linear space X. By $M(A, B, \varepsilon)$ we shall denote the maximal number n of such points $x_1, \ldots, x_n$ in A that $x_i - x_j \notin 2\varepsilon B$ for $i \neq j$. $M(A, B)$ will denote the class of all non-negative functions $\varphi(\varepsilon)$ defined for $\varepsilon > 0$ such that for sufficiently small ε

$$\varphi(\varepsilon) \geqslant M(A, B, \varepsilon).$$

Now let us suppose that X is a linear topological space. Let $\mathfrak{A}$ be the class of all open sets in X and $\mathfrak{H}$ — the class of all bounded sets in this space. The class of functions:

$$\Phi(X) = \bigcap_{U\in\mathfrak{A}} \bigcap_{B\in\mathfrak{H}} M(B, U)$$

is called the *approximative dimension* of space X (Kolmogorov [14]).

(²) For general definition of nuclear spaces see [11].

Pełczyński's considerations [21] lead to the distinction of another class, namely:

(13) $$\bigcap_{V\in\mathfrak{A}} \bigcup_{U\in\mathfrak{A}} M(U, V).$$

In the case of B_0-spaces both classes are identical [22].

We shall say that the approximative dimension of space X is equal to, less than or equal to, and less than that of Y (written $d_a X = d_a Y$, $d_a X \leqslant d_a Y$, $d_a Y < d_a Y$) if $\Phi(X) = \Phi(Y)$, $\Phi(X) \supset \Phi(Y)$, and $\Phi(X) \supsetneqq \Phi(Y)$, respectively.

It is easily seen that *the approximative dimension is an isomorphic invariant of spaces; moreover if Y is a subspace or a linear (continuous) image of X, then $d_a Y \leqslant d_a X$.*

In the case of B_0-spaces it is more convenient to deal with formula (13): instead of the family $\mathfrak{A}$ one may take an arbitrary basis (U_n) of neighbourhoods of zero.

Let $X = M(a_{\alpha n})$, with $n\in\mathcal{N}^k$, be a nuclear Köthe space and let

$$M_{\alpha\beta}(X; \varepsilon) = \prod_{n\in\mathcal{N}^k} \mathrm{E}\big(1 + a_{\alpha n}/(\varepsilon a_{\beta n})\big) \text{ (}^3\text{)}.$$

Denote by $M_{\alpha\beta}(X)$ the class of all non-negative functions $\varphi(\varepsilon)$ defined for $\varepsilon > 0$ such that $\varphi(\varepsilon) \geqslant M_{\alpha\beta}(\varepsilon)$ for sufficiently small ε.

THEOREM 1.2. *If $X = \mathcal{M}(a_{\alpha,n})$, with $n\in\mathcal{N}^k$, is a nuclear Köthe space, then $\Phi(X) = \bigcap_\alpha \bigcup_\beta M_{\alpha\beta}(X)$* [4].

THEOREM 1.3 [5]. *Let matrices $a_{\alpha n}^{(j)}$ $(j = 1, \ldots, k)$ satisfy condition* (12) *and let*

$$a_{\alpha n} = a_{\alpha n_1}^{(1)} \cdot \ldots \cdot a_{\alpha n_k}^{(k)}; \qquad N_{\alpha\beta j}(\varepsilon) = \overline{\overline{\{n: a_{\beta n}^{(j)}/a_{\alpha n}^{(j)} \geqslant \varepsilon\}}} \text{ (}^6\text{)}.$$

Then

$$\frac{1}{2}\log\frac{1}{\varepsilon}\prod_{j=1}^{k} N_{\alpha\beta j}(\sqrt[2k]{\varepsilon}) \leqslant \log M_{\alpha\beta}\big(\mathcal{M}(a_{\alpha n}); \varepsilon\big) \leqslant \log\left(1+\frac{1}{\varepsilon}\right)\prod_{j=1}^{k} N_{\alpha\beta j}(\varepsilon).$$

Proof. Write

$$P_{\alpha\beta}(\varepsilon) = \left\{n: a_{\alpha n}^{(1)}/a_{\beta n}^{(1)} \geqslant \frac{1}{\varepsilon}\right\} \times \ldots \times \left\{n: a_{\alpha n}^{(k)}/a_{\beta n}^{(k)} \geqslant \frac{1}{\varepsilon}\right\}.$$

Right-hand inequality. We have

$$\prod_{n\in\mathcal{N}^k} \mathrm{E}\big(1 + a_{\alpha n}/(\varepsilon a_{\beta n})\big) = \prod_{n\in P_{\alpha\beta}(\varepsilon)} \mathrm{E}\big(1 + a_{\alpha n}/(\varepsilon a_{\beta n})\big) \leqslant \left(1+\frac{1}{\varepsilon}\right)^{\overline{\overline{P_{\alpha\beta}(\varepsilon)}}}.$$

Since $\overline{\overline{P_{\alpha\beta}(\varepsilon)}} = \prod_{j=1}^{k} N_{\alpha\beta j}(\varepsilon)$, we obtain the required inequality.

Left-hand inequality. Since $\mathrm{E}\big(1 + a_{\alpha n}/(\varepsilon a_{\beta n})\big) \geqslant 1/\sqrt{\varepsilon}$ for $n\in P_{\alpha\beta}(\sqrt[2k]{\varepsilon})$, we have

$$\prod_{n\in\mathcal{N}^k} \mathrm{E}\big(1 + a_{\alpha n}/(\varepsilon a_{\beta n})\big) \geqslant \prod_{n\in P_{\alpha\beta}(\sqrt[2k]{\varepsilon})} \mathrm{E}\big(1 + a_{\alpha,n}/(\varepsilon a_{\beta n})\big) \geqslant (1/\sqrt{\varepsilon})^{\overline{\overline{P_{\alpha\beta}(\sqrt[2k]{\varepsilon})}}}.$$

THEOREM 1.4. *Let matrices $(a_{\alpha p})$, $(b_{\alpha q})$ satisfy condition* (12), *where $p = (n_1, \ldots, n_r)\in\mathcal{N}^r$, $q = (n_{r+1}, \ldots, n_k)\in\mathcal{N}^{k-r}$, and let $c_{\alpha n} = a_{\alpha p}\cdot b_{\alpha q}$, where $n = (n_1, \ldots, n_k)\in\mathcal{N}^k$. Then*

$$M_{\alpha\beta}\big(\mathcal{M}(c_{\alpha n}); \varepsilon\big) = \prod_{q\in\mathcal{N}^{k-r}} M_{\alpha\beta}\big(\mathcal{M}(a_{\alpha p}); \varepsilon b_{\beta q}/b_{\alpha q}\big).$$

THEOREM 1.5. *Let X_n be a sequence of B_0-spaces. If $d_a(X_i\times X_j) = d_a X_1$, then $d_a(X_1\times X_2\times\ldots)_s = d_a X_1$. In particular, if $d_a X = d_a(X\times X)$, then $d_a X = d_a(X\hat{\otimes} s)$. (Symbol $X\times Y$ denotes the Cartesian product of spaces X and Y).*

4. Spaces $\mathcal{H}_\mu(D)$. Let D be a domain in a k-dimensional Euclidean complex space. Let $\mu(\varepsilon, z)$ be a non-negative function defined for $0 < \varepsilon < 1$, $z\in D$, non-increasing with respect to ε and such that $\lim_{\varepsilon\to 0}\mu(\varepsilon, z) > 0$, for every z in D. By $\mathcal{H}_\mu(D)$ we shall denote the space of all holomorphic functions $x = x(z)$ defined on D and such that

(14) $$\|x\|_\varepsilon = \sup_{z\in D}|x(z)|\,\mu(\varepsilon, z) < +\infty$$

with the topology induced by the pseudonorms $\|\cdot\|_\varepsilon$.

Since $\mu(\varepsilon, z)$ is non-increasing with respect to ε, the topology of the space $\mathcal{H}_\mu(D)$ may be given by the sequence of pseudonorms $\|.\|_{1/n}$. Hence $\mathcal{H}_\mu(D)$ is a B_0-space.

If A_ε $(0 < \varepsilon < 1)$ is a non-increasing family of compact sets such that $\bigcup_\varepsilon A_\varepsilon = D$ and $\mu(\varepsilon, z) = \chi_A(z)$ (the characteristic function of the set A), then $\mathcal{H}_\mu(D)$ is the space of all holomorphic functions defined on D with the topology of almost uniform convergence. This space will be briefly denoted by $\mathcal{H}(D)$.

If D is the whole k-dimensional Euclidean complex space, then instead of $\mathcal{H}_\mu(D)$ we shall write $\mathcal{H}_\mu$.

The symbols C and C_0 will be reserved further for denoting the whole complex plane and the interior of the unit circle; $\bar{C}$ will denote

(3) $\mathrm{E}a$ denotes the greatest integer in a.

(4) Proofs of theorems 1.2, 1.4, and 1.5 will be published in Studia Math. in a joint paper with C. Bessaga and A. Pełczyński.

(5) cf. Mitiagin [19], theorem 1.

(6) The symbol $\overline{\overline{A}}$ denotes the cardinality (number of elements) of the set A.

the extended complex plane, i. e. $\bar{C} = C + \{\infty\}$. C^r (resp. C_0^r) will denote the Cartesian product of r copies of C (resp. C_0).

THEOREM 1.6. *Suppose we are given a space $\mathscr{H}_\mu = \mathscr{H}_\mu(C^k)$ such that the function $\mu(\varepsilon, z)$ depends only on $|z_1|, \ldots, |z_k|$, and let $z^{\boldsymbol{n}} = z_1^{n_1} \ldots z_k^{n_k}$, for $\boldsymbol{n} \in \mathscr{N}^k$. If for every $0 < \varepsilon < 1$ there exists an ε' such that*

(15) $$\sum_{\boldsymbol{n} \in \mathscr{N}^k} |z^{\boldsymbol{n}-1}|_\varepsilon / |z^{\boldsymbol{n}}|_{\varepsilon'} < \infty,$$

then the mononomials $z^{\boldsymbol{n}}$ constitute a basis of $\mathscr{H}_\mu$. The space $\mathscr{H}_\mu$ has a matrix representation $M(|z^{\boldsymbol{n}}|_{\varepsilon_a})$, where (ε_a) is an arbitrary sequence such that $0 < \varepsilon_a < 1$, $\lim_{a\to\infty} \varepsilon_a = 0$.

Proof. Every entire function, in particular every function from $\mathscr{H}_\mu$, can be uniquely represented as the sum of the series

(16) $$x(z) = \sum_{\boldsymbol{n} \in \mathscr{N}^k} c_{\boldsymbol{n}} z^{\boldsymbol{n}},$$

which is absolutely and almost uniformly convergent ([10], p. 74).

By Cauchy formula ([10], p. 24) we obtain $|c_{\boldsymbol{n}-1}| \leqslant (1/r^{\boldsymbol{n}}) \max_{|z_i|=r_i} |x(z)|$, where $r^{\boldsymbol{n}} = r_1^{n_1} \ldots r_k^{n_k}$, for every $\boldsymbol{n}$ in $\mathscr{N}^k$. Therefore

$$|c_{\boldsymbol{n}-1} z^{\boldsymbol{n}}| \leqslant \max\{|x(\xi)| : \xi = (\xi_1, \ldots, \xi_k), |\xi_i| = |z_i|\}.$$

Hence

$$\mu(\varepsilon, z) |c_{\boldsymbol{n}-1} z^{\boldsymbol{n}}| \leqslant \mu(\varepsilon, z) \max_{|\xi_i|=|z_i|} |x(\xi_i)|.$$

Now, the assumption that $\mu(\varepsilon, z)$ depends on $|z_1|, \ldots, |z_k|$ gives us

$$\|c_{\boldsymbol{n}-1} z^{\boldsymbol{n}}\|_\varepsilon \leqslant \|x\|_\varepsilon \quad \text{for every} \quad \boldsymbol{n} \in \mathscr{N}^k.$$

From this inequality we deduce by (15) that the series (16) is convergent in the topology of the space $\mathscr{H}_\mu$. Since, by (14), convergence in $\mathscr{H}_\mu$ implies uniform convergence, expansion (16) is unique also in the topology of $\mathscr{H}_\mu$, i. e. $(z^{\boldsymbol{n}})$ is a basis of $\mathscr{H}_\mu$.

The matrix representation of $\mathscr{H}_\mu$ follows from formula (15) and Theorem 1.1.

5. Tensor products. Let X and Y be B_0-spaces, X^* and Y^* — their conjugate spaces. The space of all bilinear forms defined on the Cartesian product $X^* \times Y^*$ is called the *tensor product* of spaces X and Y (written $X \hat{\hat{\otimes}} Y$) [11].

Further we shall make use of the following three facts concerning tensor products:

(17) *If $X_1 \approx X$, $Y_1 \approx Y$, then $X \hat{\hat{\otimes}} Y \approx X_1 \hat{\hat{\otimes}} Y_1$, $X \hat{\hat{\otimes}} (Y \times Z) \approx \approx (X \hat{\hat{\otimes}} Y) \times (X \hat{\hat{\otimes}} Z)$;*

(18) *If D_1 and D_2 are open sets in C^k and C^{k_1} respectively, then $\mathscr{H}(D_1 \times D_2) \approx \approx \mathscr{H}(D_1) \hat{\hat{\otimes}} \mathscr{H}(D_2)$* ([7]);

(19) *If $X \approx M(a_{ap})$, $Y \approx M(b_{aq})$, with $\boldsymbol{p} = (n_1, \ldots, n_r) \in \mathscr{N}^r$, $\boldsymbol{q} = (n_{r+1}, \ldots, n_k) \in \mathscr{N}^{k-r}$, are nuclear Köthe spaces and $c_{an} = a_{ap} b_{aq}$ for $\boldsymbol{n} = (\boldsymbol{p}, \boldsymbol{q}) = (n_1, \ldots, n_k) \in \mathscr{N}^k$, then $X \hat{\hat{\otimes}} Y \approx \mathscr{M}(c_{an})$.*

§ 2. Some isomorphic relations between spaces $\mathscr{H}(D)$

LEMMA 2.1. *Suppose we are given an analytic transformation of an open plane set D_1 onto an open plane set D_2. Then the space $\mathscr{H}(D_2)$ is isomorphic to a subspace of $\mathscr{H}(D_1)$.*

Proof. Let

$$(Tx)(z) = x(\varphi(z)).$$

We shall show that T is an isomorphism mapping of $\mathscr{H}(D_2)$ into $\mathscr{H}(D_1)$. Obviously T is a linear one-to-one mapping. To show that it is continuous let us choose $y_n \in T(\mathscr{H}(D_2))$ $(n = 1, 2, \ldots)$ such that $y_n \to 0$. Let $x_n = Ty_n$ $(n = 1, 2, \ldots)$. Suppose that $x_n \not\to 0$. Then there is a compact set $K_2 \subset D_2$ such that $\overline{\lim_n} \|x_n\|_{K_2} = \overline{\lim_n} \sup_{w \in K_2} |x_n(w)| > 0$. Since for every compact $K_2 \subset D_2$ there is a compact set $K_1 \subset D_1$ such that $\Phi(K_1) \supset K_2$, we have $\overline{\lim} \|y_n\|_{K_1} \geqslant \overline{\lim} \sup_{w \in \Phi^{-1}(K_1)} |x_n(w)| \geqslant \overline{\lim} \|x_n\|_{K_2} > 0$, which leads to a contradiction.

LEMMA 2.2. *Let D be an arbitrary k-dimensional domain. Then the space $\mathscr{H}(C \times D)$ (resp. the space $\mathscr{H}(C_0 \times D)$) is isomorphic to its cartesian square.*

Proof. First we note that the space $\mathscr{H}(C)$ is isomorphic to its Cartesian square. Let us put $hx = (x_1, x_2)$, where

$$x_1(z) = \frac{x(z) + x(-z)}{2}, \quad x_2(z) = \frac{x(z) - x(-z)}{2z},$$

for any $x \in \mathscr{H}(C)$. It is easily seen that h is the required isomorphism. In the general case, according to (17) and (18), we have

$$\mathscr{H}(C \times D) \approx \mathscr{H}(C) \otimes \mathscr{H}(D) = (\mathscr{H}(C) \times \mathscr{H}(C)) \otimes \mathscr{H}(D)$$
$$\approx (\mathscr{H}(C) \otimes \mathscr{H}(D)) \times \mathscr{H}(C) \otimes \mathscr{H}(D) \approx \mathscr{H}(C \otimes D) \times \mathscr{H}(C \otimes D).$$

The proof for the space $\mathscr{H}(C_0 \times D)$ is analogous.

([7]) This formula follows from the fact that every function $x(z)$ in $\mathscr{H}(D_1 \times D_2)$ can be almost uniformly approximated by sums of functions of form $x_1(z_1) \cdot x_2(z_2)$ with $z_1 \in D_1$, $z_2 \in D_2$ and from the example 1 in [11], p. 89-90.

THEOREM 2.1. *Let D be a finite connected domain in C. Let $Z_1, Z_2, \ldots, Z_m$ be components of the complement of D. Then*

1° $\mathscr{H}(D) \approx \mathscr{H}(C)$ *if all Z_i are points,*

2° $\mathscr{H}(D) \approx \mathscr{H}(C_0)$ *if all Z_i are continua,*

3° $\mathscr{H}(D) \approx \mathscr{H}(C_0)\times\mathscr{H}(C)$ *if among Z_i there are points and continua.*

Proof. For $m = 1$ our assertion is obvious. Let us suppose that it is true for an l-conected domain. Let D be an m-conected domain. According to Riemann theorem on conform mappings we may suppose that the component Z_m of the complement of D is (α) either the point set $\{\infty\}$ or (β) the set $\{z: |z| \geqslant 1\}$. Then in both cases there is a positive number r such that

$$x(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n} \quad \text{for} \quad x \in \mathscr{H}(D)$$

and for $|z| > r$ in the case (α) ($1-1/r < |z| < 1$ in the case (β)). It is easy to see that the correspondence $x \leftrightarrow (x_1, x_2)$, where

$$x_1(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad x_2(z) = \sum_{n=1}^{\infty} a_n z^{n-1},$$

is an isomorphism between $\mathscr{H}(D)$ and $\mathscr{H}(C)\times\mathscr{H}(D\cup Z_m)$ (between $\mathscr{H}(D)$ and $\mathscr{H}(C_0)\times\mathscr{H}(D\cup Z_m)$).

Since the domain $D\cup Z_m$ is $m-1$ connected, by the induction hypothesis and lemma 2.2 we obtain the assertion of our theorem, q. e. d.

CONJECTURE. *If D is an arbitrary finite connected domain in C, then $\mathscr{H}(D)$ is one of the standard forms: $\mathscr{H}(C)$, $\mathscr{H}(C_0)$, $\mathscr{H}(C\times C_0)$.*

THEOREM 2.2. *Let $D_1, \ldots, D_k$ be a family of finite connected plane domains. Suppose that all components of each of the spaces $C-D_i$ are points for $i = 1, \ldots, r$, continua for $i = r+1, \ldots, r+p$, points and continua for $i = r+p+1, \ldots, k$, where $1 \leqslant r \leqslant r+p \leqslant k$. Then $\mathscr{H}(D_1\times\ldots\times D_k)$ is isomorphic to the space*

$$\mathscr{H}(C^r\times C_0^{k-r})\times\mathscr{H}(C^{r+1}\times C_0^{k-r-1})\times\ldots\times\mathscr{H}(C^{k-p}\times C_0^{p}).$$

This follows immediately from theorem 2.1 and formulas (17) and (18).

Let Ω be an algebra with two operations, $+$ and $\cdot$, and one relation of "equality", defined by the following axioms:

I. Ω has two generators u, v (in other words, elements of Ω are sums of monomials $u^p v^q$ with $p, q \geqslant 0$, $p+q > 0$, where $u^p = \underbrace{u\ldots u}_{p\text{ times}}$, $v^q = \underbrace{v\ldots v}_{q\text{ times}}$).

II. $\omega_1+\omega_2 = \omega_2+\omega_1$, $\omega_1\cdot\omega_2 = \omega_2\cdot\omega_1$, $\omega_1+(\omega_2+\omega_3) = (\omega_1+\omega_2)+\omega_3$, $\omega_1\cdot(\omega_2\cdot\omega_3) = (\omega_1\cdot\omega_2)\cdot\omega_3$, $\omega_1\cdot(\omega_2+\omega_3) = \omega_1\cdot\omega_2+\omega_1\cdot\omega_3$.

III. If $p' \leqslant p$, $q' \leqslant q$, then $u^p v^q + u^{p'}v^{q'} = u^p v^q$.

Let $\omega = u^{p_1}v^{q_1}+\ldots+u^{p_n}v^{q_n}$. Denote by $\omega(X, Y)$ the space constructed from the ω by substitution $\begin{pmatrix} \cdot & + & u & v \\ \times & \hat{\hat{\otimes}} & \mathscr{H}(C) & \mathscr{H}(C_0)\end{pmatrix}$.

From theorems 2.1, 3.1 and formulas (17) and (18) follows

THEOREM 2.3. *If $\omega_1 = \omega_2$, then $\omega_1(X, Y) \approx \omega_2(X, Y)$.*

Problem 2.1. Does the relation $\omega_1(X, Y) = \omega_2(X, Y)$ imply $\omega_1 = \omega_2$?

Using theorems 2.1, 4.1, results of § 5, and formulas (17), (18) and (19) we obtain the following statements:

1° if degree $\omega_1 \neq$ degree ω_2, then $\omega_1(X, Y) \neq \omega_2(X, Y)$;

2° if ω_1 depends on one variable u, ω_2 depends on v, and ω_3 is not equal to any element of Ω depending only on one variable, then the spaces $\omega_1(X, Y)$, $\omega_2(X, Y)$, $\omega_3(X, Y)$ are not isomorphic. In particular $\mathscr{H}(C)$, $\mathscr{H}(C_0\times C)$ and $\mathscr{H}(C_0)$ are not isomorphic.

THEOREM 2.4. *Let $D = D_1\times D_2\times\ldots\times D_k$ be an arbitrary polycylinder. Then $\mathscr{H}(D)$ is isomorphic to a subspace of $\mathscr{H}(C_0^k)$.*

Proof. According to (17) and (18) it is sufficient to restrict our attention to the case where D is one-dimensional. By theorem 2.1 we may assume without loss of generality that $\bar{C}-D$ contains at least three points, for instance $\{0\}$, $\{1\}$, $\{\infty\}$. Now by a result of Poincaré ([13], p. 275) there is an analytic function φ which transforms D onto C_0. To complete the proof we apply lemma 2.2.

THEOREM 2.5. *Let $D = D_1\times D_2\times\ldots\times D_k$ be a bounded polycylinder. Then $\mathscr{H}(D)$ contains a subspace isomorphic to $\mathscr{H}(C_0^k)$.*

Proof. As in the proof of Theorem 2.4, it is enough to restrict our attention to the one-dimension case. D^* denotes the complement of the unbounded component of the complement of D. Let $r: \mathscr{H}(D^*) \to \mathscr{H}(D)$ be the restriction of functions in $\mathscr{H}(D^*)$ to functions in $\mathscr{H}(D)$, i. e. $rx(z) = x(z)$ for $z \in D$ and $x \in \mathscr{H}(D^*)$. Obviously r is a linear one-to-one operator and according to the maximum principle r^{-1} is continuous. Hence $\mathscr{H}(D^*)$ is isomorphic to a subspace of $\mathscr{H}(D)$. Since D^*, is a bounded domain, $\mathscr{H}(D^*)$ is isomorphic to $\mathscr{H}(C_0)$, q. e. d.

From theorems 2.4 and 2.5 we obtain

COROLLARY 2.1. *If $D = D_1\times\ldots\times D_k$ is a bounded polycylinder, then the spaces $\mathscr{H}(D)$ and $\mathscr{H}(C_0^k)$ have the same linear dimension in the sense of Banach* ([3], p. 193).

THEOREM 2.6. *Let D_0 be an arbitrary k-dimensional bounded domain. Then, for every k-dimensional domain D, $\mathscr{H}(D)$ is isomorphic to a subspace of the space $(\mathscr{H}(D)\times\mathscr{H}(D)\times\ldots)_s$.*

Proof. There exist points $z^{(n)}$ in C^k and positive numbers s_n such that

$$D = \bigcup_{n=1}^{\infty} D_n, \quad \text{where} \quad D_n = \left\{z \in C^k: \frac{z-z^{(n)}}{s^{(n)}} \in D_0\right\}.$$

Obviously all the spaces $\mathscr{H}(D_n)$ are isomorphic to $\mathscr{H}(D)$. Let h_n denote an isomorphism from $\mathscr{H}(D_n)$ onto $\mathscr{H}(D_0)$ and let $r_n: \mathscr{H}(D) \to \mathscr{H}(D_n)$ be the operation of restriction of functions in $\mathscr{H}(D)$ to functions in $\mathscr{H}(D_n)$. Let us put $hx = (h_n r_n x)$ for $x \in \mathscr{H}(D)$. We omit the asy checking that h is the required isomorphism.

Corollary 2.2. *Let D be an arbitrary k-dimensional domain; then $\mathscr{H}(D)$ is isomorphic to a subspace of the space $(\mathscr{H}(C_0^k)\times\mathscr{H}(C_0)\times\ldots)_s$.*

§ 3. Matrix representations of spaces $\mathscr{H}_\mu(D)$

1. Matrix form $\mathscr{M}\Big(\exp\big(\alpha(n_1^{q_1}+\ldots+n_r^{q_r})-\frac{1}{\alpha}(n_{r+1}^{q_{r+1}}+\ldots+n_k^{q_k})\big)\Big)$.

Now we prove

Theorem 3.1. *We have*

$$\mathscr{H}(C^r\times C_0^{k-r}) \approx \mathscr{M}\Big(\exp\big(\alpha(n_1+\ldots+n_r)-\frac{1}{\alpha}(n_{r+1}+\ldots+n_k)\big)\Big).$$

Proof. The sequence $z^{\boldsymbol{n}} = z_1^{n_1}\ldots z_k^{n_k}$ constitutes a basis of the space $\mathscr{H}(C^r\times C_0^{k-r})$ ([10], p. 74). We assume as pseudonorms

$$\|x\|_\alpha = \sup\{|x(z)|: |z_1|,\ldots,|z_r|\leqslant e^\alpha;\ |z_{r+1}|,\ldots,|z_k|\leqslant e^{-1/\alpha}\}.$$

Since $\sum_{\boldsymbol{n}\in\mathcal{N}^k}\|z^{\boldsymbol{n}}\|_\alpha/\|z^{\boldsymbol{n}}\|_{\alpha+1}<\infty$, by Theorem 1.1 we have $\mathscr{H}(C^r\times C_0^{k-r}) \approx \mathscr{M}(\|z^{\boldsymbol{n}}\|_\alpha)$. But $\|z^{\boldsymbol{n}}\|_\alpha = \exp\big(\alpha(n_1+\ldots+n_r)-\frac{1}{\alpha}(n_{r+1}+\ldots+n_k)\big)$, q. e. d.

Now we shall establish single matrix representations of the spaces $\mathscr{H}(C^k)$ and $\mathscr{H}(C_0^k)$. We shall use the following

Lemma 3.1. *Let $\left[\begin{smallmatrix} j\\ k\end{smallmatrix}\right]$ denote the number of all mononomials of degree j of k variables. Then $j^k/k!\leqslant\left[\begin{smallmatrix} j\\ k\end{smallmatrix}\right]\leqslant(j+1)^k$.*

Proof. It is easy to see that

$$\left[\begin{matrix} j\\ k\end{matrix}\right] = \sum_{p=0}^{j}\left[\begin{matrix} p\\ k-1\end{matrix}\right]. \tag{31}$$

Let $f_k(t) = \left[\begin{smallmatrix} \mathrm{E}t\\ k\end{smallmatrix}\right]$. By (31) we have $\int_0^{n+1} f_{k-1}(t)\,dt = f_k(n)$; therefore $\int_0^t f_{k-1}(\tau)\,d\tau\leqslant f_k(t)$. Hence, by induction, we obtain $f_{k+1}(t)\geqslant t^k/k!$; in particular $\left[\begin{smallmatrix} j\\ k+1\end{smallmatrix}\right]\geqslant j^k/k!$.

On the other hand, from (31) it follows that $\left[\begin{smallmatrix} j\\ k\end{smallmatrix}\right]\leqslant(j+1)\left[\begin{smallmatrix} j\\ k-1\end{smallmatrix}\right]$, and by induction $\left[\begin{smallmatrix} j\\ k\end{smallmatrix}\right]\leqslant(j+1)^{k-1}$, q. e. d.

Theorem 3.1′. *$\mathscr{H}(C^k)\approx\mathscr{M}(\exp\alpha\sqrt[k]{n})$ and $\mathscr{H}(C_0^k)\approx\mathscr{M}\big(\exp(-\sqrt[k]{n}/\alpha)\big)$.*

Proof. Let $\boldsymbol{p}(n) = \big(p_1(n),\ldots,p_k(n)\big)$ be an arbitrary one-to-one function from $\mathcal{N}^1$ onto $\mathcal{N}^k$ such that $|\boldsymbol{p}(n)|\leqslant|\boldsymbol{p}(n')|$ for $n\leqslant n'$, where $|\boldsymbol{p}(n)| = p_1(n)+\ldots+p_k(n)$. Write $e_n = z_1^{p_1(n)}\ldots z_k^{p_k(n)}$. By Lemma 3.1, there are positive constants A and B such that

$$\exp\alpha A\sqrt[k]{n}\leqslant\|e_n\|_\alpha\leqslant\exp\alpha B\sqrt[k]{n}\quad\text{for}\quad\mathscr{H}(C^k),$$

$$\exp(-A\sqrt[k]{n}/\alpha)\leqslant\|e_n\|_\alpha\leqslant\exp(-B\sqrt[k]{n}/\alpha)\quad\text{for}\quad\mathscr{H}(C_0^k).$$

Thus the formula $T(c_n) = \sum_{n\in\mathcal{N}^1} c_n z^{\boldsymbol{p}(n)}$ gives the required isomorphic mappings from $\mathscr{M}(\exp\alpha\sqrt[k]{n})$ onto $\mathscr{H}(C^k)$ and from $\mathscr{M}\big(\exp(-\sqrt[k]{n}/\alpha)\big)$ onto $\mathscr{H}(C_0^k)$, q. e. d.

The next two theorems will concern matrix representations of spaces of holomorphic functions defined on circular or $p_1,\ldots,p_k$-circular ([10], p. 113 and 117) domains D. For simplicity we shall formulate these theorems for the case where the centre of D coincides with $(0,\ldots,0)$.

Theorem 3.2. *Let D be a bounded domain in C^k such that*

$$\overline{aD}\subset D\quad\textit{for every}\quad|a|<1\ [8]. \tag{32}$$

Then $\mathscr{H}(D)\approx\mathscr{M}\big(\exp(-\sqrt[k]{n}/\alpha)\big)$.

Proof. Let $D_\varepsilon = (1-\varepsilon)D$. We introduce scalar-products

$$(x, y)_\varepsilon = \int_{D_\varepsilon} x(z_1,\ldots,z_k)\overline{y(z_1,\ldots,z_k)}\;dx_1\ldots dx_k dy_1\ldots dy_k,$$

where $z_j = x_j+iy_j$ for $j = 1,\ldots,k$, and the integral is taken over the domain D in a $2k$-dimensional real Euclidean space ([10], p. 119).

The pseudonorms $\|x\|_\varepsilon = \sqrt{(x,x)_\varepsilon}$ $(0<\varepsilon<1)$ give a topology equivalent to the almost uniform convergence, because

$$|D_\varepsilon|^{-1}\|x\|_\varepsilon\leqslant\sup_{z\in D_\varepsilon}|x(z)|\leqslant\pi^{-1}r_{\varepsilon\varepsilon'}^{-k}|x|_{\varepsilon'},$$

where $|D_\varepsilon|$ is the volume of D_ε and $r_{\varepsilon\varepsilon'}$ is the distance between the set D_ε and the complement of $D_{\varepsilon'}$ ([10], p. 120).

For the scalar product $(x,y)_0$ we can construct an orthonormal basis (e_n) constituted from homogeneous polynomials ([10], p. 132). The number of elements of the basis which are polynomials of degree j is equal to $\left[\begin{smallmatrix} j\\ k\end{smallmatrix}\right]$. Since e_n are homogeneous polynomials, we have $(e_r, e_s)_\varepsilon = 0$ for $r\neq s$, $(e_r,e_r)_\varepsilon = (1-\varepsilon)^j\|e_r\|_0$, where j is the degree of the polynomial e_r. Let $\varepsilon_a = 1-\exp(-1/\alpha)$. Then $\|e_n\|_{\varepsilon_a} = \|e_n\|_0\exp(-j/\alpha)$,

[8] $\bar{A}$ denotes the closure of the set A.

where j is the degree of the polynomial e_n. Now, applying Theorem 1.1 and Lemma 3.1, we obtain $\mathscr{H}(D) \approx \mathscr{M}(\exp(-\sqrt[k]{n+k}/a))$. But this space is trivially isomorphic to the space $\mathscr{M}(\exp(-\sqrt[k]{n}/a))$, q. e. d.

THEOREM 3.3. *Let D be a bounded domain in C^k such that* (33) *there are positive integers $p_1, \ldots, p_k$ such that for every real t*

$$\{(z_1 \exp ip_1 t, \ldots, z_k \exp ip_k t) : (z_1, \ldots, z_k) \in D\} \subset D,$$

$$\overline{tD} \subset D \text{ for each } 0 < t < 1. \tag{34}$$

Then $\mathscr{H}(D) \approx \mathscr{M}(\exp(-\sqrt[k]{n}/a))$.

Proof. Let $\Phi(z) = (z_1^{p_1}, \ldots, z_k^{p_k})$. It follows from (33) and (34) that the domain $D^* = \Phi^{-1}(D)$ fulfills condition (32). Let us assign to any function $x(z) \in \mathscr{H}(D)$ the function $(Ux)(z) = x(\Phi(z))$ with $z \in D^*$. It is easy to verify that U is an isomorphic mapping from $\mathscr{H}(D)$ onto a subspace X of the space $\mathscr{H}(D^*)$.

In the same way as in the proof of Theorem 3.2 we can construct a basis of X chosen from the homogeneous polynomials. The elements of this basis are sums of mononomials $z_1^{n_1} \ldots z_k^{n_k}$ such that n_i is divisible by p_i for $i = 1, 2, \ldots, k$. Let τ_j denote the number of such mononomials of degree not greater than j. As in Lemma 3.1 we can prove that there are positive constants A and B such that $Aj^k \leqslant s_j \leqslant Bj^k$. From this inequality follows the assertion of the theorem.

THEOREM 3.4. *Let* $\mu(\varepsilon, z) = \exp\left(-\sum_{j=1}^{k} (\tau_j + \varepsilon) |z_j|^{p_j}\right)$, $\tau_1, \ldots, \tau_r = 0$, $\tau_{r+1}, \ldots, \tau_k > 0$, $p_j > 0$; *then* $\mathscr{H}_\mu \approx \mathscr{M}\left(\exp\left(a(n_1 + \ldots + n_r) - \frac{1}{a}(n_{r+1} + \ldots + n_k)\right)\right)$.

The isomorphic mapping from $\mathscr{H}_\mu$ onto $\mathscr{M}\left(\exp\left(a(n_1 + \ldots + n_r) - \frac{1}{a}(n_{r+1} + \ldots + n_k)\right)\right)$ is of the form

$$T \sum_{n \in \mathscr{N}^k} c_n z^n = (d_n c_n), \text{ where } d_n = \left(\prod_{j=1}^{k} (n_j/p_j)^{n_j/p_j}\right) \exp\left(-\sum_{j=1}^{k} n_j/p_j\right). \tag{35}$$

Proof. We have

$$\|z^n\|_\varepsilon = \max_{z \in C^n} |z^n| \exp\left(-\sum_{j=1}^{k} (\tau_j + \varepsilon) |z_j|^{p_j}\right)$$
$$= \max_{t_j > 0} \exp\left(\sum_{j=1}^{k} n_j \log t_j - \sum_{j=1}^{k} (\tau_j + \varepsilon) t_j^{p_j}\right).$$

Let

$$f(t_1, \ldots, t_k) = \sum_{j=1}^{k} n_j t_j - \sum_{j=1}^{k} (\tau_j + \varepsilon) t_j^{p_j}.$$

Solving the equations $\partial f / \partial t_j = 0$ we find that the function f has its maximum for $t_j = \sqrt[p_j]{n_j/(p_j(\tau_j + \varepsilon))}$, $j = 1, \ldots, k$, whence

$$\|z^n\|_\varepsilon = \left(\prod_{j=1}^{k} \left(\frac{n_j/p_j}{\tau_j + \varepsilon}\right)^{p_j/p_j}\right) \exp\left(-\sum_{j=1}^{k} n_j/p_j\right).$$

Thus the formula $U \sum_{n \in \mathscr{N}^k} c_n z^n = (c_n)$ gives an isomorphic mapping from $\mathscr{H}_\mu$ onto $\mathscr{M}\left(d_n \exp\left(a(n_1 + \ldots + n_r) - \frac{1}{a}(n_{r+1} + \ldots + n_k)\right)\right)$, where d_n are given by formula (35). This is equivalent to the assertion of the theorem.

THEOREM 3.5. *Let* $\mu(\varepsilon, z) = \exp\left(-\sum_{j=1}^{k} (\tau_j + \varepsilon) |\log |z|_j|^{p_j}\right)$, *where* $p_j > 1$, $\tau_1, \ldots, \tau_r = 0$, $\tau_{r+1}, \ldots, \tau_k > 0$, *and let* $q_j = p_j/(p_j - 1)$, $j = 1, \ldots, k$. *Then* $\mathscr{H}_\mu \approx \mathscr{M}\left(\exp\left(a(n_1^{q_1} + \ldots + n_r^{q_r}) - \frac{1}{a}(n_{r+1}^{q_{r+1}} + \ldots + n_k^{q_k})\right)\right)$.

Proof. In the same way as in the preceding proof we obtain

$$|z^n|_\varepsilon = \prod_{j=1}^{k} A_{\varepsilon j}^{n_j^{q_j}}, \quad \text{where} \quad A_{\varepsilon j} = \exp\left(\left(\frac{1}{(\tau_j + \varepsilon) p_j}\right)^{q_j} q_j^{-1}\right).$$

If $\tau_j = 0$, then $\lim_{\varepsilon \to 0} A_{\varepsilon j} = \infty$; if $\tau_j > 0$, then $\lim_{\varepsilon \to 0} A_{\varepsilon j} = A_j$. From Theorem 1.6 it follows that $\mathscr{H}_\mu \approx \mathscr{M}(|z^n|_{\varepsilon_a})$, where (ε_a) is an arbitrary sequence tending to zero. The transformation

$$U(\xi_n) = (d_n \xi_n), \quad \text{where} \quad d_n = \prod_{j=r+1}^{k} A_j^{n_j^{q_j}} \quad \text{for} \quad n \in \mathscr{N}^k,$$

maps this space onto the space given in the assertion of Theorem 3.5.

Remark 3.1. The space $\mathscr{H}_\mu$ considered in Theorem 3.5 is isomorphic to the space $F_{p\tau}^k$ of all holomorphic functions $x(z)$ which are periodic with the period 2π with respect to every variable and such that

$$\|x\|_\varepsilon = \sup |x(z)| \exp\left(-\sum_{j=1}^{k} (\tau_j + \varepsilon) |\operatorname{Im} z_j|^{p_j}\right) < \infty.$$

The isomorphic mapping from $\mathscr{H}_\mu$ onto $F_{p\tau}^k$ is given by the formula $(Ux)(z) = x(\exp iz_1, \ldots, \exp iz_k)$.

THEOREM 3.6. *Let*

$$\mu(\varepsilon, z) = \exp\left(-\varepsilon \sum_{j=1}^{k} \left(\log \frac{1}{|z_j|}\right)^{-s_j}\right), \quad s_j > 0.$$

Then

$$\mathscr{H}_\mu(C_0^k) \approx \mathscr{M}\left(\exp\left(-\frac{1}{a}\sum_{j=1}^{k} n_j^{q_j}\right)\right), \quad \textit{where} \quad q_j = s_j/(1+s_j).$$

This isomorphism is realized by the transformation $T\left(\sum_{n\in\mathcal{N}^k} c_n z^n\right) = (c_n)$.

Proof. We have

$$\|z^n\|_\varepsilon = \exp\left(-\sum_{j=1}^{k}\left(\frac{n_j}{\varepsilon s_j}\right)^{-1/(s_j+1)}(n_j+\varepsilon)\right).$$

Hence if $|n|$ is sufficiently large, then

$$\exp\left(-2\max_x(\varepsilon s_j)^{-1/s_j+1}\sum_{j=1}^{k} n_j^{q_j}\right) \leqslant \|z^n\|_\varepsilon \leqslant \exp\left(-\tfrac{1}{2}\min_j(\varepsilon_j s)^{-1/s_j+1}\sum_{j=1}^{k} n_j^{q_j}\right).$$

This inequality gives us the assertion of the theorem [9].

Remark 3.2. Let $\mu(\varepsilon, z)$ be the function defined in Theorem 3.6 and let $\mu_1(\varepsilon, z) = \exp\left(-\varepsilon\sum_{j=1}^{k}(1-|z_j|)^{-s_j}\right)$. Then $\mathscr{H}_\mu = \mathscr{H}_{\mu_1}$.

This follows immediately from the fact that

$$\lim_{t\to 1-}\frac{1/(1-t)}{1/\log(1/t)} = 1.$$

2. Matrix form $\mathscr{M}\left((n_1^{n_1}\dots n_r^{n_r})^a\cdot(n_{r+1}^{n_{r+1}}\dots n_k^{n_k})^{-1/a}\right)$. Now we prove

Theorem 3.7. *Let* $\mu(\varepsilon, z) = \exp\left(-\sum_{j=1}^{k}|z_j|^{p_j+\varepsilon}\right)$, *where* $p_1, \dots, p_r = 0$, $p_{r+1}, \dots, p_k > 0$. *Then* $\mathscr{H}_\mu \approx \mathscr{M}\left((n_1^{n_1}\dots n_r^{n_r})^a\cdot(n_{r+1}^{n_{r+1}}\dots n_k^{n_k})^{-1/a}\right)$.

Proof. We have

$$\|z^n\|_\varepsilon = \prod_{j=1}^{k} n_j^{n_j/(p_j+\varepsilon)}(p_j+\varepsilon)^{-n_j/(p_j+\varepsilon)}\cdot\exp\left(-n_j/(p_j+\varepsilon)\right).$$

Hence for arbitrary $\eta > 0$

$$\prod_{j=1}^{k} n_j^{n_j/(p_j+\varepsilon+\eta)} \leqslant \|z^n\|_\varepsilon \leqslant \prod_{j=1}^{k} n_j^{n_j/(p_j+\varepsilon-\eta)}$$

for sufficiently large $|n|$. Therefore $\mathscr{H}_\mu \approx \mathscr{M}\left(\prod_{j=1}^{k} n_j^{n_j/(p_j+1/a)}\right)$, by Theorem 1.6. But this space may be isomorphically mapped onto the space given in the assertion of the theorem by the transformation

[9] It is enough to notice that Theorem 1.6 holds true if we replace spaces $\mathscr{H}_\mu(C)$ by $\mathscr{H}_\mu(C_0)$.

$$T(c_n) = (d_n c_n), \quad \text{where} \quad d_n = \prod_{j=r+1}^{k} n_j^{n_j/p_j}.$$

3. Matrix form $\mathscr{M}\left(\exp\left(n_1^a+\dots+n_r^a+n_{r+1}^{q_{r+1}-1/a}+\dots+n_k^{q_k-1/a}\right)\right)$. Now we prove.

Theorem 3.8. *Let* $\mu(\varepsilon, z) = \exp\left(-\sum_{j=1}^{k}\log|z_j|^{p_j+\varepsilon}\right)$, *where* $p_1, \dots, p_r = 1$, $p_{r+1}, \dots, p_k > 1$. *Then* $\mathscr{H}_\mu \approx \mathscr{M}\left(\exp\left(n_1^a+\dots+n_r^a+n_{r+1}^{q_{r+1}-1/a}+\dots+n_k^{q_k-1/a}\right)\right)$, $q_j = p_j/(p_j-1)$

Proof. We have

$$\|z^n\|_\varepsilon = \prod_{j=1}^{k}\exp\left(\left(n_j/(p_j+\varepsilon)\right)^{(p_j+\varepsilon)/(p_j+\varepsilon-1)}\cdot(p_j+\varepsilon-1)/(p_j+\varepsilon)\right).$$

Hence, for every $\eta > 0$,

$$\prod_{j=1}^{k}\exp n_j^{(p_j+\varepsilon-\eta)/(p_j+\varepsilon-\eta-1)} \leqslant \|z^n\|_\varepsilon \leqslant \exp n_j^{(p_j+\varepsilon+\eta)/(p_j+\varepsilon+\eta-1)}$$

for sufficiently large $|n|$. Hence, by Theorem 1.6, we obtain the assertion of our theorem.

§ 4. Approximative dimension

1. Space $\mathscr{M}\left(\exp\left(a(n_1^{q_1}+\dots+n_r^{q_r})-\frac{1}{a}(n_{r+1}^{q_{r+1}}+\dots+n_k^{q_k})\right)\right)$. Now we prove

Theorem 4.1. *Let* $X \approx \mathscr{M}\left(\exp\left(a(n_1^{q_1}+\dots+n_r^{q_r})-\frac{1}{a}(n_{r+1}^{q_{r+1}}+\dots+n_k^{q_k})\right)\right)$, *with* $q_i > 0$; $s = \sum_{i=1}^{k} q_i^{-1}$. *Then*

$$\Phi(X) = \begin{cases}\left\{\varphi\colon \lim\left(\log\frac{1}{\varepsilon}\right)^{s+1}\Big/\log\varphi(\varepsilon) = 0\right\}, & \textit{if} \quad r = 0,\\[2mm] \left\{\varphi\colon \overline{\lim}\left(\log\frac{1}{\varepsilon}\right)^{s+1}\Big/\log\varphi(\varepsilon) < \infty\right\}, & \textit{if} \quad r > 0.\end{cases}$$

Proof. Put $a_{an}^{(j)} = \exp(an^{q_j})$ for $j = 1, \dots, r$, $a_{an}^{(j)} = \exp(-n^{q_j}/a)$ for $j = r+1, \dots, k$. Since, for any $q > 0$,

$$\exp(an^q)/\exp(\beta n^q) \geqslant \varepsilon \text{ if and only if } n \leqslant \left(\frac{1}{\beta-a}\log\frac{1}{\varepsilon}\right)^{1/q}$$

and

$$\exp\left(-\frac{n^q}{\alpha}\right)\Big/\exp\left(-\frac{n^q}{\beta}\right) \geqslant \varepsilon \text{ if and only if } n \leqslant \left((1/\alpha-1/\beta)^{-1}\log\frac{1}{\varepsilon}\right)^{1/q},$$

we obtain

$$N_{\alpha\beta j}(\varepsilon) = \begin{cases} 1+\mathrm{E}\left(\dfrac{1}{\beta-\alpha}\log\dfrac{1}{\varepsilon}\right)^{1/q} & \text{for} \quad j = 1, \ldots, r, \\ 1+\mathrm{E}\left((1/\alpha-1/\beta)^{-1}\log\dfrac{1}{\varepsilon}\right)^{1/q} & \text{for} \quad j = r+1, \ldots, k. \end{cases}$$

Hence, for sufficiently small ε,

$$\prod_{j=1}^{k} N_{\alpha\beta j}(\sqrt[2k]{\varepsilon}) \geqslant \frac{1}{2k}(\beta-\alpha)^{-s'}(1/\alpha-1/\beta)^{-s''}\left(\log\frac{1}{\varepsilon}\right)^{s}$$

and

$$\prod_{j=1}^{k} N_{\alpha\beta j}(\varepsilon) \leqslant 2^k(\beta-\alpha)^{-s'}(1/\alpha-1/\beta)^{s''}\left(\log\frac{1}{\varepsilon}\right)^{s},$$

where $s' = 1/q_1+\ldots+1/q_r$, $s'' = 1/q_{r+1}+\ldots+1/q_k$, $s = s'+s''$. Thus, according to Theorem 1.3, for sufficiently small ε we have

$$(41) \quad \frac{1}{2k}(\beta-\alpha)^{-s'}(1/\alpha-1/\beta)^{s''}\left(\log\frac{1}{\varepsilon}\right)^{s+1} \leqslant \log M_{\alpha\beta}(\varepsilon)$$
$$\leqslant 2^{k+1}(\beta-\alpha)^{-s'}(1/\alpha-1/\beta)^{s''}\left(\log\frac{1}{\varepsilon}\right)^{s+1}.$$

In virtue of the definition of $M(A, B)$, $\varphi \in M(K_\beta, K_\alpha)$ (1) if and only if there exists $\varepsilon_0 > 0$ such that

$$\log\varphi(\varepsilon) \geqslant \log M_{\alpha\beta}(X; \varepsilon) \quad \text{for every} \quad \varepsilon < \varepsilon_0.$$

Hence, by (41),

$$2^{k+1}\overline{\lim_{\varepsilon\to 0}}\left((\beta-\alpha)^{-s'}(1/\alpha-1/\beta)^{s''}\left(\log\frac{1}{\varepsilon}\right)^{s}\right)/\log\varphi(\varepsilon) < 1$$

implies

$$\varphi \in M(K_\beta, K_\alpha),$$

which implies

$$\frac{1}{2k}\overline{\lim_{\varepsilon\to 0}}\left((\beta-\alpha)^{-s'}(1/\alpha-1/\beta)^{s''}\left(\log\frac{1}{\varepsilon}\right)^{s}\right)/\log\varphi(\varepsilon) < 1.$$

Now Theorem 1.2 gives us the assertion of the theorem.

COROLLARY 4.1. *If* $0 < r \leqslant k$, *then* $\mathrm{d}_a\mathscr{H}(C^r\times C_0^{k-r}) = \mathrm{d}_a\mathscr{H}(C^k) < \mathrm{d}_a\mathscr{H}(C_0^k)$.

(1) $K_\alpha = \{x: \|x\|_\alpha < 1\}$, $K_\beta = \{x: \|x\|_\beta < 1\}$.

COROLLARY 4.2. *If* $\mathrm{d}_a\mathscr{H}(D) = \mathrm{d}_a\mathscr{H}(D_1)$, *then* $\dim D = \dim D_1$.

COROLLARY 4.3. *If D is an arbitrary bounded domain of the k-dimensional Euclidean complex space, then*

$$\Phi(\mathscr{H}(D)) = \left\{\varphi: \lim_{\varepsilon\to 0}\left(\log\frac{1}{\varepsilon}\right)^{k+1}/\log\varphi(\varepsilon) = 0\right\}.$$

These corollaries follow from Theorems 1.5, 2.4, 2.5 and 2.6.

2. Space $\mathscr{M}\left((n_1^{n_1}\ldots n_r^{n_r})^{\alpha}\cdot(n_{r+1}^{n_{r+1}}\ldots n_k^{n_k})^{-1/\alpha}\right)$. Now we prove

THEOREM 4.2. *If* $X \approx \mathscr{M}\left((n_1^{n_1}\ldots n_r^{n_r})^{\alpha}(n_{r+1}^{n_{r+1}}\ldots n_k^{n_k})^{-1/\alpha}\right)$, *then* $\varphi \in \Phi(X)$ *if and only if*

$$\overline{\lim_{\varepsilon\to 0}}\frac{\left(\log\dfrac{1}{\varepsilon}\right)^{k+1}\Big/\left(\log\log\dfrac{1}{\varepsilon}\right)^{k}}{\log\varphi(\varepsilon)} \begin{cases} = 0 & \textit{in the case where } r = 0, \\ < +\infty & \textit{in the case where } r > 0. \end{cases}$$

Proof. Put $a_n^{(j)} = n^{\alpha n}$ for $j = 1, \ldots, r$, $a_n^{(j)} = n^{-n/\alpha}$ for $j = r, \ldots, k$. It is easy to verify that

$$N_{\alpha\beta j}(\varepsilon) = \begin{cases} \mathrm{E}\varkappa\left(\dfrac{1}{\beta-\alpha}\log\dfrac{1}{\varepsilon}\right)+1 & \text{for} \quad j \leqslant r, \\ \mathrm{E}\varkappa\left((1/\alpha-1/\beta)^{-1}\log\dfrac{1}{\varepsilon}\right)+1 & \text{for} \quad j > r, \end{cases}$$

where $\varkappa(t)$ is the function inverse to the function $f(t) = t\log t$.

Since

$$\lim_{t\to 0}\frac{\varkappa(t)}{t/\log t} = 1,$$

for sufficiently small ε, we have $\dfrac{1}{2}A_{\alpha\beta j}\log\dfrac{1}{\varepsilon}\Big/\log\log\dfrac{1}{\varepsilon} \leqslant N_{\alpha\beta j}(\varepsilon) \leqslant 2A_{\alpha\beta j}\log\dfrac{1}{\varepsilon}\Big/\log\log\dfrac{1}{\varepsilon}$, where $A_{\alpha\beta j} = 1/(\beta-\alpha)$ for $j = 1, \ldots, r$, $A_{\alpha\beta j} = (1/\alpha-1/\beta)^{-1}$ for $j = r+1, \ldots, k$. Therefore, for sufficiently small ε,

$$\prod_{j=1}^{k} N_{\alpha\beta j}(\sqrt[2k]{\varepsilon}) \geqslant \left(\frac{1}{2}\right)^{3k^2}(\beta-\alpha)^{-r}(1/\alpha-1/\beta)^{r-k}\left(\log\frac{1}{\varepsilon}/\log\log\frac{1}{\varepsilon}\right)^{k}$$

and

$$\prod_{j=1}^{k} N_{\alpha\beta j}(\varepsilon) \leqslant 2^k(\beta-\alpha)^{-r}(1/\alpha-1/\beta)^{r-k}\left(\log\frac{1}{\varepsilon}/\log\log\frac{1}{\varepsilon}\right)^{k}.$$

Now in the same way as in the preceding section one can deduce the assertion of the theorem.

3. Space $\mathscr{M}(\exp(n_1^a+\ldots+n_r^a+n_{r+1}^{q_{r+1}-1/a}+\ldots+n_k^{q_k-1/a}))$. We now prove

THEOREM 4.3. *If* $X\approx\mathscr{M}(\exp(n_1^a+\ldots+n_r^a+n_{r+1}^{q_{r+1}-1/a}+\ldots+n_k^{q_k-1/a})$, *then*

$$\Phi(X)=\{\varphi:\lim_{\varepsilon\to 0}(\log 1/\varepsilon)^{s+1+\eta}/\log\varphi(\varepsilon)=0 \text{ for some } \eta>0\},$$

where $s=1/q_{r+1}+\ldots+1/q_k$.

Proof. Let $a_n^{(j)}=\exp n^a$ for $j=1,\ldots,r$, $a_n^{(j)}=\exp n^{q_j-1/a}$ for $j=r+1,\ldots,k$. It is easily seen that, for $a<\beta$,

$$\frac{\exp n^a}{\exp n^\beta}\geqslant\varepsilon \text{ if and only if } n\leqslant\lambda_{a\beta}\left(\log\frac{1}{\varepsilon}\right),$$

and

$$\frac{\exp n^{q-1/a}}{\exp n^{q-1/\beta}}\geqslant\varepsilon \text{ if and only if } n\leqslant\lambda_{q-\frac{1}{a},q-\frac{1}{\beta}}\left(\log\frac{1}{\varepsilon}\right),$$

where $\lambda_{a\beta}(t)$ is the function inverse to the function $f(t)=t^\beta-t^a$ $(a<\beta)$. Hence

$$N_{a\beta j}(\varepsilon)=\begin{cases}\mathrm{E}\lambda_{a\beta}\left(\log\frac{1}{\varepsilon}\right) & \text{for } j\leqslant r,\\ \mathrm{E}\lambda_{q_j-\frac{1}{a},q_j-\frac{1}{\beta}}\left(\log\frac{1}{\varepsilon}\right) & \text{for } j>r.\end{cases}$$

Since $\lim_{t\to\infty}\lambda_{a\beta}(t)/t^{1/\beta}=1$, we obtain, for sufficiently small ε,

$$N_{a\beta j}(\sqrt[2k]{\varepsilon})\geqslant\frac{1}{4k}\left(\log\frac{1}{\varepsilon}\right)^{1/\beta},\quad N_{a\beta j}(\varepsilon)\leqslant 2\left(\log\frac{1}{\varepsilon}\right)^{1/\beta} \text{ for } j\leqslant r,$$

and

$$N_{a\beta j}(\sqrt[2k]{\varepsilon})\geqslant(1/2)(2k)^{-1/q_j}\left(\log\frac{1}{\varepsilon}\right)^{(q_j-1/\beta)^{-1}},$$

$$N_{a\beta j}(\varepsilon)\leqslant 2\left(\log\frac{1}{\varepsilon}\right)^{(q_j-1/\beta)^{-1}} \quad \text{for } j>r.$$

Therefore, according to Theorem 3.1, we obtain

$$A_k\left(\log\frac{1}{\varepsilon}\right)^{(r/\beta+\Sigma(q_j-1/\beta)^{-1})+1}\leqslant M_{a\beta}(\varepsilon)\leqslant B_k\left(\log\frac{1}{\varepsilon}\right)^{(r/\beta+\Sigma(q_j-1/\beta)^{-1})+1},$$

where A_k and B_k are constants.

From the last formula follows the assertion of Theorem 4.3, because

$$\lim_{\beta\to\infty}\left(r/\beta+\sum(q_j-1/\beta)^{-1}\right)=s.$$

§ 5. Power spaces. Single matrix representations. Examples of non-isomorphic spaces having the same approximative dimension

1. Definition. Nuclear Köthe spaces of the forms $\mathscr{M}(a_n^a)$ and $\mathscr{M}(b_n^{1/a})$ will be called *power spaces of infinite or finite type*, respectively.

From Theorem 1.1 it follows that in the case of power spaces

(51) *There exist* $\beta>0$ *such that* $\sum_n a_n^\beta<\infty$; *for every* $t>0$, $\sum_n b_n^t<\infty$.

Without loss of generality it may be assumed that

(52) $$1\leqslant a_0\leqslant a_1\leqslant\ldots;\qquad 1\geqslant b_0\geqslant b_1\geqslant\ldots$$

Let $\varphi(\varepsilon)$ and $\psi(\varepsilon)$ be functions defined for $\varepsilon>0$. If there exists $\varepsilon_0>0$ such that $\varphi(\varepsilon)\geqslant\psi(\varepsilon)$ (resp. $\varphi(\varepsilon)>\psi(\varepsilon)$), for $\varepsilon<\varepsilon_0$ we shall write

$$\varphi(\varepsilon)\dot{\geqslant}\psi(\varepsilon)\quad(\text{resp. } \varphi(\varepsilon)\dot{>}\psi(\varepsilon)).$$

Let $X=\mathscr{M}(a_n^a)$, $Y=\mathscr{M}(b_n^{1/a})$, with (a_n) and (b_n) satisfying conditions (51) and (52). Put

$$M_a(X;\varepsilon)=\prod_{n=0}^{\infty}\mathrm{E}\left(1+1/(\varepsilon a_n^a)\right),\qquad \tilde{M}_a(Y;\varepsilon)=\prod_{n=0}^{\infty}\mathrm{E}\left(1+b_n^{1/a}/\varepsilon\right),$$

and

$$M_a(X)=\{\varphi:\varphi(\varepsilon)\dot{\geqslant}M_a(X;\varepsilon)\},\qquad \tilde{M}_a(Y)=\{\varphi:\varphi(\varepsilon)\dot{\geqslant}\tilde{M}(Y;\varepsilon)\}$$

for $a=1,2,\ldots$

THEOREM 5.1. $\Phi(X)=\bigcup_{a=1}^{\infty}M_a(X)$, $\quad\Phi(Y)=\bigcap_{a=1}^{\infty}\tilde{M}_a(Y)$.

This theorem is a simple consequence of Theorem 1.2.

THEOREM 5.2. *Let* $X=\mathscr{M}(a_n^a)$ *and* $Y=\mathscr{M}(b_n^{1/a})$ *be power spaces of infinite type and of finite type, respectively. Then* $\mathrm{d}_aX\neq\mathrm{d}_aY$; *therefore* $X\neq Y$.

Proof. Suppose that

(53) $$\Phi(X)\supset\Phi(Y).$$

We shall prove that in this case $\Phi(X)\supsetneqq\Phi(Y)$.

From the definition of functions $M_a(X;\varepsilon)$ and $\tilde{M}_a(Y;\varepsilon)$ it follows that

(54) $$\ldots\dot{<}M_2(X;\varepsilon)\dot{<}M_1(X;\varepsilon);\qquad \tilde{M}_1(Y;\varepsilon)\dot{<}\tilde{M}_2(Y;\varepsilon)\dot{<}\ldots,$$

whence, by Theorem 5.1 and by formula (53), we have

(55) $$\tilde{M}_a(Y;\varepsilon)\dot{<}M_a(X;\varepsilon)\quad(a=1,2,\ldots).$$

Inequalities (54) and (55) imply the existence of a decreasing sequence (ε_k), with $\lim\limits_{k\to\infty}\varepsilon_k = 0$, such that

$$M_1(X;\varepsilon) > M_2(X;\varepsilon) > \ldots > M_k(X;\varepsilon) > \tilde{M}_k(Y;\varepsilon) > \tilde{M}_{k-1}(Y;\varepsilon) > \ldots > \tilde{M}_1(Y;\varepsilon) \quad \text{for} \quad \varepsilon < \varepsilon_k.$$

Put

$$\varphi(\varepsilon) = \begin{cases} 1 & \text{for} \quad \varepsilon \geqslant \varepsilon_1, \\ \frac{1}{2}\big(M_k(X;\varepsilon)+\tilde{M}_k(Y;\varepsilon)\big) & \text{for} \quad \varepsilon_k > \varepsilon \geqslant \varepsilon_{k+1}. \end{cases}$$

We have

$$M_k(X;\varepsilon) \dot{>} \varphi(\varepsilon) \dot{>} \tilde{M}_k(Y;\varepsilon);$$

this means, according to Theorem 5.1, that $\Phi(X) \neq \Phi(Y)$, q. e. d.

2. Lemma 5.1. *Let X be a nuclear space that satisfy the following condition*:

(56) $\quad \varphi \in \Phi(X)$ *if and only if* $\varphi^2 \in \Phi(X)$.

If $X = \mathscr{M}(a_n^a)$, *then* $\Phi(X) = \bigcup\limits_a \{\varphi : \log\varphi(\varepsilon)/\log\frac{1}{\varepsilon} \geqslant N(\sqrt[a]{\varepsilon})\}$;

if $X = \mathscr{M}(b_n^{1/a})$, *then* $\Phi(X) = \bigcap\limits_a \{\varphi : \log\varphi(\varepsilon)\log\frac{1}{\varepsilon} \geqslant \tilde{N}(\varepsilon^a)\}$, *where* $N(\varepsilon) = \overline{\overline{\{n : a_n \leqslant 1/\varepsilon\}}}$, $\tilde{N}(\varepsilon) = \overline{\overline{\{n : b_n \geqslant \varepsilon\}}}$.

Proof. 1° $X = \mathscr{M}(a_n)$. Theorem 1.3, for $k = 1$, gives us

$$2N(\sqrt[a]{\varepsilon})\log\frac{1}{\varepsilon} \dot{\geqslant} \log M_a(X;\varepsilon) \dot{\geqslant} \frac{1}{2}N(\sqrt[2a]{\varepsilon})\log\frac{1}{\varepsilon}.$$

Let $\varphi \in \Phi(X)$. Then, by (56), $\sqrt{\varphi} \in \Phi(X)$ and, according to Theorem 5.1, we have

$$\varphi \in \bigcup_a \Big\{\varphi : \tfrac{1}{2}\log\varphi(\varepsilon) \dot{\geqslant} \tfrac{1}{2}N(\sqrt[a]{\varepsilon})\log\frac{1}{\varepsilon}\Big\}.$$

On the other hand, if $\varphi \in \bigcup\limits_a \Big\{\varphi : 2\log\varphi(\varepsilon) \geqslant 2N(\sqrt[a]{\varepsilon})\log\frac{1}{\varepsilon}\Big\}$, then $\varphi^2 \in \Phi(X)$ and, by (56), $\varphi \in \Phi(X)$.

2° $X = M(b_n^{1/a})$. The proof in this case is based on the inequality

$$2\tilde{N}(\varepsilon^a)\log\frac{1}{\varepsilon} \dot{\geqslant} \log \tilde{M}_a(X;\varepsilon) \dot{\geqslant} \frac{1}{2}\tilde{N}(\varepsilon^{a/2})\log\frac{1}{\varepsilon}.$$

Remark 5.1. It is easily seen that in Lemma 5.1 assumption (56) may be replaced by "$\psi \in \bigcup\limits_a \{\psi : \psi(\varepsilon) \dot{\geqslant} N(\sqrt[a]{\varepsilon})$ (resp. $\psi \in \bigcup\limits_a \{\psi : \psi(\varepsilon) \dot{\geqslant} \tilde{N}(\varepsilon^a)\}$) if and only if $2\psi \in \bigcup\limits_a \{\psi : \psi(\varepsilon) \dot{\geqslant} \tilde{N}(\sqrt[a]{\varepsilon})\}$ (resp. $2\psi \in \bigcap\limits_a \{\psi : \psi(\varepsilon) \dot{\geqslant} \tilde{N}(\varepsilon^a)\}$)".

Theorem 5.3. *Let X and X_1 be power spaces satisfying condition* (56) *and let* $\mathrm{d}_a X = \mathrm{d}_a X_1$. *Then the spaces X and X_1 are isomorphic.*

Proof. 1° $X = \mathscr{M}(a_n^a)$, $X_1 = \mathscr{M}(c_n^a)$. Suppose that conditions (51) and (52) are satisfied. Let $N(\varepsilon)$ and $N_1(\varepsilon)$ be functions defined in Lemma 5.1, for the spaces X and X_1, respectively. By Lemma 5.1, we have

$$\bigcup_a \{\psi : \psi(\varepsilon) \dot{\geqslant} N(\sqrt[a]{\varepsilon})\} = \bigcup_a \{\psi : \psi(\varepsilon) \dot{\geqslant} N_1(\sqrt[a]{\varepsilon})\},$$

whence $\quad N(\varepsilon) \in \bigcup\limits_a \{\psi : \psi(\varepsilon) \dot{\geqslant} N_1(\sqrt[a]{\varepsilon})\}$ and $N_1(\varepsilon) \in \bigcup\limits_a \{\psi : \varphi(\varepsilon) \dot{\geqslant} N(\sqrt[a]{\varepsilon})\}$.

Thus there exist positive integers a_0 and β_0 such that $N(\varepsilon) \dot{\geqslant} N_1(\sqrt[a_0]{\varepsilon})$, $N_1(\varepsilon) \dot{\geqslant} N(\sqrt[\beta_0]{\varepsilon})$. Sequences (a_n), (c_n) being non-decreasing, this implies that, for sufficiently large n, $a_n \leqslant c_n^{a_0}$ and $a_n \leqslant c_n^{\beta_0}$, i. e. the identical mapping $T(\xi_n) = (\xi_n)$ is the required isomorphism from X onto X_1.

2° $X = \mathscr{M}(b_n^{1/a})$, $X_1 = \mathscr{M}(d_n^{1/a})$. Let (b_n) and (d_n) satisfy conditions (51) and (52). Denote by $\tilde{N}(\varepsilon)$ and $\tilde{N}_1(\varepsilon)$ the function defined in Lemma 5.1 for the spaces X and X_1, respectively. By Lemma 5.1

(57) $$\bigcap_a \{\psi : \psi(\varepsilon) \dot{\geqslant} \tilde{N}(\varepsilon^a)\} = \bigcap_a \{\psi : \psi(\varepsilon) \dot{\geqslant} \tilde{N}_1(\varepsilon^a)\}.$$

We shall show that there exists a positive integer a_0 such that $\tilde{N}(\varepsilon^{a_0}) \dot{\geqslant} \tilde{N}_1(\varepsilon)$.

In fact, if the last inequality held for no a, then there would exist a decreasing sequence (ε_a), with $\lim\limits_{a\to 0}\varepsilon_a = 0$, such that $\tilde{N}(\varepsilon_a^a) < \tilde{N}_1(\varepsilon_a)$.

Let

$$\psi(\varepsilon) = \begin{cases} 1 & \text{for} \quad \varepsilon > \varepsilon_1, \\ \tilde{N}(\varepsilon^a) & \text{for} \quad \varepsilon_a \geqslant \varepsilon > \varepsilon_{a+1}. \end{cases}$$

Then $\psi \in \bigcap\limits_a \{\psi : \psi(\varepsilon) \dot{\geqslant} \tilde{N}(\varepsilon^a)\}$ and $\psi \notin \bigcap\limits_a \{\psi : \psi(\varepsilon) \dot{\geqslant} \tilde{N}_1(\varepsilon)\}$, which contradicts formula (57).

Now $\tilde{N}(\varepsilon_n^{a_0}) \dot{\geqslant} \tilde{N}_1(\varepsilon)$ implies that, for sufficiently large n, $d_n \geqslant b_n^{a_0}$. The assumption of the theorem being symmetric with respect to X and X_1, we have also $b_n \geqslant d_n^{\beta_0}$ for sufficiently large n. Hence the mapping $T(\xi_n) = (\xi_n)$ is the required isomorphism from X onto X_1, q. e. d.

Since nuclear Köthe spaces of type $\mathscr{M}(a_n^a)$, $\mathscr{M}(b_n^{1/a})$ are isomorphic to power spaces, we have

COROLLARY 5.1. *If* $X = \mathscr{M}(a_n^a)$, $X_1 = \mathscr{M}(c_n^a)$, $Y = M(b_n^{1/a})$, $Y_1 = M(d_n^{1/a})$ *are nuclear Köthe spaces satisfying condition* (56) *and* $d_a X = d_a X_1$, $d_a Y = d_a Y_1$, *then* $X = X_1$, $Y = Y_1$.

Spaces considered in § 2.3, sections 1 and 2, satisfy condition (56). Hence

COROLLARY 5.2. *Let*

$$X = \mathscr{M}\Big(\exp\big(a(n_1^{q_1}+\ldots+n_r^{q_r}) - \frac{1}{a}(n_{r+1}^{q_{r+1}}+\ldots+n_r^{q_r})\big)\Big)$$

with $q_i > 0$, *and let* $q = (1/q_1+\ldots+1/q_k)^{-1}$. *Then if* $r = 0$, *then* $X \approx M(\exp - n^q/a)$; *if* $r = k$, *then* $X \approx \mathscr{M}(\exp a n^q)$.

COROLLARY 5.3. *Let* $X = \mathscr{M}\big((n_1^{n_1}\ldots n_r^{n_r})^a\cdot(n_{r+1}^{n_{r+1}}\ldots n_k^{n_k})^{-1/a}\big)$. *If* $r = 0$, *then* $X = \mathscr{M}(n^{-\sqrt[k]{n}/a})$; *if* $r = k$, *then* $X \approx \mathscr{M}(n^{a\sqrt[k]{n}})$.

To prove these two corollaries we apply Theorems 4.2 and 4.3 for an arbitrary k and for $k = 1$ and verify that suitable spaces have an equal approximative dimension.

3. In general, Köthe spaces with the same approximative dimension need not be isomorphic.

Example 5.1. *Let* $X = \mathscr{M}(\exp an)$, $Y = (X\times X\times\ldots)_s$. *Then* $d_a X = d_a Y$, *but* $X \not\approx Y$.

The quality of approximative dimensions follows from Theorem 1.5. $X \not\approx Y$, because there exist continuous (homogeneous) norms defined on X; in the space s (of all numerical sequences), which is isomorphically contained in Y, no continuous pseudonorm is a norm.

The next example will concern the spaces in which there are continuous norms.

Example 5.2. *Let* $X = \mathscr{H}(C_0)$, $Y = \mathscr{H}(C_0)\times\mathscr{H}(C)$. *Then* $d_a X = d_a Y$, *but* $X \not\approx Y$.

The equality of d_a follows from the inclusions $X \approx X\times X \supsetapprox Y \supset X$ (see Theorems 2.2 and 2.4).

To prove that X and Y are not isomorphic, we shall apply the following result of Dragilev [7]:

Every basis (e_n) *of the space* $\mathscr{H}(C_0)$ *is c-equivalent to the basis* (z^n), *i. e. there exist a sequence* (τ_n) *of positive numbers and a permutation* (p_n) *of indices such that the series* $\sum_n t_n(\tau_n e_{p_n})$ *converges if and only if the series* $\sum_n t_n z_n$ *converges.*

If $X \approx Y$ then the space $\mathscr{H}(C_0)$ would be isomorphic to the space $\mathscr{M}(c_{an})$, where $c_{an} = \exp an$ for $n = 2m-1$, $c_{an} = \exp - n/a$ for $n = 2m$. Now according to the theorem of Dragilev there would exist an increasing sequence (k_n) of non-negative integers such that $\mathscr{M}(\exp a, n) \approx \mathscr{M}(\exp - k_n/a)$, which contradicts Theorem 5.2.

4. General remarks. Let

$$\Psi(X) = \Big\{\psi(\varepsilon) = \log\varphi(\varepsilon)/\log\frac{1}{\varepsilon} : \varphi\in\Phi(X)\Big\}. \tag{58}$$

Condition (56) formulated in terms of $\Psi(X)$ has a form

(56′) $\psi\in\Psi(X)$ if and only if $2\psi\in\Psi(X)$.

It is easy to prove the following

THEOREM 5.4. *Let* X *and* Y *be power spaces satisfying condition* (56′). *Then* $\Psi(X\times Y) = \{\psi = \psi_1+\psi_2 : \psi_1\in\Psi(X),\ \psi_2\in\Psi(Y)\}$, $\Psi(X\hat{\otimes} Y) = \{\psi = \psi_1\cdot\psi_2 : \psi_1\in\Psi(X),\ \psi_2\in\Psi(Y)\}$.

All the spaces considered in § 4 are tensor products of k copies of single Köthe spaces; the results of § 4 are illustrations of Theorem of § 4.

The theorem of Dragilev cited in section 3 has been generalized by Mitiagin [20] in the following way:

In an arbitrary power space all bases are c-equivalent.

Applying this theorem, by the same consideration as in Example 5.2 we obtain

COROLLARY 5.4. *If* X *and* Y *are power spaces of finite and of infinite type, respectively, then* $X\times Y \not\approx X$, $X\times Y \not\approx Y$; $X\hat{\otimes} Y \not\approx X$, $X\hat{\otimes} Y \not\approx Y$.

In particular we have

Example 3. *Let* $X_i = \mathscr{M}\Big(\exp\big(a(n_1^{q_1}+\ldots+n_{r_i}^{q_{r_i}}) - \frac{1}{a}(n_{r_i+1}^{q_{r_i+1}}+\ldots+n_k^{q_k})\big)\Big)$, $Y_i = \mathscr{M}\big((n_1^{n_1}\ldots n_{r_i}^{n_{r_i}})^a\cdot(n_{r_i+1}^{n_{r_i+1}}\ldots n_k^{n_k})^{-1/a}\big)$ *for* $i = 1, 2$ *be spaces of type considered in* §§ 3, 4, *sections* 2 *and* 3. *If* $r_1 = 1$, $r_2 < k$, *then* $d_a X_1 = d_a X_2$, $d_a Y_1 = d_a Y_2$. *If* $r_1 = 1$, $r_2 > 1$ *or* $r_1 < k$, $r_2 = k$, *then* $X_1 \not\approx X_2$ *and* $Y_1 \not\approx Y_2$.

We do not know whether (56′) is true in the case where $1 < r_1 < r_2 < k$; in particular we do not know whether $H(C_0\times C^2) \not\approx H(C_0^2\times C)$?

This is obviously connected with the following.

Conjecture. Let X_1 and Y_1 be power spaces of infinite type, let X_2 and Y_2 be power spaces of finite type and let X_1, Y_1, X_2, Y_2 satisfy condition (56). If $X_1 \not\approx Y_1$ or $X_2 \not\approx Y_2$, then $X_1\hat{\otimes}X_2 \not\approx Y_1\hat{\otimes}Y_2$ (cf. Problem 2.1).

§ 6. Estimation of Taylor coefficients

Let $\boldsymbol{n} = (n_1, \ldots, n_k)$. We shall write $|\boldsymbol{n}| = n_1 + \ldots + n_k$.

THEOREM 6.1. *Let* $q > 0$. $(\xi_n) \in \mathcal{M}(\exp a|\boldsymbol{n}|^q)$ *if and only if* $\lim\limits_{|\boldsymbol{n}|\to\infty} \sqrt[|\boldsymbol{n}|^q]{|\xi_{\boldsymbol{n}}|} = 0$. $(\xi_n) \in \mathcal{M}(\exp - |\boldsymbol{n}|^q/a)$ *if and only if* $\overline{\lim\limits_{|\boldsymbol{n}|\to\infty}} \sqrt[|\boldsymbol{n}|^q]{|\xi_{\boldsymbol{n}}|} \leqslant 1$.

Proof. 1° The following conditions are equivalent:

(a1) $$(\xi_{\boldsymbol{n}}) \in \mathcal{M}(\exp a|\boldsymbol{n}|^q),$$

(a2) $$\lim_{|n|\to\infty} |\xi_{\boldsymbol{n}}| \exp a|\boldsymbol{n}|^q = 0 \quad \text{for} \quad a = 1, 2, \ldots,$$

(a3) $$\lim_{|n|\to\infty} (\log|\xi_{\boldsymbol{n}}| + a|\boldsymbol{n}|^q) = -\infty \quad \text{for} \quad a = 1, 2, \ldots,$$

(a4) $$\overline{\lim_{|n|\to\infty}} (\log|\xi_{\boldsymbol{n}}| + a|\boldsymbol{n}|^q) < 0 \quad \text{for} \quad a = 1, 2, \ldots,$$

(a5) $$\lim_{|n|\to\infty} \frac{\log \xi_n}{|n|^q} = -\infty,$$

(a6) $$\overline{\lim_{|n|\to\infty}} |\xi_{\boldsymbol{n}}|^{1/|n|^q} = 0.$$

2° The following conditions are equivalent:

(b1) $$(\xi_n) \in \mathcal{M}\big(\exp(-|\boldsymbol{n}|^q/a)\big),$$

(b3) $$\lim_{|\boldsymbol{n}|\to\infty} (\log|\xi_{\boldsymbol{n}}| - |\boldsymbol{n}|^q/a) = -\infty \quad \text{for} \quad a = 1, 2, \ldots,$$

(b5) $$\overline{\lim_{|\boldsymbol{n}|\to\infty}} \frac{\log \xi_{\boldsymbol{n}}}{|\boldsymbol{n}|^q} = 0,$$

(b6) $$\overline{\lim_{|\boldsymbol{n}|\to\infty}} |\xi_{\boldsymbol{n}}|^{1/|\boldsymbol{n}|^q} \leqslant 1,$$

q. e. d.

Applying Theorems 3.1, 3.4, 3.6 we obtain

COROLLARY 6.1. $\sum\limits_{\boldsymbol{n}\in\mathcal{N}^k} c_{\boldsymbol{n}} z^{\boldsymbol{n}} \in \mathcal{H}(C^k)$ *if and only if* $\lim \sqrt[|\boldsymbol{n}|]{c_{\boldsymbol{n}}} = 0$. $\sum\limits_{\boldsymbol{n}\in\mathcal{N}^k} c_{\boldsymbol{n}} z^{n} \in \mathcal{H}(C_0^k)$ *if and only if* $\overline{\lim} \sqrt[|\boldsymbol{n}|]{c_{\boldsymbol{n}}} \leqslant 1$ (*where* $\boldsymbol{n} = (n_1, \ldots, n_k)$, $z^{\boldsymbol{n}} = z_1^{n_1} \ldots z_k^{n_k}$).

COROLLARY 6.2. *Let* $\mu(\varepsilon, z) = \exp\big(-(\tau_1+\varepsilon)|z_1|^{p_1} - \ldots - (\tau_k+\varepsilon)|z_k|^{p_k}\big)$ *with* $\tau_i \geqslant 0$, $p_i > 0$. *If* $\tau_1 = \ldots = \tau_k = 0$, *then* $\sum\limits_{\boldsymbol{n}\in\mathcal{N}^k} c_{\boldsymbol{n}} z^{\boldsymbol{n}} \in \mathcal{H}_\mu$ *if and only if* $\overline{\lim} \sqrt[|\boldsymbol{n}|]{c_{\boldsymbol{n}} d_{\boldsymbol{n}}} = 0$. *If all* $\tau_i > 0$, *then* $\sum\limits_{\boldsymbol{n}\in\mathcal{N}^k} c_{\boldsymbol{n}} z^{\boldsymbol{n}} \in \mathcal{H}_\mu$ *if and only if* $\overline{\lim} \sqrt[|n|]{c_{\boldsymbol{n}} d_{\boldsymbol{n}}} \leqslant 1$, *where* $d_{\boldsymbol{n}}$ *are given by formula* (35).

COROLLARY 6.3. *Let* $\mu(\varepsilon, z) = \exp\big(-\varepsilon(\log 1/z)^{-s}\big)$, *with* $s > 0$. *Then* $\sum\limits_{n=0}^{\infty} c_n z^n \in \mathcal{H}_\mu(C_0)$ *if and only if* $\overline{\lim} \sqrt[n^q]{c_n} \leqslant 1$, *where* $q = s/(s+1)$.

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Bases, lacunary sequences and complemented subspaces in the spaces L_p

by

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In this paper we investigate the isomorphic structure (invariants of linear homeomorphisms) of subspaces of the space L_p $(1 \leqslant p < +\infty)$. We consider especially the properties of basic sequences (bases in subspaces), as well as the properties of subspaces complemented in L_p. These properties are connected with classical problems concerning lacunary series. We explain them in a more detailed way.

Let $p > 2$ and let (φ_n) be an orthonormal system. Then

$$\Big(\int_0^1 \Big|\sum_{i=1}^n t_i\varphi_i(t)\Big|^p dt\Big)^{1/p} \geqslant \Big(\int_0^1 \Big|\sum_{i=1}^n t_i\varphi_i(t)\Big|^2 dt\Big)^{1/2} = \Big(\sum_{i=1}^n |t_i|^2\Big)^{1/2}$$

for any scalars $t_1, t_2, \ldots, t_n$ $(n = 1, 2, \ldots)$.

An orthonormal system is said to be *p-lacunary* iff (1) the converse inequality

$$\Big(\int_0^1 \Big|\sum_{i=1}^n t_i\varphi_i(t)\Big|^p dt\Big)^{1/p} \leqslant C\Big(\sum_{i=1}^n |t_i|^2\Big)^{1/2}$$

holds for some C depending only on (φ_n) and for any $t_1, t_2, \ldots, t_n$ $(n = 1, 2, \ldots)$.

In the language of the functional analysis this means that there is an isomorphism (linear homeomorphism) of Hilbert space l_2 onto the closed linear manifold in L_p spanned on the functions φ_n. Under this isomorphism the unit vectors in l_2 correspond the functions φ_n, i. e. the basic sequence (φ_n) is equivalent to the unit vector basis in l_2 (see the definition in section 1). Moreover, the operator $T: x \to \big(\int_0^1 x(t)\varphi_n(t)\,dt\big)$ is a projection of L_p onto this manifold.

(1) We write “iff” instead of “if and only if”.