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On the characterization of Schwartz spaces by properties of the norm

by

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Let X be an *F-space*, i. e. a linear space with topology induced by the norm $\|x\|$ (not necessarily homogeneous) such that addition and multiplication by a number (1) are continuous with respect to both variables, but not necessarily complete (see [1], p. 35, and [8]).

If, moreover, the space X is locally convex, then there is a sequence of homogeneous pseudonorms $\|x\|_1 \leqslant \|x\|_2 \leqslant \ldots \leqslant \|x\|_m \leqslant \ldots$ (2) such that the sequence x_n tends to x if and only if the sequences $\|x_n - x\|_m$ converge to zero for all m (see [8]). These spaces are called *B_0^*-spaces.*

A set A is called *bounded* if for an arbitrary sequence of numbers $t_n \to 0$, and for an arbitrary sequence of elements $x_n \in A$, the sequence $t_n x_n \to 0$.

A B_0^*-space is called *Montel space* ([3], [5]) if every bounded set contained in X is compact (3).

A B_0^*-space X is called *Schwartz space* ([4], [5]) if for every positive integer i there is such a positive integer j that the set $A_{i+j} = \{x: \|x\|_{i+j} < 1\}$ is compact with respect to the topology induced by the pseudonorm $\|x\|_i$.

Montel and Schwartz spaces have recently been investigated by a great number of mathematicians: J. Dieudonné, I. M. Gelfand, A. Grothendieck, A. N. Kolmogorov, D. A. Raikov, L. Schwartz, G. E. Silov and many others.

The definition of Montel spaces can be extended to F^*-spaces without any change (4). Another case is presented by Schwartz spaces. Schwartz

(1) The results of this note concern multiplication by real as well as complex numbers.

(2) It means $\|tx\|_m = |t|\,\|x\|_m$ for every t, x and m.

(3) The set Z is called *compact* if every sequence $x_n \in Z$ contains a subsequence x_{n_k} such that $\|x_{n_k} - x_n\|_k$, then $k, k' \to \infty$.

(4) Till now an example of a Montel space which is not locally convex was not been known. Such an example is given at the and of this paper.

spaces are defined by quasinorms and this definition is given below, after the definition and the properties of quasinorms.

The aim of this note is the characterization of Schwartz spaces by properties of the norms.

1. We denote by $\mathfrak{A}$ the class of all open sets $A \subset X$ for which $tA \subset A$ (5) for all numbers t such that $|t| < 1$.

Let A be an arbitrary set belonging to the class $\mathfrak{A}$. The number $[x]_A = \inf\{t: t > 0,\ x/t \in A\}$ is called the *quasinorm* (see [2] and [7]) of the element x with respect to the set A.

Quasinorms have the following properties:

(a) $[tx]_A = |t|[x]_A$;

(b) the quasinorm $[x]_A$ satisfies the triangle inequality, i. e. it is a pseudonorm if and only if the set A is convex;

(c) if $A \subset B$, then $[x]_A \geqslant [x]_B$;

(d) $[x+y]_{A\oplus B} \leqslant \max([x]_A, [y]_B)$ (6).

Let the sequence $A_n \in \mathfrak{A}$ constitute a neighbourhood basis of zero ([6], p. 3).

(e) The sequence x_m converges to zero if and only if the sequences $[x_m]_{A_n}$ tend to 0 for all n.

(f) the sequence x_m is bounded if and only if there is such a sequence of numbers $N_n > 0$ that $[x_m]_{A_n} \leqslant N_n$ for all n and m.

Properties (a) and (c) are trivial.

Property (b) is the well-known property of the convex sets [8].

Property (d) results from the fact that, if we write $r = \max([x]_A, [y]_B)$, then $x \in rA$ and $y \in rB$, whence $x+y \in r(A\oplus B)$; therefore $[x+y]_{A\oplus B} \leqslant r = \max([x]_A, [y]_B)$.

We shall prove property (e). If $x_m \to 0$, then for arbitrary $\varepsilon > 0$, and n, there is such an m_0 dependent on ε and n that, for $m > m_0$, $x_m \in \varepsilon A_n$, whence $[x_m]_{A_n} \leqslant \varepsilon$. Therefore $\lim\limits_{m\to\infty} [x_m]_{A_n} = 0$ for $n = 1, 2, \ldots$ On the other hand, if $\lim\limits_{m\to\infty} [x_m]_{A_n} = 0$, then for every n there is such an m_0 dependent on n that, for $m > m_0$, $[x_m]_{A_n} \leqslant 1$, whence $x_m \in A_n$ and since A_n constitutes a neighbourhood basis of zero, then $x_m \to 0$.

Property (f) is a simple consequence of property (e). Indeed, if there is such a sequence of numbers $N_n > 0$ that $[x_m]_{A_n} \leqslant N_n$ for all m, then for an arbitrary sequence $t_m \to 0$, $[t_m x_m]_{A_n} \leqslant N_n t_m \to 0$ if $m \to \infty$, whence property (e) implies that $t_m x_m \to 0$; therefore the sequence x_m is bounded. On the other hand, if for some n there is a subsequence x_{m_k} such that

(5) By tA we denote the set of all elements tx where $x \in A$.

(6) By $A\oplus B$ we denote the set of all sums $x+y$ such that $x \in A$, $y \in B$.

$[x_{m_k}]_{A_n} \to \infty$, then the sequence

$$t_m = \begin{cases} 0 & \text{if} \quad m \neq m_k, \\ ([x_{m_k}]_{A_n})^{-1} & \text{if} \quad m = m_k \end{cases}$$

converges to zero, but the sequence $[t_m x_m]_{A_n}$ does not tend to zero, because $[t_{m_k} x_{m_k}]_{A_n} = 1$, whence property (e) implies that $t_m x_m$ does not converge to 0; therefore the sequence x_m is not bounded.

The set Z is called *compact with respect to a quasinorm* $[x]_A$ if every sequence $x_n \in Z$ contains a subsequence x_{n_k} such that $\lim\limits_{k,k'\to\infty} [x_{n_k} - x_{n_{k'}}]_A = 0$.

Definition of Schwartz spaces. Let the decreasing sequence of sets $A_n \in \mathfrak{A}$ constitute a neighbourhood basis of zero. The space X is called a *Schwartz space* if for every i there is such a j that the set A_{i+j} is compact with respect to the quasinorm $[x]_{A_i}$.

If X is a B_0^*-space, then this definition is obviously in accordance with the definition given at the beginning of this note.

It is easy to see, as in the case of B_0^*-spaces, that every Schwartz space is also a Montel space.

2. Let Y be an arbitrary F^*-space with the norm $\|x\|$, and let ε be an arbitrary positive number. We write

$$c(Y, \varepsilon, t) = \inf\{\|tx\|: x \in Y,\ \|x\| = \varepsilon\}$$

if there is such an $x \in Y$ that $\|x\| = \varepsilon$, and

$$c(Y, \varepsilon, t) = \begin{cases} \varepsilon & \text{for} \quad t \neq 0, \\ 0 & \text{for} \quad t = 0 \end{cases}$$

if, for all $x \in Y$, $\|x\| < \varepsilon$.

THEOREM 1. *Let X be a Schwartz space. Then, for every increasing sequence of finitely dimensional spaces X_n, such that the set $X^* = \bigcup\limits_{n=1}^{\infty} X_n$ is dense in X, the functions $c(X/X_n, \varepsilon, t)$ (7) are not equicontinuous in 0 for each fixed ε.*

Proof. We write $K_\eta = \{x: \|x\| < \eta\}$ and denote by K_η^* the largest set A belonging to $\mathfrak{A}$ and contained in K_η.

Suppose that the theorem does not hold. Then there is such a sequence X_n of finitely dimensional subspaces that $X_n \subset X_{n+1}$, the set $X^* = \bigcup\limits_{n=1}^{\infty} X_n$ is dense in X and the functions $c(X/X_n, \varepsilon, t)$ are equicontinuous in 0 for some ε. It is obvious that also the functions $c(X^*/X_n, \varepsilon, t)$ are equicontinuous for this ε.

(7) By X/Y we denote the quotient space (see [6], p. 18).

Let δ be an arbitrary positive number. By λ_0 we denote such a positive number that $c(X^*/X_n, \varepsilon, \lambda) < \delta/2$ for $0 < \lambda \leqslant \lambda_0$. We choose by induction sequences of positive integers k_n and of elements x_n in the following way:

1. $x_1 \in X_1$,
2. $\|x_n\| < \delta$ $(n = 1, 2, \ldots)$,
3. $x_n \in X_{k_n}$,
4. $[x_n - x_i]_{K^*_{\varepsilon/2}} > \lambda_0$ for $i = 1, 2, \ldots, n-1$.

This is possible. Let us suppose that for some n we have chosen the elements $x_1, \ldots, x_n$ satisfying conditions 1-4. By hypothesis there is in the space X^*/X_{K_n} such a residue class Z that $\|Z\| = \varepsilon$ and $\|\lambda_0 Z\| < \delta/2$. By x_{n+1} we denote an arbitrary element of $\lambda_0 Z$ such that $\|x_{n+1}\| < \delta$, and by $X_{k_{n+1}}$ we denote a subspace containing the element x_{n+1} (it is possible since $X^* = \bigcup_{n=1}^{\infty} X_n$). Since, for every $x \in X_{K_n}$, $(x_{n+1} - x) \in \lambda_0 Z$, $(x_{n+1} - x)/\lambda_0 \notin K_{\varepsilon/2}$, we have $[x_{n+1} - x_i]_{K^*_{\varepsilon/2}} > \lambda_0$ for $i = 1, 2, \ldots, n-1, n$.

Thus the sequence x_n is not compact with respect to the quasinorm $[x]_{K^*_{\varepsilon/2}}$, whence the arbitrariness of δ implies that the space X is not a Schwartz space.

I do not know whether the inverse theorem to the theorem 1 is true in the general case.

THEOREM 2. *Let X be an F^*-space for which there is an $\varepsilon_0 > 0$ such that for every $x \in X$ there is such a number t_x that $\|t_x X\| = \varepsilon_0$. If there is such a sequence of finitely dimensional subspaces X_n that the functions $c(X/X_n, \varepsilon, t)$ are not equicontinuous in 0 for any fixed ε, then the space X is a Schwartz space.*

The proof of this theorem is based on the following

LEMMA. *Let Z be an arbitrary set contained in X. If for every number $\delta > 0$, there is a finite number of elements $x_1, \ldots, x_n \in X$ such that for each $x \in Z$ there is such an x_i $(1 \leqslant i \leqslant n)$ that $[x - x_i]_A < \delta$, then the set Z is compact with respect to the quasinorm $[x]_{A \oplus A}$.*

Proof of the lemma. Let x_n be an arbitrary sequence of elements of the set Z. We define by induction the sequences x_n^k in the following way:

1. $x_n^1 = x_n$,
2. the sequence x_n^{i+1} is a subsequence of the sequence x_n^i,
3. for each sequence x_n^i there is such an element $x_i' \in X$ that $[x_n^i - x_i']_A < 1/i$ for $n = 1, 2, \ldots$

The existence of such sequences trivially follows from the assumption of the set Z. The sequence x_n^n is the required sequence. Indeed, property

(d) of quasinorms implies that $[x_n^n - x_m^m]_{A \oplus A} < 1/n$ if $n < m$, because x_n^n and x_m^m are elements of the sequence x_k^n (n being fixed) and there is such an x_n' that $[x_n^n - x_n']_A < 1/n$, $[x_m^m - x_n']_A < 1/n$, q. e. d.

Proof of theorem 2. We write $K_\eta' = \bigcup_{|t|<1} tK_\eta$. Let $0 < \varepsilon < \varepsilon_0/2_u$ Suppose that there is such a sequence of finitely dimensional subspaces X. that the functions $c(X/X_n, \varepsilon, t)$ are not equicontinuous in 0, i. e. that there are a sequence n_k of positive integers and a number $\delta > 0$ such that for each $\mu > 0$ there are a k_0 and a sequence μ_k, $0 < \mu_k < \mu$ for which $c(X/X_{n_k}, \varepsilon, \mu_k) > \delta$ provided that $k > k_0$. It means that, in the space $X/X_{n_k}\mu_k$, $K_\varepsilon' \supset K_\delta$, whence also $\mu_k K_2' \supset K_\delta'$ and thus $\mu K_\varepsilon' \supset K_\delta'$. Let $x \in X$ and $\|x\| < \delta$. By Z we denote the residue class containing x. Since $\|Z/\mu_k\| < \varepsilon$, there is an element $x_0 \in Z$, such that $\|x_0/\mu_k\| < \varepsilon$, whence $[x_0]_{K_\varepsilon'} \leqslant \mu_k < \mu$.

We can write $x = x_0 + (x - x_0)$. We have $(x - x_0) \in X_{n_k}$, $(x - x_0) \in K_\delta \oplus \oplus \mu_k K_\varepsilon \in \mu_k K_\varepsilon \oplus K_\varepsilon \in \mu_k K_{2\varepsilon}$ (8). Since X_{n_k} has a finite dimension and $2\varepsilon < \varepsilon_0$, there is such a finite number of elements $y_1, \ldots, y_m$ that for each x, $\|x\| < \delta$, there is an y_i for which $[(x - x_0) - y_i]_{K_\varepsilon'} < \mu$. The property (d) of the quasinorms implies that $[x - y_i]_{K_\varepsilon'^2} < \mu$. Hence by the lemma the set K_δ is compact with respect to the quasinorm $[x]_{K_\varepsilon'^4}$. And since the sets $K_\varepsilon'^4$ constitute a neighbourhood basis of zero, the space X is a Schwartz space, q. e. d.

COROLLARY. *Let X be an F^*-space with basis e_n* (see [1], p. 110) *and suppose that there exists such an $\varepsilon_0 > 0$ that for every $x \in X$ there is such a number tx that $\|t_k x\| = 0$. The space X is a Schwartz space if and only if the functions $c(X_n', \varepsilon, t)$, where X_n' is the space generated by the elements $e_{n+1}, e_{n+2}, \ldots$, are not equicontinuous in 0 for any fixed ε.*

A simple consequence of this corollary is an example of a Schwartz space which is not locally convex.

Let X be the space of all sequences $x = \{\xi_n\}$ such that

$$\|x\| = \sum_{n=1}^{\infty} |\xi_n|^{1/n} < +\infty$$

with the topology induced by the norm $\|x\|$. It is easy to see that it is an F-space and the sequence $e_n = \{0, 0, \ldots, \underset{n\text{-place}}{0, 1}, 0, \ldots, 0\}$ is a basis in this space. This space is not locally convex. Really, let ε be an arbitrary positive number.

(8) We denote $Z \oplus Z$ by Z^2, $Z^n \oplus Z$ by Z^{n+1}.

By x_p^ε we denote $x_p^\varepsilon = \{\xi_i^{p,\varepsilon}\}$, where

$$\xi_i^{p,\varepsilon} = \begin{cases} 0 & \text{for} \quad i \neq p, \\ \varepsilon^p & \text{for} \quad i = p. \end{cases}$$

Obviously $\|x_p^\varepsilon\| = \varepsilon$, but

$$\left\| \frac{x_1^\varepsilon + \ldots + x_n^\varepsilon}{n} \right\| = \varepsilon \sum_{p=1}^{n} \left(\frac{1}{n}\right)^{1/p} \geqslant \varepsilon \left(\frac{1}{n} + \sum_{p=2}^{n} \left(\frac{1}{n}\right)^{1/2} \right) \geqslant \varepsilon \frac{(n-1)}{\sqrt{n}} \to \infty.$$

On the other hand, it is easy to verify that

$$c(X_n', \varepsilon, t) = \begin{cases} \varepsilon t^{1/n} & \text{for} \quad |t| \leqslant 1, \\ \varepsilon & \text{for} \quad |t| > 1, \end{cases}$$

whence X is a Schwartz space.

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Anerkennung der Priorität zu meinem "Beitrag zur Theorie des Maßringes mit Faltung"

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Satz 3 aus meinem Beitrag [3] ist implizit in [5], [1] und [2] enthalten. Die Autoren haben nämlich bewiesen (z. B. Theorem 3 in [2], S. 189), daß im Ring aller Funktionen von beschränkter Schwankung in $(-\infty, \infty)$ eine Funktion f immer dann eine Reziproke (im Sinne der Faltung) hat, wenn ihre Fouriertransformierte F, ihre sprungartige Komponente h und ihre stetige singuläre Komponente s folgenden Ungleichungen genügen:

(1) $$|F(x)| > c > 0 \qquad (-\infty < x < \infty),$$

(2) $$\operatorname*{Var}_{-\infty<t<\infty} s(t) < \inf_x \left| \int_{-\infty}^{\infty} e^{ixt} dh(t) \right|.$$

Überträgt man diesen Satz von der reellen Achse auf den Kreis, was keine Änderung der Beweismethode erfordert, so erhält man ein Ergebnis, das sich vom Satz 3 aus [3] dadurch unterscheidet, daß anstatt $s(t) \equiv 0$ das dem Kreis angepaßte Analogon von (2) angenommen wird und daß jede Lokalisierung der Sprungstellen der Reziproken von f fehlt. In logischer Hinsicht sind diese Aussagen unvergleichbar, sie sind aber nicht wesentlich verschieden.

Meine Beweismethode ist der in [2] ganz analog, insofern sie das Dichtliegen der stetigen Charaktere in der Charaktergruppe der „diskreten Zahlenachse" (Theorem 2 in [2], S. 177) ausnutzt. Somit ist Satz 1 aus [3] auch teilweise als bekannt zu betrachten, nämlich fur den Fall, wenn die Gruppe G die reelle Achse oder (durch naheliegende Modifizierung) die Kreisgruppe ist. In dieser Beziehung muß ich den Einwand des Referenten in Mathematical Reviews [4] dankend anerkennen und mich durch den erschwerten Zugang zu der von ihm zitierten Literatur (aus den der Kriegszeit dicht benachbarten Jahren) zu rechtfertigen suchen.

Andererseits kann ich dem Referenten der Mathematical Reviews nicht vorbehaltslos beistimmen, wenn er [3] als ein bloßes Wiederholen