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Integrals of distributions

by

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It is a great advantage of the theory of distributions that differentiation is always feasible and that many classical theorems can be formulated in a much simpler way and under weaker hypotheses than in the classical Analysis. Therefore calculation in the theory of distributions is easier and more automatical than in the classical Analysis. To simplify the calculation on distributions, it is very convenient to use a notation similar to that used for functions, and to formulate definitions in the same way, if possible, as in the classical Analysis. The original theory of Schwartz [7] does not satisfy these conditions. It was a good idea of Mikusiński [5] to use for distributions the same notation as for functions (this notation has systematically been used in our papers [6]). The aim of this paper is to show that the notion of convolution and of the Fourier transform of distributions can be introduced formally in the same way as in the classical Analysis. For this purpose I introduce a distributional substitute of the notion of the definite integral of functions. Using this notion I define the improper integrals of distributions and I define convolution and the Fourier transform of distributions by the same formulas as in the classical Analysis. I also show how to deduce fundamental theorems on convolution and the Fourier transform from the definition adopted. The idea of defining the convolution and the Fourier transform of distributions by this method was suggested to me by Mikusiński.

It is irrelevant in this paper which definition of distributions is adopted: the original functional definition of Soboleff [14] and Schwartz [7], the sequential definition of Mikusiński [4] or [5] (see also Korevaar [3] and Mikusiński and Sikorski [6]), the differential definition of Halperin [1] and Sikorski [8] (see also König [2], Słowikowski [11, 12, 13]), or the axiomatic definition suggested by Sikorski [8] and Silva [10].

Only the sequential topology in the space of distributions is investigated.

§ 1. Terminology and notation. If $x = (\xi_1, \ldots, \xi_q)$ and $y = (\eta_1, \ldots, \eta_q)$ are points of the q-dimensional space, we write

$$x < y \text{ iff } \xi_j \leqslant \eta_j \text{ for } j = 1, \ldots, q,$$

$$x \leqslant y \text{ iff } \xi_j \leqslant \eta_j \text{ for } j = 1, \ldots, q.$$

Thus, if a and b are points in the q-dimensional space, then the inequalities $a < x < b$ and $a \leqslant x \leqslant b$ define open and closed intervals respectively. By an *interval* we shall always understand an open finite interval. An interval $a < x < b$ is said to be *inside* an open subset O of the q-dimensional space if its closure $a \leqslant x \leqslant b$ is a subset of O.

Following [5], [6], distributions will be denoted by symbols $f(x)$, $g(x), \ldots$, or by $f(\xi_1, \ldots, \xi_q)$, $g(\xi_1, \ldots, \xi_q), \ldots$ (in general the substitution of concrete points for the symbolic variables is not feasible). Incidentally the same symbols denote continuous functions. The symbols $F(x), G(x), \ldots$ denote only continuous functions, and the symbol $\omega(x)$—infinitely derivable functions.

If $u = (\xi_1, \ldots, \xi_p)$ and $v = (\xi_{p+1}, \ldots, \xi_q)$, then (u, v) denotes of course the point $x = (\xi_1, \ldots, \xi_q)$, and $f(u, v)$, $F(u, v)$, $\omega(u, v), \ldots$ is another notation for $f(x)$, $F(x)$, $\omega(x), \ldots$ respectively.

If $f(x)$ is a distribution, and $k = (\varkappa_1, \ldots, \varkappa_q)$ is a sequence of non-negative integers (k being usually called *order*, for brevity), then $f^{(k)}(x)$ is its distributional *derivative of order* k, i. e.

$$f^{(k)}(x) = \frac{\partial^{\varkappa_1}}{\partial \xi_1^{\varkappa_1}} \cdots \frac{\partial^{\varkappa_q}}{\partial \xi_q^{\varkappa_q}} f(x).$$

We use the notation

$$e_1 = (1, 0, 0, \ldots, 0),$$
$$e_2 = (0, 1, 0, \ldots, 0),$$
$$\ldots\ldots\ldots\ldots$$
$$e_q = (0, 0, \ldots, 0, 1),$$
$$e = (1, 1, \ldots, 1),$$
$$0 = (0, 0, \ldots, 0).$$

Thus $f^{(e_j)}(x)$ is an abbreviation for $\partial f(x)/\partial \xi_j$.

By a well-known theorem, if $f(x)$ is a distribution in an open set O, then for every interval I inside O there exist an order k and a continuous function $F(x)$ such that $f(x) = F^{(k)}(x)$ in I.

We write $F_n(x) \rightrightarrows F(x)$ if $F_n(x)$ converges uniformly to $F(x)$.

We write $f_n(x) \to f(x)$ in O (or $f(x) = \lim\limits_{n\to\infty} f_n(x)$ in O) if the sequence $f_n(x)$ of distributions converges distributionally in O to a distribution $f(x)$, i. e. if for every interval I inside O there exist an order k and functions $F_n(x)$, $F(x)$ such that in I

$$F_n^{(k)}(x) = f_n(x), \quad F^{(k)}(x) = f(x), \quad F_n(x) \rightrightarrows F(x).$$

A similar notation is applied to the convergence of distributions depending on a continuous parameter α:

$$f_\alpha(x) \to f(x) \text{ in } O \text{ for } \alpha \to \alpha_0 \quad (\text{or } f(x) = \lim_{\alpha\to\alpha_0} f_\alpha(x)).$$

This denotes that for every interval I inside O there exist an order k and functions $F_\alpha(x)$ (defined for α near α_0) and $F(x)$ such that

$$F_\alpha^{(k)}(x) = f_\alpha(x), \quad F^{(k)}(x) = f(x), \quad F_\alpha(x) \rightrightarrows F(x) \quad \text{for } \alpha \to \alpha_0.$$

If $k = (\varkappa_1, \ldots, \varkappa_q)$ is an order, λ is a number, and $x = (\xi_1, \ldots, \xi_q)$ is a point, then, by definition,

$$\lambda x = (\lambda\xi_1, \ldots, \lambda\xi_q),$$

$$\lambda^k = \lambda^{\varkappa_1 + \ldots + \varkappa_q}, \quad \lambda^{-k} = \frac{1}{\lambda^k}, \quad x^k = \xi_1^{\varkappa_1} \cdot \ldots \cdot \xi_q^{\varkappa_q}.$$

If $x = (\xi_1, \ldots, \xi_q)$, $y = (\eta_1, \ldots, \eta_q)$, then

$$x+y = (\xi_1+\eta_1, \ldots, \xi_q+\eta_q), \quad x-y = (\xi_1-\eta_1, \ldots, \xi_q-\eta_q),$$
$$xy = \xi_1\eta_1 + \ldots + \xi_n\eta_n.$$

Consequently $x^2 = \xi_1^2 + \ldots + \xi_q^2$. If $k = (\varkappa_1, \ldots, \varkappa_q)$ and $l = (\lambda_1, \ldots, \lambda_q)$ are orders, $l \leqslant k$, then $\binom{k}{l} = \binom{\varkappa_1}{\lambda_1} \cdot \ldots \cdot \binom{\varkappa_q}{\lambda_q}$.

§ 2. The integral of a distribution. If O is an open subset of the q-dimensional space, let O' denote the set of all points $(\xi_1, \ldots, \xi_{q-1}, \eta, \zeta)$ such that each of the inequalities

$$\eta \leqslant \xi_q \leqslant \zeta, \quad \zeta \leqslant \xi_q \leqslant \eta$$

implies $(\xi_1, \ldots, \xi_q) \in O$. Obviously, O' is an open subset of the $(q+1)$-dimensional space.

Let $f(x)$ be a distribution in O. The integral

$$\int_\eta^\zeta f(\xi_1, \ldots, \xi_q)\, d\xi_q \quad \left(\text{or simply: } \int_\eta^\zeta f(x)\, d\xi_q\right)$$

is a distribution defined in O' such that

$$(1) \qquad \frac{\partial}{\partial \zeta} \int_\eta^\zeta f(\zeta_1, \ldots, \zeta_q)\, d\xi_q = f(\xi_1, \ldots, \xi_{q-1}, \zeta),$$

$$(2) \qquad \int_\eta^\zeta f(\xi_1, \ldots, \xi_q)\, d\xi_q = -\int_\zeta^\eta f(\xi_1, \ldots, \xi_q)\, d\xi_q.$$

Observe that (1) and (2) imply

$$\frac{\partial}{\partial\eta}\int_{\eta}^{\zeta} f(\xi_1, \ldots, \xi_q)\, d\xi_q = -f(\xi_1, \ldots, \xi_{q-1}, \eta). \tag{3}$$

Conditions (1), (2) determine uniquely the integral $\int_{\eta}^{\zeta} f(x)\, d\xi_q$. It suffices to prove it only in the case where $f(x)$ is the zero distribution. If $g(\xi_1, \ldots, \xi_{q-1}, \eta, \zeta)$ is a distribution such that

$$\frac{\partial}{\partial\zeta} g(\xi_1, \ldots, \xi_{q-1}, \eta, \zeta) = 0, \tag{4}$$

$$g(\xi_1, \ldots, \xi_{q-1}, \eta, \zeta) = -g(\xi_1, \ldots, \xi_{q-1}, \zeta, \eta), \tag{5}$$

then also

$$\frac{\partial}{\partial\eta} g(\xi_1, \ldots, \xi_{q-1}, \eta, \zeta) = 0. \tag{6}$$

By (4) and (6) $g(\xi_1, \ldots, \xi_{q-1}, \eta, \zeta)$ depends on $\xi_1, \ldots, \xi_{q-1}$ only, i. e. locally $g(\xi_1, \ldots, \xi_{q-1}, \eta, \zeta) = h(\xi_1, \ldots, \xi_{q-1})$. By (5), $h(\xi_1, \ldots, \xi_{q-1}) = -h(\xi_1, \ldots, \xi_{q-1})$, i. e. $h(\xi_1, \ldots, \xi_{q-1})$ is the zero distribution. Consequently $g(\xi_1, \ldots, \xi_{q-1}, \eta, \zeta)$ is the zero distribution, i. e. the equations (4), (5) have only one solution: the zero distribution.

The integral $\int_{\eta}^{\zeta} f(x)\, d\xi_q$ exists in O'. In fact, in any interval I inside O there exists a distribution $g(\xi_1, \ldots, \xi_q)$ such that

$$\frac{\partial}{\partial\xi_q} g(\xi_1, \ldots, \xi_q) = f(\xi_1, \ldots, \xi_q).$$

The formula

$$\int_{\eta}^{\zeta} f(\xi_1, \ldots, \xi_q)\, d\xi_q = g(\xi_1, \ldots, \xi_{q-1}, \zeta) - g(\xi_1, \ldots, \xi_{q-1}, \eta) \tag{7}$$

defines a distribution satisfying (1) and (2) in the corresponding interval I' inside O'. It follows from the uniqueness that if I_1, I_2 are intervals inside O, and the corresponding intervals I_1', I_2' intersect, then the integrals $\int_{\eta}^{\zeta} f(x)\, d\xi_q$ defined by (7) in I_1', I_2' coincide in $I_1' \cap I_2'$. Thus all the distributions (7) defined in intervals I' determine together the distribution $\int_{\eta}^{\zeta} f(x)\, d\xi_q$ in the whole open set O'.

Naturally, if a fixed definition of distributions is adopted, we can give a simpler constructive definition of $\int_{\eta}^{\zeta} f(x)\, d\xi_q$. For instance, in the sequential theory of distributions (see Mikusiński and Sikorski [6]), the distribution $f(x)$ under consideration can be represented by a fundamental sequence $f_n(x)$ of functions. Then $\int_{\eta}^{\zeta} f(x)\, d\xi_q$ is the distribution represented by the fundamental sequence of the functions $\int_{\eta}^{\zeta} f_n(x)\, d\xi_q$. In the functional theory of Soboleff [14] and Schwartz [7], the integral $\int_{\eta}^{\zeta} f(x)\, d\xi_q$ can be defined by the equation

$$\Big\langle \int_{\eta}^{\zeta} f(x)\, d\xi_q, \varphi(\xi_1, \ldots, \xi_{q-1})\psi_1(\zeta)\psi_2(\eta) \Big\rangle$$
$$= \Big\langle f(\xi_1, \ldots, \xi_q), \varphi(\xi_1, \ldots, \xi_{q-1}) \cdot \int_{-\infty}^{\xi_q} d\eta \int_{\xi_q}^{\infty} \big(\psi_1(\zeta)\psi_2(\eta) - \psi_1(\eta)\psi_2(\zeta)\big)\, d\zeta \Big\rangle$$

where the carriers of the indefinitely derivable functions φ, ψ_1, ψ_2 satisfy obvious conditions.

It follows from (1), (2) (or from (7)) that

$$\int_{\eta}^{\zeta} f(x)\, d\xi_q + \int_{\zeta}^{\xi} f(x)\, d\xi_q = \int_{\eta}^{\xi} f(x)\, d\xi_q, \tag{8}$$

$$\int_{\eta}^{\zeta} \big(f_1(x) + f_2(x)\big)\, d\xi_q = \int_{\eta}^{\zeta} f_1(x)\, d\xi_q + \int_{\eta}^{\zeta} f_2(x)\, d\xi_q, \tag{9}$$

$$\int_{\eta}^{\zeta} \lambda f(x)\, d\xi_q = \lambda \int_{\eta}^{\zeta} f(x)\, d\xi_q, \tag{10}$$

$$\frac{\partial}{\partial\xi_j}\int_{\eta}^{\zeta} f(x)\, d\xi_q = \int_{\eta}^{\zeta} \frac{\partial}{\partial\xi_j} f(x)\, d\xi_q \quad \text{for} \quad 1 \leqslant j < q. \tag{11}$$

Substituting in (7) $g(x) = \omega(x)f(x)$ we get the formula on integration by parts for any distribution $f(x)$ and any infinitely derivable function $\omega(x)$:

$$\int_{\eta}^{\zeta} \omega(x)\frac{\partial}{\partial\xi_q} f(x)\, d\xi_q + \int_{\eta}^{\zeta} \frac{\partial}{\partial\xi_q}\omega(x) f(x)\, d\xi_q = \omega(x)f(x)\Big|_{\xi_q=\eta}^{\xi_q=\zeta} \tag{12}$$
$$= \omega(\xi_1, \ldots, \xi_{q-1}, \zeta)f(\xi_1, \ldots, \xi_{q-1}, \zeta) - \omega(\xi_1, \ldots, \xi_{q-1}, \eta)f(\xi_1, \ldots, \xi_{q-1}, \eta).$$

Just as in the classical Analysis, we get from (7) the following formula for integration by substitution:

$$\int_{\sigma(\eta)}^{\sigma(\zeta)} f(\xi_1, \ldots, \xi_q)\, d\xi_q = \int_{\eta}^{\zeta} f\big(\xi_1, \ldots, \xi_{q-1}, \sigma(\xi_q)\big)\sigma'(\xi_q)\, d\xi_q \tag{13}$$

where $\sigma(\xi_q)$ is an indefinitely derivable function with a non-vanishing derivative. The distribution on the left side of (13) is of course the result of the substitution of $\sigma(\eta)$ and $\sigma(\zeta)$ for η and ζ respectively in the distribution $\int_{\eta}^{\zeta} f(x)\,d\xi_q$. Such substitution is always feasible (see Mikusiński and Sikorski [6], (II)).

We also have the following theorem:

$$\text{(14)} \qquad \text{if} \quad f_n(x) \to f(x), \quad \text{then} \quad \int_{\eta}^{\zeta} f_n(x)\,d\xi_q \to \int_{\eta}^{\zeta} f(x)\,d\xi_q.$$

In the same way we define the integrals $\int_{\eta}^{\zeta} f(x)\,d\xi_j$ $(1 \leqslant j \leqslant q)$ and the iterated integrals $\int_{\eta_r}^{\zeta_r} \ldots \int_{\eta_1}^{\zeta_1} f(x)\,d\xi_{j_1} \ldots d\xi_{j_r}$ $(j_n \neq j_m$ for $n \neq m)$.

It is easy to verify that the arrangement of integration in iterated integrals is of no consequence. To simplify the notation, if $u = (\xi_1, \ldots, \xi_q)$, $t = (\xi_{p+1}, \ldots, \xi_q)$, $r = q-p$, $v = (\zeta_1, \ldots, \zeta_r)$, $w = (\eta_1, \ldots, \eta_r)$, we write

$$\text{(15)} \qquad \int_{w}^{v} f(u, t)\,dt$$

instead of $\int_{\eta_r}^{\zeta_r} \ldots \int_{\eta_1}^{\zeta_1} f(u, t)\,d\xi_{p+1} \ldots d\xi_q$.

It is evident that, in the case where the distribution $f(x)$ is a continuous function, the integrals just defined coincide with the corresponding integrals from the classical Analysis.

§ 3. Distributions continuous in some variables. Let $f(x) = f(u, t)$ be a distribution defined in an open subset O of the q-dimensional space, $u = (\xi_1, \ldots, \xi_p)$, $t = (\xi_{p+1}, \ldots, \xi_q)$. The distribution $f(u, t)$ is said to be *continuous in the variable* t iff, for every interval I inside O, there exist an order $k = (\varkappa_1, \ldots, \varkappa_p, 0, \ldots, 0)$ and a continuous function $F(u, t)$ such that

$$\text{(1)} \qquad F^{(k)}(u, t) = f(u, t).$$

The order k can be replaced here, if necessary, by any order $l = (\lambda_1, \ldots, \lambda_p, 0, \ldots, 0)$, $l \geqslant k$.

It is easy to see that if $f(u,t)$ and $g(u,t)$ are distributions continuous in t, then so are $f(u,t)+g(u,t)$, $f(u,t)-g(u,t)$, $cf(u,t)$, and $f^{(m)}(u,t)$ for every order $m = (\mu_1, \ldots, \mu_p, 0, \ldots, 0)$.

It is a little more difficult to verify that the product $\omega(u,t)f(u,t)$ of an infinitely derivable function $\omega(u,t)$ and a distribution $f(u,t)$ continuous in t is a distribution continuous in t. For if (1) holds, then

$$\omega(x)f(x) = \sum_{l \leqslant k} (-1)^l \binom{k}{l} \left(F(x)\,\omega^{(l)}(x)\right)^{(k-l)},$$

and the differentation on the right side is taken with respect to the variables $\xi_1, \ldots, \xi_p$ only. Observe that the last formula enables us to extend the notion of the product $\omega(x)f(x)$ to the case where $\omega(u,t)$ is indefinitely derivable in u only, i. e. all the derivatives $\omega^{(m)}(x)$ where $m = (\mu_1, \ldots, \mu_p, 0, \ldots, 0)$ are continuous functions.

If $f(u,t)$ is any distribution, then $f(u, v+t)$ and $f(u, v-t)$ are continuous in t. Observe that they are also continuous in v but, in general, they are not continuous in (v, t).

The following notion of convergence is adequate for distributions $f(u,t)$ continuous in t. A sequence $f_n(u,t)$ of distributions continuous in t is said to *converge* in O to a distribution $f(u,t)$ *almost uniformly in* t iff for every interval I inside O there exist an order $k = (\varkappa_1, \ldots, \varkappa_p, 0, \ldots, 0)$ and continuous functions $F_n(u,t)$, $F(u,t)$ such that in I:

$$F_n^{(k)}(x) = f_n(x), \qquad F^{(k)}(x) = f(x), \qquad F_n(x) \rightrightarrows F(x).$$

Of course, if $f_n(u,t) \to f(u,t)$ almost uniformly in t, then $f(u,t)$ is a distribution continuous in t, and $f_n(u,t) \to f(u,t)$ in the sense defined in § 1, p. 120.

Let t_0 be a point such that the open set O_{t_0} of all points u such that $(u, t_0) \in O$ is not empty. If $f(u,t)$ is a distribution defined in O and continuous in t, we can substitute for the variable t the constant value t_0. Then we get a new distribution $f(u, t_0)$ in O_{t_0}. The exact definition of $f(u, t_0)$ can be formulated for instance as follows: If $f(u,t)$ is a distribution continuous in t, then there exists a sequence $f_n(u,t)$ of continuous functions such that $f_n(u,t) \to f(u,t)$ distributionally but almost uniformly in t. The sequence of functions $f_n(u, t_0)$ converges distributionally to a distribution which does not depend on the choice of $f_n(u,t)$. The limit distribution is the distribution $f(u, t_0)$.

It can easily be proved that if $f_n(u,t)$ are distributions continuous in t, and $f_n(u,t) \to f(u,t)$ almost uniformly in t, then $f_n(u, t_0) \to f(u, t_0)$ in the sense defined in § 1, p. 120.

§ 4. Integration of distributions continuous in some variables. It is easy to verify that if $f(u,t)$ is a distribution continuous in t, then the distribution $\int_{w}^{v} f(u,t)\,dt$ is continuous in (w, v), and consequently we can substitute for w and v some constant values w_0 and v_0. Thus we get the distribution

$$\text{(1)} \qquad \int_{w_0}^{v_0} f(u,t)\,dt$$

depending on the variable u only. For the same reason, if $f(u,t)$ is continuous in t, we get the integrals

$$\text{(2)} \qquad \int_{w_0}^{v} f(u,t)\,dt \quad \text{and} \quad \int_{w}^{v_0} f(u,t)\,dt$$

depending on the variable point (u, v) and (u, w) respectively, the points w_0, v_0 being fixed.

However, if $f(u, t)$ is continuous in t, we can also introduce more complicated integrals

$$(3) \qquad \int_A f(u, t)\,dt$$

where A is a bounded measurable set. If $f(u, t)$ is defined in an open set O, the integral (3) is a distribution defined in the open set O' of all points u such that the closure of the set $\{(u, v)\colon v \in A\}$ is contained in O. The most convenient definition of (3) can be given in the sequential theory of distributions. In fact, there exists a sequence $f_n(u, t)$ of continuous functions such that $f_n(u, t) \to f(u, t)$ distributionally but almost uniformly in t. The sequence $\int_A f_n(u, t)\,dt$ then converges distributionally to a distribution which does not depend on the choice of $f_n(u, t)$. The limit distribution is denoted by $\int_A f(u, t)\,dt$.

Integral (3) has many properties analogous to the integral of a continuous function. If A is the interval $w_0 \leqslant t \leqslant v_0$, then (3) coincides with (1).

Integral (3) can also be taken with respect to other measures than Lebesgue measure. In particular it is possible to define integrals extended over some hypersurfaces of any dimension $\leqslant q-p$ with respect to the area of the hypersurface (always under the hypothesis that the integrated function is continuous in the variable of integration). We have for instance the formula

$$\delta_1(\sigma(x)) = \int_S |\operatorname{grad}\sigma(t)|^{-1}\,\delta_q(x-t)\,dt$$

where $S = \{x\colon \sigma(x) = 0\}$, $\delta_r(x)$ is the r-dimensional Dirac delta distribution, and $\sigma(x)$ is an infinitely derivable function whose gradient does not vanish on S (for details, see Sikorski [9]).

Integral (3) can also be extended over a larger class of distributions locally integrable in t, which will not be examined here.

Observe that, without any additional hypotheses with regard to $f(u, t)$, neither (1) nor (2) has any sense. However, without any additional hypothesis regarding $f(u, t)$, we can always consider the integrals

$$\int_a^b f(u, v+t)\,dt, \qquad \int_A f(u, v+t)\,dt \qquad (a, b\text{ — fixed})$$

because $f(u, v+t)$ is continuous in t. These integrals are distributions of the variable (u, v).

It is easy to verify that, for every distribution $f(u, t)$, we have the identity

$$(4) \qquad \int_w^v f(u, t)\,dt = \int_0^{v-w} f(u, w+t)\,dt.$$

In general, $\int_w^v f(u, t)\,dt$ is continuous neither in v, nor in w. It follows from (4) that if $\int_w^v f(u, t)\,dt$ is interpreted as a distribution of $(u, w, v-w)$, it is continuous in the variable $y = v-w$.

Identity (4) shows that the difference between the integral $\int_w^v f(u, t)\,dt$ and the integral $\int_a^b f(u, w+t)\,dt$ is not essential.

§ 5. The value of a distribution at infinity. Let $f(u, v)$ be defined in the open set $\{(u, v)\colon u \in O\}$ where O is an open subset of the p-dimensional space.

It is easy to verify that if the distributional limit $\lim_{a\to-\infty} f(u, v+a)$ exists, it is a distribution independent of the variable v. Therefore the limit can be interpreted as a distribution of the variable u only, defined in O. We shall denote it by $f(u, -\infty)$ or by $\lim_{v\to-\infty} f(u, v)$.

Similarly we define $f(u, \infty)$ or $\lim_{v\to\infty} f(u, v)$ as the distributional limit $\lim_{b\to\infty} f(u, v+b)$ interpreted as a distribution of the variable u only.

More generally for any distribution $f(u, v, w)$ by $f(u, -\infty, \infty)$ or $\lim_{\substack{w\to-\infty\\ v\to\infty}} f(u, w, v)$ we shall understand the distributional limit

$$\lim_{\substack{a\to-\infty\\ b\to\infty}} f(u, w+a, v+b)$$

interpreted as the distribution of the variable u only.

Observe that if $\lim_{\substack{w\to-\infty\\ v\to\infty}} f(u, w, v)$ exists, then for every order m the limit $\lim_{\substack{w\to-\infty\\ v\to\infty}} f^{(m)}(u, w, v)$ exists also, and

$$(1) \qquad \lim_{\substack{w\to-\infty\\ v\to\infty}} f^{(m)}(u, w, v) = \big(\lim_{\substack{w\to-\infty\\ v\to\infty}} f(u, w, v)\big)^{(m)}.$$

Since the differentation is here commutative with "lim", we can use the symbol $f^{(m)}(u, -\infty, \infty)$ to denote distribution (1), without any ambiguity.

§ 6. Improper integrals. According to § 5, we define the *improper integral*

$$\int_{-\infty}^{\infty} f(u, t)\,dt$$

as the distributional limit

$$\lim_{\substack{w\to-\infty\\ v\to\infty}} \int_{w}^{v} f(u,t)\,dt = \lim_{\substack{a\to-\infty\\ b\to\infty}} \int_{a+w}^{b+v} f(u,t)\,dt$$

interpreted as a distribution of the p-dimensional variable u only.

It is evident that, in the case where the distribution $f(u,t)$ is a continuous function and the ordinary improper integral $\int_{-\infty}^{\infty} f(u,t)\,dt$ exists, and converges almost uniformly, the distributional improper integral $\int_{-\infty}^{\infty} f(u,t)\,dt$ just defined also exists and they coincide.

The following theorems hold for any distributions $f(u,t)$, $g(u,t)$:

(i) *If the integrals* $\int_{-\infty}^{\infty} f(u,t)\,dt$, $\int_{-\infty}^{\infty} g(u,t)\,dt$ *exist, then the integrals* $\int_{-\infty}^{\infty} \big(f(u,t)+g(u,t)\big)\,dt$, $\int_{-\infty}^{\infty} \big(f(u,t)-g(u,t)\big)\,dt$ *also exist, and*

$$\int_{-\infty}^{\infty} \big(f(u,t)+g(u,t)\big)\,dt = \int_{-\infty}^{\infty} f(u,t)\,dt + \int_{-\infty}^{\infty} g(u,t)\,dt, \tag{1}$$

$$\int_{-\infty}^{\infty} \big(f(u,t)-g(u,t)\big)\,dt = \int_{-\infty}^{\infty} f(u,t)\,dt - \int_{-\infty}^{\infty} g(u,t)\,dt. \tag{2}$$

(ii) *If the integral* $\int_{-\infty}^{\infty} f(u,t)\,dt$ *exists, then the integrals* $\int_{-\infty}^{\infty} \lambda f(u,t)\,dt$, $\int_{-\infty}^{\infty} f(u,t_0+t)\,dt$, $\int_{-\infty}^{\infty} f^{(m)}(u,t)\,dt$ $(m=(\mu_1,\ldots,\mu_p,0,\ldots,0))$ *also exist and*

$$\int_{-\infty}^{\infty} \lambda f(u,t)\,dt = \lambda \int_{-\infty}^{\infty} f(u,t)\,dt, \tag{3}$$

$$\int_{-\infty}^{\infty} f(x,t_0+t)\,dt = \int_{-\infty}^{\infty} f(u,t)\,dt, \tag{4}$$

$$\int_{-\infty}^{\infty} f^{(m)}(u,t)\,dt = \Big(\int_{-\infty}^{\infty} f(u,t)\,dt\Big)^{(m)}. \tag{5}$$

§ 7. The convolution of distributions. Let $f(x)$ and $g(x)$ be distributions defined in the whole q-dimensional space. The product $f(x)g(t)$ is a distribution defined in $2q$-dimensional space. Since the linear transformation $\sigma(x,t)=(x-t,t)$ has a non-vanishing jacobian, we can replace in the distribution $f(x)g(t)$ the point (x,t) by $(x-t,t)$. We thus get the distribution $f(x-t)g(t)$.

By the *convolution* of distributions $f(x)$, $g(x)$ we understand the improper integral

$$f(x)*g(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt$$

provided it exists in the whole q-dimensional space.

If $f(x)$ and $g(x)$ are continuous functions and the ordinary convolution $f(x)*g(x)$ exists and converges almost uniformly, then the distributional convolution $f(x)*g(x)$ just defined also exists and they coincide.

The following elementary properties of the convolution of any distributions $f(x)$, $g(x)$, $h(x)$ follow immediately from known properties of improper integrals:

(i) *If* $f(x)*g(x)$ *and* $f(x)*h(x)$ *exist, then* $f(x)*\big(g(x)\pm h(x)\big)$ *exists also, and*

$$f(x)*\big(g(x)\pm h(x)\big) = f(x)*g(x) \pm f(x)h(x).$$

Similarly, if $f(x)*h(x)$ *and* $g(x)*h(x)$ *exist, then* $\big(f(x)+g(x)\big)*h(x)$ *also exists and*

$$\big(f(x)\pm g(x)\big)*h(x) = f(x)*h(x) \pm g(x)*h(x).$$

(ii) *If* $f(x)*g(x)$ *exists, then* $\big(\lambda f(x)\big)*g(x)$ *and* $f(x)*\big(\lambda g(x)\big)$ *also exist, and*

$$\big(\lambda f(x)*g(x)\big) = \lambda\big(f(x)*g(x)\big) = f(x)*\big(\lambda g(x)\big).$$

It is a little more difficult to prove that

(iii) *If* $f(x)*g(x)$ *exists, then, for every order* m, $f^{(m)}(x)*g(x)$ *and* $f(x)*g^{(m)}(x)$ *also exist, and*

$$f^{(m)}(x)*g(x) = \big(f(x)*g(x)\big)^{(m)} = f(x)*g^{(m)}(x).$$

(iv) *If* $f(x)*g(x)$ *exists, then* $g(x)*f(x)$ *also exists, and* $f(x)*g(x) = g(x)*f(x)$.

The associative law will be proved only in the following two particular cases to be investigated in this section:

(v) *If one of the distributions* $f(x)$, $g(x)$ *vanishes outside an interval, then the convolution* $f(x)*g(x)$ *exists.*

(vi) *If a distribution* $f(x)$ *vanishes outside an infinite interval* $x \geqslant c_1$, *and a distribution* $g(x)$ *vanishes outside an infinite interval* $x \geqslant c_2$, *then the convolution* $f(x)*g(x)$ *exists and vanishes outside the infinite interval* $x \geqslant c_1+c_2$.

The existence of $f(x)*g(x)$ follows from the fact that, in both cases, for any q-dimensional interval I there exist points a_0 and b_0 such that

$$\int_{w+a}^{v+b} f(x-t)g(t)\,dt = \int_{w+a_0}^{v+b_0} f(x-t)g(t)\,dt$$

$$\text{for all}\quad a \leqslant a_0,\quad b \geqslant b_0 \text{ and } (x,w,v)\in I\times I\times I.$$

Hence it follows that

$$(1)\qquad \int_{-\infty}^{\infty} f(x-t)g(t)\,dt = \int_{w+a_0}^{v+b_0} f(x-t)g(t)\,dt \quad \text{for} \quad x\in I,\ w\in I,\ v\in I.$$

The interval I being arbitrary, we infer that this improper integral exists in the whole q-dimensional space.

The last remark in theorem (vi) follows from the fact that in the complement of the set $\{x: x \geqslant c_1+c_2\}$ the integrated distribution vanishes.

(vii) *If any two of the distributions $f(x)$, $g(x)$, $h(x)$ vanish outside an interval, or if all the three distributions vanish outside an infinite interval $x \geqslant c_0$, then*

$$f(x)*\big(g(x)*h(x)\big) = \big(f(x)*g(x)\big)*h(x).$$

The proof is the same as the proof of the corresponding theorem for functions. The proof is based on a change of the order of integration, which is feasible in the cases under consideration because the integrals defining convolutions are indeed proper integrals (see (1)).

(viii) *If $f_n(x)\to f(x)$, $g_n(x)\to g(x)$, and all the distributions $g_n(x)$ vanish outside a fixed interval, then $f_n(x)*g_n(x)\to f(x)*g(x)$.*

(ix) *If $f_n(x)\to f(x)$, $g_n(x)\to g(x)$ and all the distributions $f_n(x)$, $g_n(x)$ vanish outside a fixed infinite interval $x\geqslant c_0$, then $f_n(x)*g_n(x)\to f(x)*g(x)$.*

Theorems (viii), (ix) follow immediately from (1) and the analogous formulas for $f_n(x)*g_n(x)$ since $f_n(x-t)g_n(t)\to f(x-t)g(t)$ and consequently

$$\int_{w+a_0}^{v+a_0} f_n(x-t)g_n(t)\,dt \to \int_{w+a_0}^{v+a_0} f(x-t)g(t)\,dt.$$

Let $\delta(x)$ be the q-dimensional Dirac delta distribution.

(x) *For any distribution $f(x)$,*

$$f(x)*\delta(x) = f(x).$$

Let $\delta_n(x)$ be a sequence of non-negative continuous functions vanishing outside the intervals $-e/n \leqslant x \leqslant e/n$, such that $\int_{-\infty}^{\infty}\delta_n(x)\,dx = 1$. By definition, $\delta_n(x)\to\delta(x)$, and consequently $f(x)*\delta_n(x)\to f(x)*\delta(x)$ by (viii).

If $f(x)$ is any continuous function, then $f(x)*\delta_n(x) \rightrightarrows f(x)$ in every finite interval, and consequently $f(x)*\delta_n(x)\to f(x)$. This proves (x) in the case where $f(x)$ is a continuous function.

If $f(x)$ is any distribution, there exists a sequence of continuous functions $f_n(x)\to f(x)$. Hence $f(x)*\delta(x) = \lim\limits_{n\to\infty} f_n(x)*\delta(x) = \lim\limits_{n\to\infty} f_n(x) = f(x)$.

§ 8. Distributions slowly increasing and rapidly decreasing. All functions and distributions considered in this section are defined in the whole q-dimensional space. We introduce the following classification of functions and distributions from the point of view of their behaviour at infinity.

$F(x)$ is said to be a *slowly increasing function* iff there exists an integer $\varkappa > 0$ such that the function $F(x)/(1+x^2)^\varkappa$ is bounded. The class of all slowly increasing functions is linear. Any indefinite integral

$$\int_a^{\xi_j} F(\xi_1, \ldots, \xi_{j-1}, \tau, \xi_{j+1}, \ldots, \xi_q)\,d\tau$$

of a slowly increasing function is a slowly increasing function.

$\omega(x)$ is said to be a *function slowly increasing with all its derivatives* iff, for every order m, $\omega^{(m)}(x)$ is a slowly increasing function. The class of all functions slowly increasing with all its derivatives is a linear subset of the class of all infinitely derivable functions.

$f(x)$ is said to be a *slowly increasing distribution* iff there exist an order k and a slowly increasing function $F(x)$ such that

$$(1)\qquad f(x) = F^{(k)}(x) \text{ in the whole space.}$$

The order k can be replaced here, if necessary, by any order $l\geqslant k$ (replace $F(x)$ by a suitable iterated indefinite integral!) The class of all slowly increasing distributions is linear. The product $\omega(x)f(x)$ of a function $\omega(x)$ slowly increasing with all its derivatives and a slowly increasing distribution $f(x)$ is a slowly increasing distribution, for if (1) holds, we have

$$(2)\qquad \omega(x)f(x) = \sum_{l\leqslant k}(-1)^l\binom{k}{l}\big(F(x)\,\omega^{(l)}(x)\big)^{(k-l)}.$$

$f(x)$ is said to be a *bounded distribution* if it is a finite sum of derivatives of bounded continuous functions, i. e. if there exist bounded continuous functions $F_1(x), \ldots, F_r(x)$ and orders $k_1, \ldots, k_r$ such that

$$f(x) = F_1^{(k_1)}(x)+\ldots+F_r^{(k_r)}(x).$$

The class of all bounded distributions is linear.

$g(x)$ is said to be a *rapidly decreasing distribution* iff $(1+x^2)^\varkappa g(x)$ is a bounded distribution for every integer $\varkappa\geqslant 0$. The class of all rapidly decreasing distributions is linear.

$g(x)$ is a rapidly decreasing distribution iff, for every integer $\varkappa\geqslant 0$, $g(x) = G_1^{(k_1)}(x)+\ldots+G_r^{(k_r)}$ where $k_1, \ldots, k_r$ are some orders, and all the continuous functions $(1+x^2)^\varkappa G_1(x), \ldots, (1+x^2)^\varkappa G_r(x)$ are bounded.

In fact, if $g(x)$ is rapidly decreasing, then

$$(1+x^2)^\varkappa g(x) = F_1^{(k_1)}(x)+\ldots+F_r^{(k_r)}(x)$$

where all the functions $F_1(x), \ldots, F_r(x)$ are bounded. Hence

$$g(x) = \sum_{j=1}^{r} (1+x^2)^{-\varkappa} F_j^{(k_j)}(x)$$
$$= \sum_{j=1}^{r} \sum_{l\leqslant k_j} (-1)^l \binom{k_j}{l} \Big(F_j(x)\big((1+x^2)^{-\varkappa}\big)^{(l)}\Big)^{(k_j-l)}$$

and the functions $F_j(x)\big((1+x^2)^{-\varkappa}\big)^{(l)}(1+x^2)^{\varkappa}$ are bounded. Conversely, if $g(x) = \sum_{j=1}^{r} G_j^{(k_j)}(x)$, and $G_j(x)(1+x^2)^{\varkappa}$ are bounded, then

$$(1+x^2)^{\varkappa} g(x) = \sum_{j=1}^{r} \sum_{l\leqslant k_j} (-1)^l \binom{k_j}{l} \Big(G_j(x)\big((1+x^2)^{\varkappa}\big)^{(l)}\Big)^{(k_j-l)}$$

and all the functions $G_j(x)\big((1+x^2)^{\varkappa}\big)^{(l)}$ are bounded. Thus $(1+x^2)^{\varkappa} g(x)$ is a bounded distribution for every $\varkappa \geqslant 0$.

Every rapidly decreasing distribution is of course a slowly increasing distribution.

If $f(x)$ is a slowly increasing distribution (or a rapidly decreasing distribution), then so is $f^{(m)}(x)$ for every order m.

(i) *If $f(x)$ is a slowly increasing distribution, and $g(x)$ is a rapidly decreasing distribution, then the convolution $f(x)*g(x)$ exists and is a slowly increasing distribution.*

We have $f(x) = F^{(k)}(x)$ where $(1+x^2)^{-\varkappa}F(x)$ is a bounded function, $\varkappa \geqslant 0$, and $g(x) = G_1^{(k_1)}(x)+\ldots+G_r^{(k_r)}(x)$ where $(1+x^2)^{\varkappa+q}G_i(x)$ are bounded functions. By a simple calculation, the convolutions $F(x)*G_j(x)$ exist and are slowly increasing functions. Hence $f(x)*g(x) = \big(F(x)*G_1(x)\big)^{(k+k_1)}+ +\ldots+\big(F(x)*G_r(x)\big)^{(k+k_r)}$ exists on account of § 7 (i) and (iii).

By the same method we can prove that

(ii) *If one of the distributions $f(x), g(x), h(x)$ is slowly increasing and the two remaining ones are rapidly decreasing, then*

$$\big(f(x)*g(x)\big)*h(x) = f(x)*\big(g(x)*h(x)\big).$$

A sequence $f_n(x)$ of slowly increasing distributions is said to *converge strongly* to a distribution $f(x)$ iff there exist slowly increasing functions $F_n(x)$, $F(x)$, an order k and an integer $\varkappa \geqslant 0$ such that, in the whole space,

$$(3) \quad F_n^{(k)}(x) = f_n(x), \quad F^{(k)}(x) = f(x), \quad (1+x^2)^{-\varkappa}F_n(x) \rightrightarrows (1+x^2)^{-\varkappa}F(x).$$

The limit $f(x)$ is then also a slowly increasing distribution.

Replacing $\varkappa$ by a greater integer we can always assume that the functions $(1+x^2)^{-\varkappa}F_n(x)$ are commonly bounded. Strong convergence implies of course the convergence defined in § 1, p. 120.

The following two lemmas follow immediately from the definition of strong convergence.

(iii) *If $F_n(x)$, $F(x)$ are continuous functions, and for some integer $\varkappa \geqslant 0$, $(1+x^2)^{-\varkappa}F_n(x) \rightrightarrows (1+x^2)^{-\varkappa}F(x)$ and the functions $(1+x^2)^{-\varkappa}F_n(x)$ are commonly bounded, then $F_n(x) \to F(x)$ strongly.*

(iv) *If $f_n(x) \to f(x)$ strongly, then $f_n^{(m)}(x) \to f^{(m)}(x)$ strongly for every order m.*

The next lemma follows immediately from (iii), (iv) and identity (2).

(v) *If $f_n(x) \to f(x)$ strongly, then $\omega(x)f_n(x) \to \omega(x)f(x)$ strongly for every function $\omega(x)$ slowly increasing with all its derivatives.*

Now we shall prove that

(vi) *If $f_n(x) \to f(x)$ strongly, and $g(x)$ is a rapidly decreasing distribution, then $f_n(x)*g(x) \to f(x)*g(x)$ strongly.*

Suppose (3) holds, $(1+x^2)^{-\varkappa}F_n(x)$ are commonly bounded, and $g(x) = G_1^{(k_1)}(x)+\ldots+G_r^{(k_r)}(x)$ where $(1+x^2)^{\varkappa+q}G_j(x)$ are bounded functions. It follows that the functions $(1+x^2)^{-\varkappa}\big(F_n(x)*G_j(x)\big)$, $(1+x^2)^{-\varkappa}\big(F(x)*G_j(x)\big)$ satisfy the hypotheses of (iii) and consequently $F_n(x)*G_j(x) \to F(x)*G_j(x)$ strongly. By differentation and addition we get (vi).

§ 9. The Fourier transform. By the *Fourier transform* $\mathcal{F}f(x)$ *of* a distribution $f(x)$ (in the q-dimensional space) we understand the improper integral (over the q-dimensional space)

$$(1) \qquad \mathcal{F}f(x) = \int_{-\infty}^{\infty} f(t)\exp(-2\pi ixt)\,dt$$

provided the integral converges in the whole q-dimensional space. By definition, $\mathcal{F}f(x)$ is the distributional limit

$$\lim_{\substack{w\to-\infty\\ v\to\infty}} \int_w^v f(t)\exp(-2\pi ixt)\,dt = \lim_{\substack{a\to-\infty\\ b\to\infty}} \int_{w+a}^{v+b} f(t)\exp(-2\pi ixt)\,dt.$$

Observe that if a continuous function $f(x)$ has the Fourier transform in the ordinary sense and (1) converges almost uniformly, then its distributional Fourier transform just defined also exists, and they coincide.

The following properties of the Fourier transforms of any distributions $f(x), g(x)$ follow immediately from the definition.

(i) *If $\mathcal{F}f(x)$ and $\mathcal{F}g(x)$ exist, then $\mathcal{F}\big(f(x)\pm g(x)\big)$, $\mathcal{F}\big(\lambda(x)\big)$ also exist, and*

$$\mathcal{F}\big(f(x)\pm g(x)\big) = \mathcal{F}\big(f(x)\pm\mathcal{F}g(x)\big), \quad \mathcal{F}\big(\lambda f(x)\big) = \lambda\mathcal{F}f(x).$$

In the sequel, τ_j will denote the j-th coordinate of the point t, and ξ_j — the j-th coordinate of x.

(ii) *If a distribution $f(x)$ has the properties:*

(a) *for any system $j_1, \ldots, j_p$ of different positive integers $\leqslant q$, $p>0$, the integral*

$$\int_{-\infty}^{\infty} \ldots \int f(t)\exp(-2\pi i x t)\, d\tau_{j_1}\ldots d\tau_{j_p}$$

converges everywhere;

(b) *for any system $j_1, \ldots, j_{p+r}$ of different positive integers $\leqslant q$ and any $p \geqslant 0$, $r>0$,*

$$\lim_{\substack{\eta_1,\ldots,\eta_p\to-\infty\\ \zeta_1,\ldots,\zeta_p\to\infty\\ |\tau_{j_{p+1}}|,\ldots,|\tau_{j_{p+r}}|\to\infty}} \int_{\eta_1}^{\zeta_1} \ldots \int_{\eta_p}^{\zeta_p} f(t)\exp(-2\pi i x t)\, d\tau_{j_1}\ldots d\tau_{j_p} = 0;$$

then all the derivatives $f^{(e_j)}(x)$ also have these properties. Moreover

$$\mathcal{F} f^{(e_j)}(x) = 2\pi i \xi_j \mathcal{F} f(x). \tag{2}$$

If j is one of the integers $j_1, \ldots, j_p$, say $j = j_p$, then integration by parts yields

$$\int_{\eta_1}^{\zeta_1} \ldots \int_{\eta_p}^{\zeta_p} f^{(e_j)}(t)\exp(-2\pi i x t)\, d\tau_{j_1}\ldots d\tau_{j_p}$$

$$= \int_{\eta_1}^{\zeta_1} \ldots \int_{\eta_{p-1}}^{\zeta_{p-1}} f(t_\zeta)\exp(-2\pi i x t_\zeta)\, d\tau_{j_1}\ldots d\tau_{j_{p-1}} -$$

$$- \int_{\eta_1}^{\zeta_1} \ldots \int_{\eta_{p-1}}^{\zeta_{p-1}} f(t_\eta)\exp(-2\pi i x t_\eta)\, d\tau_{j_1}\ldots d\tau_{j_{p-1}} +$$

$$+ 2\pi i \xi_j \int_{\eta_1}^{\zeta_1} \ldots \int_{\eta_p}^{\zeta_p} f(t)\exp(-2\pi i x t)\, d\tau_{j_1}\ldots d\tau_{j_p}$$

where t_ζ, t_η are points whose coordinates, except the j_p-th coordinate, are all equal to the corresponding coordinates of t, the j_p-th coordinate of t_ζ being equal to ζ_p, and the j_p-th coordinate of t_η being equal to η_p. It immediately follows from the equality just obtained that $f^{(e_j)}(x)$ also has properties (a) and (b). Moreover, in the case $p = q$, we obtain (2) by passing to improper integrals.

If j is none of the integers $j_1, \ldots, j_p$, then

$$\int_{\eta_1}^{\zeta_1} \ldots \int_{\eta_p}^{\zeta_p} f^{(e_j)}(t)\exp(-2\pi i x t)\, d\tau_{j_1}\ldots d\tau_{j_p}$$

$$= \frac{\partial}{\partial \tau_j}\left(\int_{\eta_1}^{\zeta_1} \ldots \int_{\eta_p}^{\zeta_p} f(t)\exp(-2\pi i x t)\, d\tau_{j_1}\ldots d\tau_{j_p}\right) +$$

$$+ 2\pi i \xi_j \int_{\eta_1}^{\zeta_1} \ldots \int_{\eta_p}^{\zeta_p} f(t)\exp(-2\pi i x t)\, d\tau_{j_1}\ldots d\tau_{j_p}.$$

It immediately follows from these formulas that $f^{(e_j)}(x)$ also has properties (a) and (b).

(iii) *If $\mathcal{F} f(x)$ exists, then $\mathcal{F}\xi_j f(x)$ also exists, and*

$$\mathcal{F}\xi_j f(x) = -(2\pi i)^{-1}\big(\mathcal{F} f(x)\big)^{(e_j)}. \tag{3}$$

Moreover, if $f(x)$ has properties (a), (b), *$\xi_j f(x)$ also has them.*

The first part of (iii) can be obtained by the differentiation of (1) The second part follows analogously from the identity

$$\int_{\eta_1}^{\zeta_1} \ldots \int_{\eta_p}^{\zeta_p} \tau_j f(t)\exp(-2\pi i x t)\, d\tau_{j_1}\ldots d\tau_{j_p}$$

$$= -(2\pi i)^{-1}\frac{\partial}{\partial \xi_j}\int_{\eta_1}^{\zeta_1} \ldots \int_{\eta_p}^{\zeta_p} f(t)\exp(-2\pi i x t)\, d\tau_{j_1}\ldots d\tau_{j_p}.$$

(iv) *For every slowly increasing distribution $f(x)$, the Fourier transform $\mathcal{F} f(x)$ exists and satisfies the identities*

$$\mathcal{F} f^{(k)}(x) = (2\pi i)^k x^k \mathcal{F} f(x), \qquad \mathcal{F} x^k f(x) = (-2\pi i)^{-k}\big(\mathcal{F} f(x)\big)^{(k)}. \tag{4}$$

Moreover, every slowly increasing distribution has properties (a) *and* (b).

This immediately follows from (ii) and (iii) since $f(x) = \big((1+x^2)^\varkappa F(x)\big)^{(k)}$ where $F(x)$ is a continuous function such that $\int_{-\infty}^{\infty}(1+x^2)^q |F(x)|\,dx < \infty$, and consequently $F(x)$ has properties (a), (b).

(v) *The Fourier transform of a slowly increasing distribution is a slowly increasing distribution.*

If $\mathcal{F} f(x)$ is a slowly increasing distribution, then so are $\mathcal{F}\xi_j f(x)$ and $\mathcal{F} f^{(e_j)}(x)$ by (3) and (2). If $F(x)$ is a continuous function such that $(1+x^2)^q F(x)$ is bounded, then $\mathcal{F} F(x)$ is a bounded function (by a classical theorem), and consequently $\mathcal{F} F(x)$ is a slowly increasing distribution. This implies that $\mathcal{F} f(x)$ is a slowly increasing distribution for every slowly increasing distribution $f(x)$ since $f(x)$ can be represented in the form $f(x) = \big((1+x^2)^\varkappa F(x)\big)^{(k)}$ where $(1+x^2)^q F(x)$ is a bounded continuous function.

(vi) *The Fourier transform of a rapidly decreasing distribution is a function slowly increasing with all its derivatives.*

Any rapidly decreasing distribution is of the form $f(x) = G_1^{(k_1)}(x) + \ldots + G_r^{(k_r)}(x)$ where $(1+x^2)^q G_j(x)$ are bounded continuous functions. Thus $\mathcal{F} G_j(x)$ are bounded continuous functions and consequently $\mathcal{F} f(x) = (2\pi i)^{k_1} x^{k_1}\mathcal{F} G_1(x) + \ldots + (2\pi i)^{k_r} x^{k_r}\mathcal{F} G_r(x)$ (see (4) and (i)) is a slowly increasing function. Since $x^k f(x)$ is also a rapidly decreasing distribution, we infer that $\big(\mathcal{F} f(x)\big)^{(k)} = (-2\pi i)^k \mathcal{F} x^k f(x)$ (see (4)) is a slowly increasing function for every order k.

(vii) *The Fourier transform of a function slowly increasing with all its derivatives is a rapidly decreasing distribution.*

We precede the proof of (vii) by the following remark: For every slowly increasing distribution $f(x)$ and every integer $\varkappa \geqslant 0$

$$(5)\qquad \mathcal{F}(1+x^2)^\varkappa f(x) = (1-\Delta)^\varkappa \mathcal{F} f(x), \qquad \mathcal{F}(1-\Delta)^\varkappa f(x) = (1+x^2)^\varkappa \mathcal{F} f(x)$$

where Δ is an abbreviation for the differential operator

$$\Delta = (2\pi)^{-2}\left(\frac{\partial^2}{\partial \xi_1^2}+\ldots+\frac{\partial^2}{\partial \xi_q^2}\right).$$

Formulas (5) are an immediate consequence of (2) and (3).

To prove (vii), suppose that $f(x)$ is a function slowly increasing with all its derivatives. Since $f(x)$ is slowly increasing, we have $f(x) = (1+x^2)^\varkappa F(x)$ where $(1+x^2)^q F(x)$ is a bounded continuous function. Consequently, by a classical theorem, $\mathcal{F}F(x)$ is a bounded continuous function and, by (5), $\mathcal{F}f(x)$ is a bounded distribution. Since $f^{(k)}(x)$ is also a slowly increasing function for every order k, we infer from (5) that $(1+x^2)^\varkappa \mathcal{F}f(x) = \mathcal{F}(1-\Delta)^\varkappa f(x)$ is a bounded distribution for every $\varkappa \geqslant 0$. This proves that $\mathcal{F}f(x)$ is a rapidly decreasing distribution.

(viii) *If $f(x)$ is a slowly increasing distribution, and $g(x)$ is a rapidly decreasing distribution, then*

$$(6)\qquad \mathcal{F}\big(f(x)*g(x)\big) = \mathcal{F}f(x)\cdot\mathcal{F}g(x).$$

By a classical theorem, (6) holds if $f(x)$ and $g(x)$ are continuous functions such that $(1+x^2)^q f(x)$, $(1+x^2)^q g(x)$ are bounded.

Hence it follows that (6) holds if $(1+x^2)^q f(x)$ is a bounded continuous function and $g(x)$ is a rapidly decreasing distribution. In fact, $g(x) = G_1^{(k_1)}(x)+\ldots+G_r^{(k_r)}(x)$ where $(1+x^2)^q G_1(x), \ldots, (1+x^2)^q G_r(x)$ are bounded continuous functions.

We have

$$\mathcal{F}\big(f(x)*g(x)\big) = \sum_{j=1}^{r}\mathcal{F}\big(f(x)*G_j^{(k_j)}(x)\big) = \sum_{j=1}^{r}\mathcal{F}\Big(\big(f(x)*G_j(x)\big)^{(k_j)}\Big)$$

$$= \sum_{j=1}^{r}(2\pi i)^{k_j}x^{k_j}\mathcal{F}\big(f(x)*G_j(x)\big) = \sum_{j=1}^{r}\mathcal{F}f(x)\cdot(2\pi i)^{k_j}x^{k_j}\mathcal{F}G_j(x)$$

$$= \sum_{j=1}^{r}\mathcal{F}f(x)\cdot\mathcal{F}G_j^{(k_j)}(x) = \mathcal{F}f(x)\cdot\mathcal{F}g(x).$$

To complete the proof, it suffices to show that if (6) holds for a slowly increasing distribution $f(x)$ and every rapidly decreasing distribution $g(x)$, then (6) holds also for the distributions $f^{(e_j)}(x)$, $g(x)$ and for the distributions $\xi_j f(x)$, $g(x)$.

The first statement can be proved as follows:

$$\mathcal{F}\big(f^{(e_j)}(x)*g(x)\big) = \mathcal{F}\Big(\big(f(x)*g(x)\big)^{(e_j)}\Big) = 2\pi i\xi_j\mathcal{F}\big(f(x)*g(x)\big)$$

$$= 2\pi i\xi_j\mathcal{F}f(x)\cdot\mathcal{F}g(x) = \mathcal{F}f^{(e_j)}(x)\cdot\mathcal{F}g(x).$$

The second statement follows from the identity

$$(7)\qquad (-2\pi i)^{-1}\Big(\mathcal{F}\big(f(x)*g(x)\big)\Big)^{(e_j)} = \mathcal{F}\Big(\big(\xi_j f(x)\big)*g(x)\Big)+\mathcal{F}\Big(f(x)*\big(\xi_j g(x)\big)\Big).$$

Since $\xi_j g(x)$ is also a rapidly decreasing distribution, we have $\mathcal{F}\Big(f(x)*\big(\xi_j g(x)\big)\Big) = \mathcal{F}f(x)\cdot\mathcal{F}\big(\xi_j g(x)\big)$. Consequently

$$\mathcal{F}\Big(\big(\xi_j f(x)\big)*g(x)\Big) = (-2\pi i)^{-1}\big(\mathcal{F}f(x)\cdot\mathcal{F}g(x)\big)^{(e_j)} - \mathcal{F}f(x)\cdot\mathcal{F}\xi_j g(x)$$

$$= (-2\pi i)^{-1}\Big(\big(\mathcal{F}f(x)\big)^{(e_j)}\cdot\mathcal{F}g(x)+\mathcal{F}f(x)\cdot\big(\mathcal{F}g(x)\big)^{(e_j)}\Big) - \mathcal{F}f(x)\cdot\mathcal{F}\xi_j g(x)$$

$$= \mathcal{F}\xi_j f(x)\cdot\mathcal{F}g(x)$$

on account of (3).

By the *adjoint Fourier transform* of a distribution $f(x)$ we understand the distribution

$$(1')\qquad \overline{\mathcal{F}}f(x) = \int_{-\infty}^{\infty} f(t)\exp(2\pi i x t)\,dt$$

provided the integral converges in the whole space. By definition,

$$\overline{\mathcal{F}}f(x) = \mathcal{F}f(-x).$$

Hence it follows that the adjoint Fourier transform has, roughly speaking, the same properties as the Fourier transform. More exactly, theorems (i)-(viii) hold for the adjoint transform, but (2), (3), (4), (5), (6) should now be replaced by

$$(2')\qquad \overline{\mathcal{F}}f^{(e_j)}(x) = -2\pi i\xi_j\overline{\mathcal{F}}f(x),$$

$$(3')\qquad \overline{\mathcal{F}}\xi_j f(x) = (2\pi i)^{-1}\big(\overline{\mathcal{F}}f(x)\big)^{(e_j)},$$

$$(4')\qquad \overline{\mathcal{F}}f^{(k)}(x) = (-2\pi i)^k x^k\overline{\mathcal{F}}f(x), \qquad \overline{\mathcal{F}}x^k f(x) = (2\pi i)^{-k}\big(\overline{\mathcal{F}}f(x)\big)^{(k)},$$

$$(5')\qquad \overline{\mathcal{F}}(1+x^2)^\varkappa f(x) = (1-\Delta)^\varkappa\overline{\mathcal{F}}f(x), \quad \overline{\mathcal{F}}(1-\Delta)^\varkappa f(x) = (1+x^2)^\varkappa\overline{\mathcal{F}}f(x),$$

and

$$(6')\qquad \overline{\mathcal{F}}f\big((x)*g(x)\big) = \overline{\mathcal{F}}f(x)\cdot\overline{\mathcal{F}}g(x)$$

respectively.

(ix) *For every slowly increasing distribution* $f(x)$:

$$\overline{\mathcal{F}}\mathcal{F}f(x) = f(x), \quad \mathcal{F}\overline{\mathcal{F}}f(x) = f(x).$$

In fact, by a classical theorem, the formulas $\overline{\mathcal{F}}\mathcal{F}F(x) = F(x)$, $\mathcal{F}\overline{\mathcal{F}}F(x) = F(x)$ hold if $(1+x^2)^q F(x)$ is a bounded continuous function. Any slowly increasing distribution is of the form $f(x) = \big((1+x^2)^\varkappa F(x)\big)^{(k)}$ where $(1+x^2)^q F(x)$ is a bounded continuous function. Hence, by (4), (5), (4′), (5′), $\overline{\mathcal{F}}\mathcal{F}f(x) = \overline{\mathcal{F}}\mathcal{F}\big((1+x^2)^\varkappa F(x)\big)^{(k)} = \overline{\mathcal{F}}\big((2\pi i)^k x^k (1-\Delta)^\varkappa \mathcal{F}F(x)\big) = \big((1+x^2)^\varkappa \overline{\mathcal{F}}\mathcal{F}F(x)\big)^{(k)} = f(x)$.

The second identity can be proved analogously.

The following theorem is a supplement to (v), (vi), (vii):

(x) *Every slowly increasing distribution* (*every rapidly decreasing distribution, every function slowly increasing with all its derivatives*) $f(x)$ *is a Fourier transform of a slowly increasing distribution* (*a function slowly increasing with all its derivatives, a rapidly decreasing distribution*) $f_0(x)$.

For it suffices to assume $f_0(x) = \overline{\mathcal{F}}f(x)$.

(xi) *For every slowly increasing distribution* $f(x)$ *and for every function* $g(x)$ *slowly increasing with all its derivatives,*

$$\mathcal{F}\big(f(x)g(x)\big) = \mathcal{F}f(x) * \mathcal{F}g(x).$$

Let $f_0(x) = \overline{\mathcal{F}}f(x)$, $g_0(x) = \mathcal{F}g(x)$, i. e. $f(x) = \mathcal{F}f_0(x)$, $g(x) = \overline{\mathcal{F}}g_0(x)$. We have by (6′)

$$f(x)g(x) = \overline{\mathcal{F}}f_0(x)\cdot\overline{\mathcal{F}}g_0(x) = \overline{\mathcal{F}}\big(f_0(x) * g_0(x)\big).$$

Hence, by (ix),

$$\mathcal{F}\big(f(x)g(x)\big) = f_0(x) * g_0(x) = \mathcal{F}f(x) * \mathcal{F}g(x).$$

(xii) *If* $f_n(x) \to f(x)$ *strongly, then* $\mathcal{F}f_n(x) \to \mathcal{F}f(x)$ *strongly.*

Suppose that $F_n(x)$, $F(x)$ are slowly increasing functions such that, for an order k and an integer $\varkappa \geqslant 0$,

$$F_n^{(k)}(x) = f_n(x), \qquad F^{(k)}(x) = f(x), \qquad (1+x^2)^{-\varkappa}F_n(x) \rightrightarrows (1+x^2)^{-\varkappa}F(x),$$

and all the functions $(1+x^2)^{-\varkappa}F_n(x)$ are commonly bounded. Hence it follows that

$$\mathcal{F}\big((1+x^2)^{-(\varkappa+q)}F_n(x)\big) \rightrightarrows \mathcal{F}\big((1+x^2)^{-(\varkappa+q)}F(x)\big)$$

and the functions on the left side are commonly bounded. By § 8 (iii), the uniform convergence $\rightrightarrows$ can be replaced here by the strong distributional convergence $\to$. Consequently, by § 8 (iv) and (5),

$$\mathcal{F}F_n(x) = (1-\Delta)^{\varkappa+q}\mathcal{F}\big((1+x^2)^{(\varkappa+q)}F_n(x)\big) \to \mathcal{F}F(x) \text{ strongly.}$$

Multiplying by $(2\pi i)^k x^k$ we get by (4)

$$\mathcal{F}f_n(x) = (2\pi i)^k x^k \mathcal{F}F_n(x) \to \mathcal{F}f(x) \text{ strongly.}$$

We finish this section by proving that the q-dimensional Dirac delta distribution $\delta(x)$ is the Fourier transform of the function identically equal to 1. In fact,

$$\mathcal{F}\,1 = \int_{-\infty}^{\infty} \exp(-2\pi ixt)\,dt = \Big(\prod_{j=1}^{q}\int_{-\infty}^{\infty}\frac{\exp(-2\pi i\xi_j\tau_j)-1}{-2\pi i\tau_j}\,d\tau_j\Big)^{(e)}$$

$$= \Big(\prod_{j=1}^{q}\tfrac{1}{2}\operatorname{sgn}\xi_j\Big)^{(e)} = \delta(x).$$

The theory of the partial Fourier transform (i. e. the Fourier transform with respect to some variables $\xi_{j_1}, \ldots, \xi_{j_p}$ only) can be developed in a similar way.

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