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# Joint probability distributions of observables in quantum mechanics

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**1. Introduction.** To every physical system in quantum mechanics there corresponds a Hilbert space $\boldsymbol{H}$ of complex-valued functions $\psi = \psi(x_1, x_2, \ldots, x_n)$. The variables $x_1, x_2, \ldots, x_n$ may be chosen in several ways, each giving rise to a consistent description equivalent to all others; here they will be taken to be space coordinates, for this gives rise to the form of quantum mechanics most commonly used, namely Schrödinger's. The number $n$ of variables is associated with the dimension of configuration space of the classical analogue of the system in question. The inner product on $\boldsymbol{H}$ is defined by the formula

$$(\psi_1, \psi_2) = \int\limits_X \psi_1(x_1, x_2, \ldots, x_n)\psi_2^*(x_1, x_2, \ldots, x_n)\,dx_1 dx_2 \ldots dx_n,$$

where $dx_1 dx_2 \ldots dx_n$ is the volume in the configuration space $X$ (usually $n$-dimensional Lebesgue measure) and $*$ denotes the complex conjugate. Every function $\psi$ belonging to the unit sphere of $\boldsymbol{H}$ is called a *state* of the physical system in question. In quantum theory to every physical quantity or *observable* there corresponds a self-adjoint (not necessarily bounded) linear operator on $\boldsymbol{H}$. For example, consider a system with one degree of freedom. Let $x$ be the Cartesian coordinate describing the position of a particle with the unit mass on a straight line ($-\infty < x < \infty$). Then $\boldsymbol{H}$ is the space of all square-integrable complex-valued functions on the real line. The position operator $P$ is defined by the formula $(P\psi)(x) = x\psi(x)$ on the linear manifold of all functions $\psi$ for which

$$\int\limits_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx$$

is finite. Further, the operator of the linear momentum $M$ is assumed to be $-i\hbar\dfrac{d}{dx}$, where $\hbar$ is an abbreviation for Planck's constant divided by $2\pi$. The linear momentum operator is defined on the linear manifold of all absolutely continuous functions from $\boldsymbol{H}$ with a square-integrable derivative. The operator $E_0$ of the total energy of a simple harmonic oscillator, i. e., the Hamiltonian operator of a simple harmonic oscillator, is of the form

$$(E_0\psi)(x) = -\frac{\hbar^2}{2}\frac{d^2}{dx^2}\psi(x)+\frac{\omega^2}{2}x^2\psi(x),$$

where $\omega$ is a parameter connected with the classical frequency of the oscillator. Other examples of operators associated with observables are given in [4], [6], [7] and [11].

A *spectral measure* is a function $\Pi$ whose domain is the Borel field of subsets of the real line $\mathcal{R}$ and whose values are projections on $\boldsymbol{H}$, such that $\Pi(\mathcal{R})$ is the unit operator and $\Pi\left(\bigcup_{n=1}^{\infty}\mathcal{E}_n\right) = \sum_{n=1}^{\infty}\Pi(\mathcal{E}_n)$ whenever $\mathcal{E}_1, \mathcal{E}_2, \ldots$ is a sequence of disjoint sets (for the properties of spectral measures see [5], p. 58). Let $A$ be a self-adjoint operator on $\boldsymbol{H}$ associated with an observable. Then there exists one and only one spectral measure $\Pi_A$ such that

$$A = \int_{-\infty}^{\infty}\lambda\Pi_A(d\lambda)$$

(cf. [12], p. 318, [13], p. 180). Let $\psi$ be a state of our physical system and let $\mathcal{E}$ be a Borel subset of the real line. The probability $p_A^{\psi}(\mathcal{E})$ that the observable whose operator is $A$ has at the state $\psi$ a value belonging to $\mathcal{E}$ is given by the formula

$$p_A^{\psi}(\mathcal{E}) = (\Pi_A(\mathcal{E})\psi, \psi).$$

This is the basic postulate of the quantum mechanics.

Now we shall quote well-known examples of probability distributions of observables.

The spectral measure of the position operator $P$ is given by the formula $\Pi_P(\mathcal{E})\psi = \chi_{\mathcal{E}}\psi$, where $\chi_{\mathcal{E}}$ denotes the indicator of the set $\mathcal{E}$, i. e., $\chi_{\mathcal{E}}(x) = 1$ or $0$, according to $x\in\mathcal{E}$ or $x\notin\mathcal{E}$. Thus

$$(1)\qquad p_P^{\psi}(\mathcal{E}) = \int_{\mathcal{E}}|\psi(x)|^2dx.$$

Further, the position operator and the differential operator $D = -i\dfrac{d}{dx}$ are unitarily equivalent. Namely, $D = F^{-1}PF$, where $F$ is the Fourier-Plancherel operator

$$(F\psi)(x) = \underset{T\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-T}^{T}\psi(t)e^{-ixt}dt.$$

Hence we get the unitary equivalence of the spectral measures $\Pi_D$ and $\Pi_P$: $\Pi_D(\mathcal{E}) = F^{-1}\Pi_P(\mathcal{E})F$ for every Borel set $\mathcal{E}$ (see [13], p. 242). Consequently,

$$(2)\qquad (\Pi_D(\mathcal{E})\psi, \psi) = (F^{-1}\Pi_P(\mathcal{E})F\psi, \psi) = (\Pi_P(\mathcal{E})F\psi, F\psi) = \int_{\mathcal{E}}|(F\psi)(x)|^2dx.$$

Hence, by virtue of the equality $M = \hbar D$, we get the probability distribution of the linear momentum at a state $\psi$

$$(3)\qquad p_M^{\psi}(\mathcal{E}) = \frac{1}{\hbar}\int_{\mathcal{E}}\left|(F\psi)\left(\frac{x}{\hbar}\right)\right|^2dx.$$

The spectrum of the energy operator $E_0$ of a harmonic oscillator consists only of proper values

$$\lambda_n = (n+\tfrac{1}{2})\hbar\omega \qquad (n = 0, 1, 2, \ldots)$$

corresponding to the proper functions

$$\psi_n(x) = \sqrt[4]{\frac{\omega}{\hbar}}\,H_n\left(\sqrt{\frac{\omega}{\hbar}}\cdot x\right) \qquad (n = 0, 1, 2, \ldots),$$

where $H_n$ are Hermite's functions

$$H_n(x) = \frac{(-1)^n}{\sqrt[4]{\pi}\,\sqrt{2^n\cdot n!}}\exp\left(\frac{x^2}{2}\right)\frac{d^n}{dx^n}\exp(-x^2) \qquad (n = 0, 1, 2, \ldots).$$

Since the family of Hermite's functions is complete, we have the equality

$$\Pi_{E_0}(\mathcal{E})\psi = \sum_{\lambda_n\in\mathcal{E}}(\psi, \psi_n)\psi_n,$$

whence the formula

$$p_{E_0}^{\psi}(\mathcal{E}) = \sum_{\lambda_n\in\mathcal{E}}|(\psi, \psi_n)|^2$$

follows.

In the present paper we shall define and study the notion of joint probability distribution of a system of observables.

**II. Joint probability distribution of observables.** Let $p$ be a probability measure of the Borel field of subsets of the $N$-dimensional Euclidean space $\mathcal{R}^n$. To every system $a_1, a_2, \ldots, a_N$ of real numbers there corresponds the family of parallel hyperplanes $S_t^{a_1,a_2,\ldots,a_N}$ given by the equations $\sum_{j=1}^{N}a_jx_j = t$ $(-\infty < t < \infty)$. Letting for every Borel subset $\mathcal{E}\subset\mathcal{R}$

$$p_{a_1,a_2,\ldots,a_N}(\mathcal{E}) = p\Big(\bigcup_{t\in\mathcal{E}}S_t^{a_1,a_2,\ldots,a_N}\Big)$$

we get a probability measure on $\mathscr{R}$. It is well-known (see [3], p. 291) that there is a one-to-one correspondence between the $N$-dimensional probability measure $p$ and the family of all induced one-dimensional probability measures $p_{a_1,a_2,\ldots,a_N}$ $(-\infty < a_1, a_2, \ldots, a_N < \infty)$. In the language of random variables this result is equivalent to the following: the family of probability distributions of all linear combinations $\sum_{j=1}^{N} a_j \xi_j$ of random variables $\xi_1, \xi_2, \ldots, \xi_N$ uniquely determine the joint probability distribution of these random variables. This remark leads to the following definition of the joint probability distribution of observables.

Given a Hilbert space $\boldsymbol{H}$ corresponding to a physical system, we consider a system of observables associated with self-adjoint operators $A_1, A_2, \ldots, A_N$. We suppose that for all systems $a_1, a_2, \ldots, a_N$ of real numbers the linear combinations $\sum_{j=1}^{N} a_j A_j$ are self-adjoint operators on $\boldsymbol{H}$. Consequently, for every system $a_1, a_2, \ldots, a_N$ of real numbers and every state $\psi \in \boldsymbol{H}$ the probability distribution $p^{\psi}_{a_1A_1+\ldots+a_NA_N}$ is well defined.

Given a state $\psi \in \boldsymbol{H}$, a probability measure $p$ on $\mathscr{R}^N$ is said to be *the joint probability distribution* of observables associated with the operators $A_1, A_2, \ldots, A_N$, if for every system $a_1, a_2, \ldots, a_N$ of real numbers, the projection $p_{a_1,a_2,\ldots,a_N}$ is equal to $p^{\psi}_{a_1A_1+\ldots+a_NA_N}$. Of course, the joint probability distribution is uniquely determined, provided it exists. In the sequel, we shall denote by $p^{\psi}_{A_1,A_2,\ldots,A_N}$ the joint probability distribution at a state $\psi$ of the system of observables associated with the operators $A_1, A_2, \ldots, A_N$. Generalized joint probability distributions (not necessarily positive) of the position and the linear momentum were first studied by E. Wigner [14] and J. E. Moyal [10].

It is very easy to formulate the necessary and sufficient condition for the existence of the joint probability distribution by means of characteristic functions. Namely, denoting by $\Phi^{\psi}_{a_1,a_2,\ldots,a_N}$ the characteristic function of the probability distribution $p^{\psi}_{a_1A_1+\ldots+a_NA_N}$:

$$\Phi^{\psi}_{a_1,a_2,\ldots,a_N}(t) = \int_{-\infty}^{\infty} e^{i\lambda t} p^{\psi}_{a_1A_1+\ldots+a_NA_N}(d\lambda),$$

we have the following statement:

*A system of observables associated with operators $A_1, A_2, \ldots, A_N$ has the joint probability distribution at a state $\psi \in \boldsymbol{H}$ if and only if the function of $N$ variables $\Phi^{\psi}_{t_1,t_2,\ldots,t_N}(1)$ is a characteristic function of an $N$-dimensional probability distribution. Moreover, the characteristic function of the joint probability distribution is equal to $\Phi^{\psi}_{t_1,t_2,\ldots,t_N}(1)$.*

Further, taking into account the obvious equality $\Phi^{\psi}_{0,0,\ldots,0}(t) = 1$ and applying Bochner's theorem on the representation of positive definite functions (see [1], p. 58, Theorem 3.2.3), we can write the last assertion in the following form:



*A system of observables associated with operators $A_1, A_2, \ldots, A_N$ has the joint probability distribution at a state $\psi \in \boldsymbol{H}$ if and only if $\Phi^{\psi}_{t_1,t_2,\ldots,t_N}(1)$ is a continuous positive definite function of $N$ variables.*

In classical physics it is possible in principle to determine the values of physical quantities simultaneously. In quantum theory the situation is quite different. There are physical quantities which do not permit the simultaneous definability of their values. For instance, according to the famous Heisenberg uncertainty principle the measurement of a particle's position disturbs its momentum, and vice versa, so that when one quantity is ascertained with precision, the other loses it. Thus in quantum theory we have two kinds of systems of observables. The simultaneous measurement of observables of the first kind yields definite values for them. Mathematically, to these systems of observables there correspond systems of operators $A_1, A_2, \ldots, A_N$ which commute with one another; i. e., denoting their spectral measures by $\Pi_{A_1}, \Pi_{A_2}, \ldots, \Pi_{A_N}$, for every pair $\mathscr{E}_1, \mathscr{E}_2$ of Borel sets we have the relation

$$\Pi_{A_i}(\mathscr{E}_1)\Pi_{A_j}(\mathscr{E}_2) = \Pi_{A_j}(\mathscr{E}_2)\Pi_{A_i}(\mathscr{E}_1) \quad (i, j = 1, 2, \ldots, N).$$

To systems of observables of the second kind there correspond non-commuting systems of operators and, in general, it is impossible to measure their values simultaneously. This classification is of great significance for contemporary physics. The concept of joint probability distributions of observables leads to a new classification of systems of physical quantities. Namely, a system of observables associated with operators $A_1, A_2, \ldots, A_N$ is said to be *probabilistically definite* if for every state $\psi \in \boldsymbol{H}$ the joint probability distribution $p^{\psi}_{A_1,A_2,\ldots,A_N}$ exists. Further, the system of observables is said to be *probabilistically indefinite* if for every state $\psi \in \boldsymbol{H}$ the joint probability distribution $p^{\psi}_{A_1,A_2,\ldots,A_N}$ does not exist. Systems of observables which are neither probabilistically definite nor indefinite will be called *mixed*. For a mixed system of observables associated with operators $A_1, A_2, \ldots, A_N$ we can find a pair $\psi_1, \psi_2$ of states in such a way that $p^{\psi_1}_{A_1,A_2,\ldots,A_N}$ exists and $p^{\psi_2}_{A_1,A_2,\ldots,A_N}$ does not exist. In the sequel we shall show that there exist systems of observables of all three kinds.

**III. Probabilistically definite systems.** We shall show that the probabilistic definability of observables is closely connected with the commutation of their operators.

THEOREM 1. *Every system of observables associated with commuting operators is probabilistically definite.*

Proof. Let $A_1, A_2, \ldots, A_N$ be a system of commuting self-adjoint operators corresponding to a system of observables. There exists then a self-adjoint operator $B$ such that all the operators $A_1, A_2, \ldots, A_N$ are functions of the operator $B$. Precisely, there exist real-valued functions $f_1, f_2, \ldots, f_N$ defined on the real line such that

$$A_j = \int_{-\infty}^{\infty} f_j(\lambda)\Pi_B(d\lambda) \qquad (j = 1, 2, \ldots, N),$$

where $\Pi_B$ is the spectral measure of $B$ (see [11], p. 355). Setting $g(\lambda) = \sum_{j=1}^{N} a_j f_j(\lambda)$, we have the equalities

$$\sum_{j=1}^{N} a_j A_j = \int_{-\infty}^{\infty} g(\lambda)\Pi_B(d\lambda) \quad \text{and} \quad \Pi_{a_1A_1+\ldots+a_NA_N}(\mathscr{E}) = \Pi_B(g^{-1}(\mathscr{E})).$$

Consequently, for every state $\psi \in \boldsymbol{H}$ we obtain the relation

$$\begin{aligned}\Phi^{\psi}_{a_1,a_2,\ldots,a_N}(t) &= \int_{-\infty}^{\infty} e^{i\lambda t} p^{\psi}_{a_1A_1+\ldots+a_NA_N}(d\lambda)\\ &= \int_{-\infty}^{\infty} e^{i\lambda t}\big(\Pi_B(g^{-1}(d\lambda))\psi, \psi\big) = \int_{-\infty}^{\infty} e^{ig(\lambda)t}\big(\Pi_B(d\lambda)\psi, \psi\big)\\ &= \int_{-\infty}^{\infty} \exp\big[it\big(a_1f_1(\lambda)+\ldots+a_Nf_N(\lambda)\big)\big] p^{\psi}_B(d\lambda).\end{aligned}$$

Hence we infer that, for every state $\psi \in \boldsymbol{H}$, $\Phi^{\psi}_{t_1,t_2,\ldots,t_N}$ (1) is a characteristic function. Thus our system of observables is probabilistically definite.

Now we shall prove the converse implication under an additional assumption concerning the spectrum of operators. We do not know whether this assumption is essential.

A self-adjoint operator $A$ is said to have a *purely point spectrum* if its spectral measure $\Pi_A$ is purely atomic. The spectrum of such an operator consists only of proper values.

THEOREM 2. *If all operators corresponding to a system of probabilistically definite observables have purely point spectra, then they commute with one another.*

Proof. Since every subsystem of a probabilistically definite system is the same one, it is sufficient to prove our theorem for a pair of observables. Let $A$ and $B$ be operators corresponding to them. Let $\lambda_1, \lambda_2, \ldots$ and $\mu_1, \mu_2, \ldots$ be sequences of all proper values of $A$ and $B$ respectively. It is obvious that for all states $\psi \in \boldsymbol{H}$ the probability measures $p^{\psi}_A$ and $p^{\psi}_B$ are purely atomic and concentrated on the sets $\{\lambda_j: j = 1, 2, \ldots\}$ and $\{\mu_j: j = 1, 2, \ldots\}$ respectively. Thus the joint probability distribution $p^{\psi}_{A,B}$ is concentrated on the subset $\{\langle\lambda_i, \mu_j\rangle: i, j = 1, 2, \ldots\}$ of the plane. Since all numbers of the form

$$\frac{\mu_j - \mu_r}{\lambda_i - \lambda_s} \qquad (i \neq s;\ i, j, s, r = 1, 2, \ldots)$$

form an at most denumerable set, we can find a real number $a$ which differs from these numbers. The probability distribution $p^{\psi}_{aA+B}$ is concentrated on the set $\{a\lambda_i + \mu_j: i, j = 1, 2, \ldots\}$. Moreover, the equality $a\lambda_i + \mu_j = a\lambda_s + \mu_r$ holds if and only if $\lambda_i = \lambda_s$ and $\mu_j = \mu_r$. Hence we get the equality

$$(4) \qquad p^{\psi}_{aA+B}(\{a\lambda_i+\mu_j\}) = p^{\psi}_{A,B}(\{\langle\lambda_i, \mu_j\rangle\}) \qquad (i, j = 1, 2, \ldots).$$

Further, since for all states $\psi \in \boldsymbol{H}$ the probability distributions $p^{\psi}_{aA+B}$ are concentrated on the same set $\{a\lambda_i+\mu_j: i, j = 1, 2, \ldots\}$, the operator $aA+B$ has a purely point spectrum contained in this set. Let $\boldsymbol{H}^A_i$, $\boldsymbol{H}^B_j$ and $\boldsymbol{H}^{aA+B}_{i,j}$ be subspaces of $\boldsymbol{H}$ spanned by all proper functions corresponding to the proper values $\lambda_i$, $\mu_j$ and $a\lambda_i+\mu_j$ of the operators $A$, $B$ and $aA+B$, respectively. Taking a state $\psi$ from $\boldsymbol{H}^{aA+B}_{i,j}$ we have $p^{\psi}_{aA+B}(\{a\lambda_i+\mu_j\}) = 1$ and, by virtue of (4), $p^{\psi}_{A,B}(\{\langle\lambda_i, \mu_j\rangle\}) = 1$, which implies the relations $p^{\psi}_A(\{\lambda_i\}) = 1$ and, $p^{\psi}_B(\{\mu_j\}) = 1$. Hence, according to the equalities $p^{\psi}_A(\{\lambda_i\}) = (\Pi_A(\{\lambda_i\})\psi, \psi)$, $p^{\psi}_B(\{\mu_j\}) = (\Pi_B(\{\mu_j\})\psi, \psi)$ and $(\psi, \psi) = 1$, we obtain the relations $\Pi_A(\{\lambda_i\})\psi = \psi$ and $\Pi_B(\{\mu_j\})\psi = \psi$. Consequently, $\psi \in \boldsymbol{H}^A_i \cap \boldsymbol{H}^j_B$ and $\boldsymbol{H}^A_i = \boldsymbol{H}^{aA+B}_{i,1} \oplus \boldsymbol{H}^{aA+B}_{i,2} \oplus \ldots$, $\boldsymbol{H}^B_j = \boldsymbol{H}^{aA+B}_{1,j} \oplus \boldsymbol{H}^{aA+B}_{2,j} \oplus \ldots$, where $\oplus$ denotes the direct sum of subspaces. Since $\Pi_A(\{\lambda_i\})$ and $\Pi_B(\{\mu_j\})$ are projections on the subspaces $\boldsymbol{H}^A_i$ and $\boldsymbol{H}^B_j$ respectively, we infer, in view of the decompositions of $\boldsymbol{H}^A_i$ and $\boldsymbol{H}^B_j$ into direct sums, that $\Pi_A(\{\lambda_i\})\Pi_B(\{\mu_j\}) = \Pi_B(\{\mu_j\})\Pi_A(\{\lambda_i\})$ $(i, j = 1, 2, \ldots)$. Consequently, $A$ and $B$ commute with one another. The theorem is thus proved.

Given a self-adjoint operator $A$, a real-valued function $f$ defined on the real line is said to be *admissible* if the operator $f(A)$ defined by means of the spectral integral

$$f(A) = \int_{-\infty}^{\infty} f(\lambda)\Pi_A(d\lambda)$$

is also self-adjoint. Since for every system of commuting operators $A_1, A_2, \ldots, A_N$ and every system of admissible functions $f_1, f_2, \ldots, f_N$ the operators $f_1(A_1), f_2(A_2), \ldots, f_N(A_N)$ commute with one another, we get from Theorem 1 the following

COROLLARY 1. *Let $A_1, A_2, \ldots, A_N$ be commuting self-adjoint operators. For every system $f_1, f_2, \ldots, f_N$ of admissible functions the system of observables associated with operators $f_1(A_1), f_2(A_2), \ldots, f_N(A_N)$ is probabilistically definite.*

The converse implication is also true. For any self-adjoint operator $A$ the indicator of an arbitrary Borel set is an admissible function and

$$\Pi_A(\mathscr{E}) = \int_{-\infty}^{\infty} \chi_{\mathscr{E}}(\lambda)\Pi_A(d\lambda).$$

Moreover, projection operators have purely point spectra contained in the two-point set $\{0,1\}$. Thus as a direct consequence of the Theorem 2 we get the following

COROLLARY 2. *Let $A_1, A_2, \ldots, A_N$ be self-adjoint operators. If for every system $f_1, f_2, \ldots, f_N$ of admissible functions the observables associated with the operators $f_1(A_1), f_2(A_2), \ldots, f_N(A_N)$ are probabilistically definite, then the operators $A_1, A_2, \ldots, A_N$ commute with one another.*

**IV. Probabilistically indefinite systems.** In the one-dimensional motion of a particle with the unit mass the Hamiltonian operator $E$ of the total energy is given by the formula

$$(E\psi)(x) = -\frac{\hbar^2}{2}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x),$$

where $V$ is the potential energy of the classical analogue of the system in question. According to the physical meaning of the function $V$, we assume that it is bounded below and locally integrable. It is well-known that under these assumptions the operator $E$ is self-adjoint (see [2]).

THEOREM 3. *If for every positive number $c$ the potential energy $V$ satisfies the condition*

$$\lim_{|x|\to\infty} \int_x^{x+c} V(t)\,dt = \infty = \lim_{|x|\to\infty} \int_x^{x+c} \big(V(t)+t\big)\,dt$$

*and the function $V(x)+x$ is bounded below on the whole line, then the total energy and the position form a probabilistically indefinite system.*

Since the potential energy $V(x) = \frac{1}{2}\omega^2 x^2$ of a simple harmonic oscillator fulfills the conditions of the theorem, we infer that the total energy and the position of a harmonic oscillator are probabilistically indefinite observables.

The proof of the Theorem will be carried out in a Lemma.

LEMMA. *Let $A$ and $B$ be operators associated with a pair of observables. If $\psi$ is such a state that the probability distributions $p_A^\psi$ and $p_{A+B}^\psi$ are purely atomic and the probability distribution $p_B^\psi$ is absolutely continuous, then the joint probability distribution $p_{A,B}^\psi$ does not exist.*

Proof. Contrary to this statement let us suppose that the joint probability distribution $p_{A,B}^\psi$ exists. Put $\mathscr{E} = \{\lambda: \lambda \neq c_i - a_j;\, i,j = 1,2,\ldots\}$, where $\{a_j: j = 1,2,\ldots\}$ and $\{c_j: j = 1,2,\ldots\}$ are the sets on which the probability distributions $p_A^\psi$ and $p_{A+B}^\psi$ are concentrated respectively. Obviously, we have the inequality

$$p_{A,B}^\psi(a_k\times\mathscr{E}) \leqslant p_{A+B}^\psi(a_k+\mathscr{E}) \leqslant p_{A+B}^\psi(\mathscr{D}) = 0 \qquad (k = 1,2,\ldots),$$

where $\mathscr{D}$ is the complement of the set $\{c_j: j = 1,2,\ldots\}$, $a_k\times\mathscr{E} = \{\langle a_k,\lambda\rangle: \lambda\in\mathscr{E}\}$ and $a_k+\mathscr{E} = \{a_k+\lambda: \lambda\in\mathscr{E}\}$. Hence we get the equality $p_B^\psi(\mathscr{E}) = \sum_{k=1}^{\infty} p_{A,B}^\psi(a_k\times\mathscr{E}) = 0$, which implies that the probability distribution $p_B^\psi$ is concentrated on the complement of $\mathscr{E}$. But this complement is at most denumerable, which contradicts the absolute continuity of $p_B^\psi$. The Lemma is thus proved.

Proof of theorem 3. We already know that for every state $\psi\in\boldsymbol{H}$ the probability distribution of the position $p_P^\psi$ is absolutely continuous (see formula (1)). Now we shall prove that for every state $\psi\in\boldsymbol{H}$ the probability distributions $p_E^\psi$ and $p_{E+P}^\psi$ are purely atomic. To prove this it is sufficient to show that both operators $E$ and $E+P$ have purely point spectra. But this fact follows from the following general theorem on differential operators of the second order (see [9], p. 175): if the function $q$ is locally integrable, bounded below and

$$\lim_{|x|\to\infty} \int_x^{x+c} q(t)\,dt = \infty \quad \text{for all} \quad c > 0,$$

then the operator

$$(A\psi)(x) = -\frac{d^2}{dx^2}\psi(x) + q(x)\psi(x)$$

has a discrete and, consequently, purely point spectrum. Thus, by the Lemma, for every state $\psi\in\boldsymbol{H}$ the point probability distribution $p_{E,P}^\psi$ does not exist.

**V. Mixed systems.** It may occur that for given non-commuting self-adjoint operators $A$ and $B$ with purely point spectra there exists a common proper function, say $\psi_0$. Of course, by Theorem 2, the system of observables associated with operators $A$ and $B$ is not probabilistically definite. On the other hand, it is very easy to prove that this system of observables has the joint probability distribution at the state $\psi_0$. In fact, denoting by $\lambda_0$ and $\mu_0$ the proper values of the operators $A$ and $B$, respectively, corresponding to the common proper function $\psi_0$, we infer that $\psi_0$ is the proper function of the operator $aA+bB$ corresponding to the proper value $a\lambda_0+b\mu_0$. Thus $p_{aA+bB}^{\psi_0}(\mathscr{E}) = \chi_{\mathscr{E}}(a\lambda_0+b\mu_0)$, where $\chi_{\mathscr{E}}$ denotes the indicator of $E$. Hence, by simple computations, $\Phi_{a,b}^{\psi_0}(t) = \exp\big(i(a\lambda_0+b\mu_0)t\big)$ and, consequently, $\Phi_{1,t_2}^{\psi_0}(1) = \exp\big(i(\lambda_0 t_1+\mu_0 t_2)\big)$. But the last function is a characteristic function, which gives the existence of the joint probability distribution at the state $\psi_0$. Thus the system of observables in question is mixed.

Now we shall present a less trivial example of mixed systems of observables associated with operators having no common proper function. We shall use previously introduced notation $P$ and $M$ for the operators corresponding to the position and the linear momentum respectively in a one-dimensional motion of a particle. First of all we shall prove the following

THEOREM 4. *The joint probability distribution $p^{\psi}_{M,P}$ exists at a state $\psi$ if and only if the inequality*

$$\int_{-\infty}^{\infty} e^{itx}\psi(y+t)\psi^*(y-t)\,dt \geqslant 0 \tag{5}$$

*holds for all $x$ and $y$. Moreover, the joint probability distribution $p^{\psi}_{M,P}$ is always absolutely continuous and its density function $g^{\psi}_{M,P}$ is given by the formula*

$$g^{\psi}_{M,P}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\psi\left(y+\frac{\hbar}{2}t\right)\psi^*\left(y-\frac{\hbar}{2}t\right)dt\ (^1). \tag{6}$$

Proof. First let us consider the operator $aM+bP$ with the positive coefficient $a$. It is well-known that the study of any self-adjoint differential operator of the first order may be reduced to the study of the operator $D = -i\frac{d}{dx}$ (see [13], p. 425, Theorem 10.6). Applying this result to the operator $aM+bP$, we get the unitary equivalence of the operators $aM+bP$ and $D$. More precisely, the formula

$$(U\psi)(x) = \sqrt{a\hbar}\exp\left(\tfrac{1}{2}iab\hbar x^2\right)\psi(a\hbar x) \tag{7}$$

defines a unitary operator on $\boldsymbol{H}$ such that $aM+bP = U^{-1}DU$. This equality implies the unitary equivalence of spectral measures $\Pi_{aM+bP}(\mathscr{E}) = U^{-1}\Pi_D(\mathscr{E})U$ (see [13], p. 242). Hence and from (2) for every state $\psi\in\boldsymbol{H}$ we obtain the probability distribution

$$p^{\psi}_{aM+bP}(\mathscr{E}) = \left(\Pi_{aM+bP}(\mathscr{E})\psi,\psi\right) = \left(U^{-1}\Pi_D(\mathscr{E})U\psi,\psi\right)$$
$$= \left(\Pi_D(\mathscr{E})U\psi, U\psi\right) = \int_{\mathscr{E}} |(FU\psi)(x)|^2dx.$$

Set $\psi_n(x) = \psi(x)$ or $0$ according to $|x|\leqslant a\hbar n$ or $|x|>a\hbar n$ $(n = 1,2,\ldots)$. The convergence $\operatorname*{l.i.m}_{n\to\infty}\psi_n = \psi$ is evident. Hence for every Borel set $\mathscr{E}$ we get the convergence

$$\lim_{n\to\infty}\left(\Pi_D(\mathscr{E})U\psi_n, U\psi_n\right) = \left(\Pi_D(\mathscr{E})U\psi, U\psi\right) = p^{\psi}_{aM+bP}(\mathscr{E}).$$

Thus the characteristic function $\Phi^{\psi}_{a,b}$ of the probability distribution $p^{\psi}_{aM+bP}$ can be written in the form

$$\Phi^{\psi}_{a,b}(t) = \lim_{n\to\infty}\int_{-\infty}^{\infty} e^{i\lambda t}|(FU\psi_n)(\lambda)|^2d\lambda. \tag{8}$$

Further, according to (7), we have the equality

$$\int_{-\infty}^{\infty} e^{i\lambda t}|(FU\psi_n)(\lambda)|^2d\lambda = \frac{a\hbar}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda t}\left|\int_{-n}^{n}\exp\left(-ix\lambda+\frac{1}{2}iab\hbar x^2\right)\psi(a\hbar x)\,dx\right|^2d\lambda$$
$$= \lim_{\Lambda\to\infty}\frac{a\hbar}{2\pi}\int_{-\Lambda}^{\Lambda}\int_{-n}^{n}\int_{-n}^{n}\exp\Big(i\lambda t - iy\lambda+\frac{1}{2}iab\hbar y^2+ix\lambda - \frac{1}{2}iab\hbar x^2\Big)\psi(a\hbar y)\psi^*(a\hbar x)\,dx\,dy\,d\lambda.$$

Since the integrand in the last expression is absolutely integrable over the domain $-\Lambda\leqslant\lambda\leqslant\Lambda$, $-n\leqslant x\leqslant n$, $-n\leqslant y\leqslant n$, we may change the order of integrations, which leads to the equality

$$\int_{-\infty}^{\infty} e^{i\lambda t}|(FU\psi_n)(\lambda)|^2d\lambda = \lim_{\Lambda\to\infty}\frac{a\hbar}{\pi}\int_{-n}^{n}\int_{-n}^{n}\exp\left(\frac{1}{2}iab\hbar(y^2-x^2)\right)\frac{\sin\Lambda(t+x-y)}{t+x-y}\psi(a\hbar y)\psi^*(a\hbar x)\,dx\,dy.$$

Hence, by well-known properties of the kernel $(\sin\Lambda z)/z$, we get the formula

$$\int_{-\infty}^{\infty} e^{i\lambda t}|(FU\psi_n)(\lambda)|^2d\lambda = a\hbar\exp\left(\tfrac{1}{2}iab\hbar t^2\right)\int_{-\infty}^{\infty} e^{iab\hbar xt}\psi_n\big(a\hbar(x+t)\big)\psi_n^*(a\hbar x)\,dx,$$

which together with (8) implies the equality

$$\Phi^{\psi}_{a,b}(t) = a\hbar\exp\left(\tfrac{1}{2}iab\hbar t^2\right)\int_{-\infty}^{\infty} e^{iab\hbar xt}\psi\big(a\hbar(x+t)\big)\psi^*(a\hbar x)\,dx.$$

(1) This expression for the density function was discovered by L. Szilard and E. Wigner ([14], p. 750).

The substitution $ah x+\frac{1}{2}ah t = z$ reduces the last expression to the form

$$\Phi^{\psi}_{a,b}(t) = \int_{-\infty}^{\infty} e^{ibzt}\psi\left(z+\frac{ah}{2}t\right)\psi^*\left(z-\frac{ah}{2}t\right)dz. \tag{9}$$

We have proved this equality under the assumption $a>0$. Now we shall show that (9) holds for every real $a$. For negative $a$ this a direct consequence of the equality $\Phi^{\psi}_{a,b}(t) = \left(\Phi^{\psi}_{-a,-b}(t)\right)^*$. If $a=0$ and $b\neq 0$, then, by formula (1),

$$p^{\psi}_{bP}(\mathcal{E}) = \frac{1}{|b|}\int_{\mathcal{E}}\left|\psi\left(\frac{x}{b}\right)\right|^2 dx$$

and, consequently,

$$\Phi^{\psi}_{0,b}(t) = \int_{-\infty}^{\infty} e^{i\lambda t}p^{\psi}_{bP}(d\lambda) = \int_{-\infty}^{\infty} e^{ibzt}|\psi(z)|^2 dz.$$

Finally, $\Phi^{\psi}_{0,0}(t)\equiv 1$. Thus formula (9) holds for the arbitrary pair $a$ and $b$.

From (9) we obtain the equality

$$\Phi^{\psi}_{t_1,t_2}(1) = \int_{-\infty}^{\infty} e^{it_2 z}\psi\left(z+\frac{h}{2}t_1\right)\psi^*\left(z-\frac{h}{2}t_1\right)dz.$$

Now it is very easy to verify that $\Phi^{\psi}_{t_1,t_2}(1)$ is the Fourier transform of the continuous function $g^{\psi}_{M,P}$ defined by formula (6). Thus $\Phi^{\psi}_{t_1,t_2}(1)$ is a characteristic function if and only if $g^{\psi}_{M,P}(x,y)\geqslant 0$ for all $x$ and $y$. But this inequality is equivalent to inequality (5). Moreover, $g^{\psi}_{M,P}$ is the density function of the joint probability distribution, provided it exists. The Theorem is thus proved.

From Theorem 4 we get the following

COROLLARY. *The linear momentum and the position of a particle in a one-dimensional motion form a mixed system of observables.*

In fact, setting

$$\psi_1(x) = H_1(x) = \frac{\sqrt{2}\,x}{\sqrt[4]{\pi}}\exp\left(-\frac{x^2}{2}\right),$$

we have the inequality

$$\int_{-\infty}^{\infty} e^{itx}\psi_1(y+t)\psi_1^*(y-t)\,dt = \left(\frac{x^2}{2}+2y^2-1\right)\exp\left(-\frac{x^2}{4}-y^2\right) < 0$$



for all $x$ and $y$ for which $\frac{1}{2}x^2+2y^2<1$. Thus, by Theorem 4 the joint probability distribution $p^{\psi_1}_{M,P}$ does not exist. Further, setting

$$\psi_2(x) = H_0(x) = \frac{1}{\sqrt[4]{\pi}}\exp\left(-\frac{x^2}{2}\right), \tag{10}$$

we have the inequality

$$\int_{-\infty}^{\infty} e^{itx}\psi_2(y+t)\psi_2^*(y-t)\,dt = \exp\left(-\frac{x^2}{4}-y^2\right) > 0,$$

and, consequently, by Theorem 4, the joint probability distribution $p^{\psi_2}_{M,P}$ exists. Thus the linear momentum and the position form a mixed system of observables. We remark that the operators $M$ and $P$ have no common proper function.

We say that observables associated with operators $A$ and $B$ are *independent at a state* $\psi\in\boldsymbol{H}$ if the joint probability distribution $p^{\psi}_{A,B}$ exists and is a direct product of the marginal probability distributions $p^{\psi}_A$ and $p^{\psi}_B$, i. e., for every pair $\mathcal{E}_1,\mathcal{E}_2$ of Borel sets $p^{\psi}_{A,B}(\mathcal{E}_1\times\mathcal{E}_2) = p^{\psi}_A(\mathcal{E}_1)p^{\psi}_B(\mathcal{E}_2)$, where $\mathcal{E}_1\times\mathcal{E}_2$ is the Cartesian product of $\mathcal{E}_1$ and $\mathcal{E}_2$.

By (1) and (3) the probability distributions of the position and the linear momentum are absolutely continuous. Denoting by $g^{\psi}_P$ and $g^{\psi}_M$ their density functions at a state $\psi$, we have the equalities

$$g^{\psi}_P(x) = |\psi(x)|^2,\qquad g^{\psi}_M(x) = \frac{1}{h}\left|(F\psi)\left(\frac{x}{h}\right)\right|^2.$$

Hence for the state $\psi_2$ defined by formula (10) we have the equalities

$$g^{\psi_2}_P(x) = \frac{1}{\sqrt{\pi}}\exp(-x^2),\qquad g^{\psi_2}_M(x) = \frac{1}{\sqrt{\pi}\,h}\exp\left(-\frac{x^2}{h^2}\right)$$

and, according to (6),

$$g^{\psi_2}_{M,P}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\psi_2\left(y+\frac{h}{2}t\right)\psi_2^*\left(y-\frac{h}{2}t\right)dt$$

$$= \frac{1}{\pi h}\exp\left(-\frac{x^2}{h^2}-y^2\right).$$

Thus $g^{\psi_2}_{M,P}(x,y) = g^{\psi_2}_M(x)g^{\psi_2}_P(y)$ and, consequently, the linear momentum and the position are independent at the state $\psi_2$. Now we shall find all such states.

THEOREM 5. *The linear momentum and the position are independent at a state $\psi$ if and only if the equality*

(11) $$\psi(x) = c\sqrt[4]{\frac{2a}{\pi}}\exp\left(-ax^2+bx-\frac{(\operatorname{Re}b)^2}{4a}\right),$$

*where $a>0$, $b$ and $c$ are complex numbers and $|c|=1$, holds almost everywhere.*

Proof. Sufficiency. The identical computations as in the case of the state (10) leads for every state $\psi$ of form (11) to the formulae

$$\int_{-\infty}^{\infty} e^{itx}\psi(y+t)\psi^*(y-t)\,dt$$
$$= \exp\left(-2ay^2+2\operatorname{Re}by-\frac{(x+2\operatorname{Im}b)^2}{8a}-\frac{(\operatorname{Re}b)^2}{2a}\right),$$
$$g^{\psi}_{M,P}(x,y) = \frac{1}{\pi\hbar}\exp\left(-\frac{(\operatorname{Im}b\hbar-x)^2}{2a\hbar^2}-\left(\sqrt{2a}y-\frac{\operatorname{Re}b}{\sqrt{2a}}\right)^2\right),$$
$$g^{\psi}_{M}(x) = \frac{1}{\hbar\sqrt{2a\pi}}\exp\left(-\frac{(\operatorname{Im}b\hbar-x)^2}{2a\hbar^2}\right),$$
$$g^{\psi}_{P}(x) = \sqrt{\frac{2a}{\pi}}\exp\left(-\left(\sqrt{2a}x-\frac{\operatorname{Re}b}{\sqrt{2a}}\right)^2\right).$$

Thus the joint probability distribution exists and, moreover, $g^{\psi}_{M,P}(x,y) = g^{\psi}_{M}(x)g^{\psi}_{P}(y)$. In other words, the linear momentum and the position are independent at any state $\psi$ of form (11).

Necessity. Now let us suppose that the linear momentum and the position are independent at a state $\psi$. Taking into account equalities (1) and (6), we have the equality

$$g^{\psi}_{M}(x)|\psi(y)|^2 = g^{\psi}_{M}(x)g^{\psi}_{P}(y) = g^{\psi}_{M,P}(x,y)$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\psi\left(y+\frac{\hbar}{2}t\right)\psi^*\left(y-\frac{\hbar}{2}t\right)dt$$

almost everywhere. Since the right side of this equality is continuous with respect to $y$, we may assume without loss of generality of our considerations that the function $|\psi(y)|^2$ is continuous. Further, from the last equality, by means of the Fourier transformation with respect to the variable $x$, we get the following one:

(12) $$\Phi(t)|\psi(y)|^2 = \psi\left(y+\frac{\hbar}{2}t\right)\psi^*\left(y-\frac{\hbar}{2}t\right)$$



for almost all $t$ and $y$, where $\Phi$ is the characteristic function of the probability distribution $p^{\psi}_{M}$. By substitution $y+\frac{1}{2}\hbar t = u$, $y-\frac{1}{2}\hbar t = v$ into (12), we obtain the equality

(13) $$\Phi\left(\frac{u-v}{\hbar}\right)\left|\psi\left(\frac{u+v}{2}\right)\right|^2 = \psi(u)\psi^*(v)$$

for almost all pairs $u, v$. Since the left side of the laste quality is a continuous function, the function $\psi$ is equivalent to a continuous function and, consequently, may be supposed to be continuous. Then, of course, equality (13) holds for all $u$ and $v$. Now we shall prove that

(14) $$\psi(u)\neq 0 \quad \text{for every} \quad u.$$

Contrary to this let us suppose that $\psi(u_0)=0$ for a point $u_0$. Since $\Phi$ is the characteristic function, we can find a number $T$ such that $\Phi(t)\neq 0$ whenever $|t|\leqslant T$. Substituting into (13) $u=u_0$ and $v=u_0+2z$, where $|z|\leqslant \hbar T/4$ we have $\Phi(4z/\hbar)|\psi(u_0+z)|^2 = 0$. Thus the function $\psi$ vanishes on the whole interval $u_0-\hbar T/4\leqslant u\leqslant u_0+\hbar T/4$. The iteration of this procedure leads to the equality $\psi(u)=0$ for all $u$, which contradicts the equality $(\psi,\psi)=1$. Relation (14) is thus proved.

Setting $v=-u$ into (13), we get the formula

(15) $$\Phi\left(\frac{2u}{\hbar}\right) = \frac{\psi(u)\psi^*(-u)}{|\psi(0)|^2}$$

which together with (13) gives the equality

(16) $$\psi\left(\frac{u-v}{2}\right)\psi^*\left(\frac{v-u}{2}\right)\left|\psi\left(\frac{u+v}{2}\right)\right|^2 = |\psi(0)|^2\psi(u)\psi^*(v).$$

We introduce the auxiliary function

$$F(x) = \frac{|\psi(x)|^2|\psi(-x)|^2}{|\psi(0)|^4}.$$

By (16) is satisfies the equation

(17) $$F^2\left(\frac{u-v}{2}\right)F^2\left(\frac{u+v}{2}\right) = F(u)F(v).$$

Moreover, by (15),

(18) $$\left|\Phi\left(\frac{4u}{\hbar}\right)\right|^2 = F(u).$$

We now proceed to the problem of determining the form of the function $F$.

First by induction on $n$ we shall prove the formula

$$F(nt) = F^{n^2}(t) \quad (n = 0, 1, 2, \ldots). \tag{19}$$

For $n = 0$ and 1 it is evident. Suppose that our assertion is true for all non-negative integers smaller than $n$ $(n \geqslant 2)$. By substitution $u = nt$, $v = (n-2)t$ into (17) we get the equality $F^2(t)F^2((n-1)t) = F(nt)F((n-2)t)$. Hence

$$F(nt) = \frac{F^2(t)F^2((n-1)t)}{F((n-2)t)} = \frac{F^2(t)F^{2(n-1)^2}(t)}{F^{(n-2)^2}(t)} = F^{n^2},$$

which completes the proof of (19).

Substituting $t = m/n$ $(m = 0, 1, 2, \ldots)$ into (19) we have the equality $F(m) = F^{n^2}(m/n)$. But, on the other hand, $F(m) = F^{m^2}(1)$, which implies the equality $F(m/n) = F^{(m/n)^2}(1)$ $(n, m = 1, 2, \ldots)$. Since the function $F$ is continuous and even, the last equality implies the following one: $F(u) = F^{u^2}(1)$ $(-\infty < u < \infty)$. Hence, by (18), we get the formula $|\Phi(u)|^2 = \exp(-a_0u^2)$, where $a_0$ is a real constant. Since, by (15), the function $\Phi$ is integrable over the whole line, this constant must be positive. In other words, we have proved that the product $\Phi\cdot\Phi^*$ is the characteristic function of a Gaussian distribution. Applying Cramer's Theorem on decomposition of Gaussian distributions (cf. [8], p. 271) we infer that $\Phi$ is the same one, i. e., $\Phi(t) = \exp(ia_1t - a_2t^2)$, where $a_1$, $a_2$ are real constants and $a_2 > 0$. By the last equality and letting

$$G(x) = \frac{\psi(x)}{\psi(0)}\exp\left(-\frac{ia_1x}{\hbar} + \frac{2a_2x^2}{\hbar^2}\right) \tag{20}$$

we have, according to (13), the equation

$$\left|G\left(\frac{u+v}{2}\right)\right|^2 = G(u)G^*(v). \tag{21}$$

Hence, in view of the equality $G(0) = 1$, we obtain the relation $|G(u/2)|^2 = G(u)$. Thus $G$ is a positive function and equality (21) can be written in the form $G(u+v) = G(u)G(v)$. It is well-known that all continuous positive solutions of the last equation are of the form $G(u) = e^{a_3u}$, where $a_3$ is a real constant. Comparing this result with (20), we conclude that the function $\psi$ is of the form $\psi(x) = \psi(0)\exp(-ax^2+bx)$, where $a > 0$ and $b$ is a complex constant. From the normalization condition $\int_{-\infty}^{\infty}|\psi(x)|^2dx = 1$ we get the formula

$$|\psi(0)| = \sqrt[4]{\frac{2a}{\pi}}\exp\left(-\frac{(\mathrm{Re}\,b)^2}{4a}\right),$$

which completes the proof of the theorem.



Denoting by $\sigma_\psi^2(M)$ and $\sigma_\psi^2(P)$ the variances at a state $\psi$ of the linear momentum and the position respectively, we have Heisenberg's uncertainty relation

$$\sigma_\psi^2(M)\sigma_\psi^2(P) \geqslant \tfrac{1}{4}\hbar^2.$$

It gives an indication of how far one can ascribe values simultaneously to the position of a particle and to its momentum. It is well-known that the necessary and sufficient condition for equality in Heisenberg's relation is simply that the state $\psi$ is of the form (11) (see [11], p. 124). Thus as a direct consequence of Theorem 5 we get the following

COROLLARY. *The linear momentum and the position of a particle are independent at a state $\psi$ if and only if at this state the equality in Heisenberg's relation holds.*

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