Extinguishing a class of functions

By

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Let $E$ be a set of real positive numbers. By $L(E)$ we shall denote the family of all intervals of the form

$$ I = \{(x, y): \alpha x + y = t, \ x > 0, \ y > 0\}, $$

where $\alpha E$ and $0 < t < \infty$. A complex-valued continuous function $\varphi$ of two variables defined on the first quadrant is said to be extinguished by the set $E$ if $\int_{I \in L(E)} f(x, y) \, ds = 0$ for any interval $I \in L(E)$. It is well known (12), p. 63) that

(*) The unique function extinguished by the right half-line is the function

$$ \varphi(x, y) = \sum_{n=1}^{\infty} f_n(x) g_n(y), $$

where all the functions $f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_n$ are continuous on the right half-line. By $\mathcal{E}_n$, we shall denote the class of all sets $E$ of positive numbers such that the unique function belonging to $\mathcal{E}_n$ and extinguished by $E$ is the function identically equal to 0. From Titchmarsh’s Theorem on convolution ([3], p. 327) it follows that all one-point sets belong to $\mathcal{E}_1$. Indeed, if a function $\varphi$ is extinguished by a set $\{a\}$ and $\varphi(x, y) = f(x)g(y)$, then we have the equality

$$ \int_{y=a} f(x)g(y) \, ds = 0 \quad (t > 0). $$

Hence for any positive $t$ we get the equality

$$ \int_{x=a}^{t} f(x)g(x-t-x) \, dx = 0. $$
which, according to Titchmarsh’s Theorem, implies either \( f(x) = 0 \) for \( x \geq 0 \) or \( g(y) = 0 \) for \( y \geq 0 \). Thus \( \varphi(x, y) \) vanishes in the whole first quadrant.

Let \( P_n \) denote the least power of sets belonging to \( C_n \), i.e. \( P_n = \min E \), where \( E \) is the power of the set \( E \). We have proved above that \( P_1 = 1 \). The aim of our note is to prove the inequality

\[
(n + 2) < P_n < \frac{1}{2}(n^2 - n + 4) \quad (n \geq 2),
\]

which for \( n = 2 \) implies the equality \( P_1 = 3 \).

In the proof of inequality (**) Mikusiński’s Operational Calculus will be used [1].

Let us consider the set of all complex-valued continuous functions defined on the right half-line. This set is a commutative ring with respect to usual addition and convolution as multiplication:

\[
(f * g)(t) = \int_0^t f(u)g(t-u)du.
\]

By Titchmarsh’s Theorem on convolution the ring in question has no divisors of zero. Therefore it can be extended to a quotient field. The elements of that quotient field are called operators.

For any positive number \( a \) we put

\[
f^a(t) = f(at).
\]

Let us introduce a family of transformations \( T^a \) \((0 < a < \infty)\) defining them for every operator \( a = \frac{f}{g} \), where \( f \) and \( g \) are continuous functions, by the equality

\[
T^a b = \frac{f}{g}.
\]

It is easy to verify that this definition does not depend on the choice of the representation of the operator by a quotient of continuous functions. Moreover, we have the equalities

\[
T^a b = T^a(T^b a) = T^a(T^b a),
\]

\[
T^a a = a,
\]

\[
T^a(ab) = T^a a \cdot T^b b
\]

for all operators \( a \) and \( b \) and all positive numbers \( a \) and \( b \).

A system \( a_1, a_2, \ldots, a_n \) of positive numbers is said to be independent if from the equality \( a_1^m a_2^m \cdots a_n^m = 1 \), where \( m_1, m_2, \ldots, m_n \) are integers, follows the equality \( m_1 = m_2 = \cdots = m_n = 0 \).

Lemma 1. The only invariant operators under two transformations

\[ T^a \text{ and } T^b \]

where \( a \) and \( b \) are independent, are constant operators.

Proof. Let us assume that an operator \( a \) satisfies the equalities

\[ T^a b = a \quad (k, s = 0, \pm 1, \pm 2, \ldots, \ldots) \]

Writing the operator \( a \) in the form \( \frac{f}{g} \), where \( f \) and \( g \) are continuous functions and \( g \) is not identically equal to 0, and using notation (1) we have, according to (4), the following equalities:

\[
\frac{f^s}{g^{s+1}} = \frac{f}{g} \quad (k, s = 0, \pm 1, \pm 2, \ldots),
\]

or

\[
g^{s+1} - f^s = 0 \quad (k, s = 0, \pm 1, \pm 2, \ldots) .
\]

It is easy to see that for any continuous function \( h \) the convergence of \( \gamma \) to \( \gamma \) of a sequence \( \gamma, \gamma_2, \ldots \) of positive numbers implies the convergence to \( \gamma \), uniform in every finite interval of the sequence \( \gamma, \gamma_2, \ldots \). Since for independent \( a \) and \( b \) the set \( \{ a^s b^t \} \), \( k = 0, \pm 1, \pm 2, \ldots \) is dense on the right half-line, we have according to (5) \( \gamma g^t - f^t = 0 \) for each positive number \( \gamma \). This means that

\[
\int \left( \frac{g(x)}{f(x)} \right)^t dx = 0
\]

for all positive \( t \). Introducing the auxiliary function

\[
\varphi(x, y) = \frac{g(x)}{f(x)} - \frac{f(x)}{g(x)},
\]

we have, according to (6),

\[
\int_{x+y=t} \varphi(x, y) dx = 0
\]

for every positive \( t \). In other words, the function \( \varphi \) is extinguished by the right half-line. Thus, by theorem (a),

\[
\varphi(x, y) = 0 \quad \text{in the first quadrant.}
\]

We have assumed that the function \( g \) is not identically equal to 0. Let \( y_0 \) be a positive number for which \( g(y_0) \neq 0 \). From (7) and (8) we get the equality \( \varphi(x) = \frac{f(y_0)}{g(y_0)} g(x) \) for any non-negative \( x \). Thus, \( a = \frac{f(y_0)}{g(y_0)} \),

which proves that \( a \) is a constant operator.
For every system \( a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \) of operators we shall denote by \( A(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \) the set of all positive numbers \( \lambda \) for which the equality \( \sum a_i T^\lambda b_j = 0 \) holds. Further, for any pair \( a \) and \( b \) of positive numbers we put \( E_0(a, b) = \{0\} \) and \( E_0(a, b) = \{a^b, b^a\} : k \geq 0 \), \( s \geq 0, k + s \leq n - 2 \) or \( s = 0, k = n - 1 \) and \( s = 0, k = n - 1 \) if \( n \geq 2 \). For example, \( E_0(1, a) = \{1, a\} \), \( E_0(a, 1) = \{a, 1, a^2, a^3, \ldots\} \).

**Lemma 2.** If \( E_0(a, \beta) \subset A(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \) and \( n \geq 2 \), then both \( a^\beta b \) and \( a^\beta b \) belong to \( A(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \).

**Proof.** If \( n = 2 \) our assertion is obvious because \( a^\beta b = a \epsilon E_0(a, \beta) \) and \( a^\beta b = \beta \epsilon E_0(a, \beta) \). Therefore we may suppose that \( n \geq 3 \). Moreover, if \( a_1 = a_2 = \ldots = a_n = 0 \), then every positive number belongs to \( A(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \). Consequently, we may assume that at least one operator \( a_1, a_2, \ldots, a_n \) is different from 0. Hence it follows that the rank of the matrix \( \{a_i^j b_k\} \) (\( j = 1, 2, \ldots, n; n \), \( k \epsilon A(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \)) is not greater than \( n - 1 \).

First let us assume that the rank of the matrix \( \{T^\lambda b_j\} \) (\( j = 1, 2, \ldots, n \)) is equal to \( n - 1 \). Since for every \( \mu \epsilon A(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \), the rank of the matrix \( \{T^\lambda b_j\} \) (\( j = 1, 2, \ldots, n; \lambda = 1, a, a^2, \ldots, a^{n-1}, \mu \)) is also \( n - 1 \), there is a system of operators \( a_1, a_2, \ldots, a_{n-1} \) such that

\[
T^\lambda b_j = \sum_{i=1}^{n-1} c_i T^{a_i^\lambda} b_j \quad (j = 1, 2, \ldots, n).
\]

Hence we get the equality

\[
T^{a_1^\lambda} b_j = \sum_{i=1}^{n-1} c_i T^{a_i^\lambda} b_j \quad (j = 1, 2, \ldots, n),
\]

which implies

\[
\sum_{i=1}^{n} q_i T^{a_i^\lambda} b_j = \sum_{i=1}^{n} T^{a_i^\lambda} c_i T^{a_i^\lambda} b_j = 0
\]

because \( a_1, a_2, \ldots, a_{n-1} \) belong to \( E_0(a_1, b_1, b_2, \ldots, b_n) \). In other words, we have got the relation \( \mu \epsilon A(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \), which implies \( a_1^\lambda b \) and \( a_2^\lambda b \) belong to \( E_0(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \), because for \( n \geq 3 \), \( a^\lambda b \) and \( b^\lambda b \) belong to \( E_0(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \). By symmetry it follows that \( a^\lambda b \) and \( b^\lambda b \) also belong to \( A(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \) if the rank of the matrix \( \{T^\lambda b_j\} \) (\( j = 1, 2, \ldots, n; \lambda = 0, 1, \ldots, n - 2 \)) is equal to \( n - 1 \). Now let us assume that the rank of the matrices \( \{T^\lambda b_j\} \) (\( j = 1, 2, \ldots, n; \lambda = 0, 1, \ldots, n - 2 \)) is smaller than \( n - 1 \). By symmetry it suffices to show that

\[
\beta a^\lambda b \epsilon A(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n).
\]

Since the rank of \( \{T^\lambda b_j\} \) (\( j = 1, 2, \ldots, n; \lambda = 0, 1, \ldots, n - 2 \)) is smaller than \( n - 1 \), there is an index \( k \leq n - 2 \) and a system of operators \( a_1, a_2, \ldots, a_k \), where \( a_k \neq 0 \), such that

\[
\sum_{i=1}^{k} d_i T^{a_i^\lambda} b_j = 0 \quad (j = 1, 2, \ldots, n).
\]

Hence we get the equality

\[
T^{a_1^\lambda} b_j = -\sum_{i=1}^{k-1} d_i T^{a_i^\lambda} b_j \quad (j = 1, 2, \ldots, n).
\]

Further, taking into account the inequalities \( 0 \leq \lambda < k - s \), \( 0 \leq n - k \), \( 0 \leq k \leq n - 2 \), and the definition of \( E_0(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \), we have the relation \( \beta a^\lambda b \epsilon E_0(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \) for \( 0 \leq k \leq n - 2 \) and \( 0 \leq k \leq n - 1 \). Hence and from (9) we get the equality

\[
\sum_{i=1}^{n} q_i T^{a_i^\lambda} b_j = -\sum_{i=1}^{k-1} d_i T^{a_i^\lambda} b_j \quad (j = 1, 2, \ldots, n).
\]

Thus \( a^\lambda b \epsilon A(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \), which completes the proof of the Lemma.

Since for \( n \geq 2 \), \( a \epsilon E_0(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \), \( a \epsilon A(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \), we get, as a direct consequence of Lemma 2, the following

**Corollary.** If \( E_0(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \) and \( n \geq 2 \), then \( a \epsilon A(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \) for any \( a \epsilon A(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \).

**Lemma 3.** Let \( a \) and \( b \) be a pair of independent positive numbers. If the operators \( b_1, b_2, \ldots, b_n \) are linearly independent with respect to the field of complex numbers and \( E_0(a, b) \subset A(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) \), then \( a_1 = a_2 = \ldots = a_n = 0 \).

**Proof.** We shall prove our Lemma by induction with respect to the index \( n \). For \( n = 1 \) our statement is a direct consequence of Titchmarsh’s Theorem. Now let us suppose that \( n \geq 2 \) and for all indices smaller than \( n \) the statement of our Lemma is true. Further, let us suppose that not all operators \( a_1, a_2, \ldots, a_n \) vanish. Without loss of generality of our considerations we may assume that \( a_n \neq 0 \). From the linear independence
of \( b_1, b_2, \ldots, b_n \), we infer that \( b_n \neq 0 \). Putting
\[
\delta_j = \frac{b_j}{a_n}, \quad \beta_j = \frac{b_j}{b_n} \quad (j = 1, 2, \ldots, n-1),
\]
we have the equalities
\[
\sum_{j=1}^{n-1} a_j T^{a_j} \beta_j + T^{a_n} b_n = 0, \\
\sum_{j=1}^{n-1} \delta_j T^{a_j} \beta_j + 1 = 0
\]
for any \( \lambda \in A(a, a_n) \). Hence, by the Corollary to Lemma 2, we get the equalities
\[
\sum_{j=1}^{n-1} a_j T^{a_j} \beta_j + 1 = 0, \\
\sum_{j=1}^{n-1} \delta_j T^{a_j} \beta_j + 1 = 0
\]
for any \( \lambda \in E_{a-1}(a, a_n) \). Applying the transformations \( T^{a_j} \) and \( T^{a_n} \) to equations (12) and (13) respectively, we get the following system of equations:
\[
\sum_{j=1}^{n-1} T^{a_j} \beta_j + 1 = 0, \quad \sum_{j=1}^{n-1} T^{a_j} \beta_j + 1 = 0
\]
for any \( \lambda \in A(a, a_n) \). Hence and from (11) we obtain the equations
\[
\sum_{j=1}^{n-1} (T^{a_j} \beta_j - \beta_j) T^{a_n} \beta_j + 0 = 0, \\
\sum_{j=1}^{n-1} (T^{a_j} \beta_j - \beta_j) T^{a_n} \beta_j = 0
\]
for every \( \lambda \in A(a, a_n) \). Thus, by the linear independence of \( b_1, b_2, \ldots, b_{n-1} \) and the induction assumption, we have the equalities \( T^{a_j} \beta_j = \beta_j = T^{a_n} \beta_j \) (\( j = 1, 2, \ldots, n-1 \)). The numbers \( 1/a \) and \( 1/\beta \) are independent. Consequently, in view of Lemma 1, all the operators \( \delta_j, \beta_j, \ldots, \beta_{n-1} \) are constant, i.e. are complex numbers. Hence and from (10) follows the linear dependence of the operators \( b_1, b_2, \ldots, b_n \), which is impossible. The Lemma is thus proved.

Proof of inequality (**). It is very easy to verify that \( E_{a}(a, b) = \frac{1}{2}(n^2 - n + 4) \) for independent \( a \) and \( b \) and \( n \geq 2 \). Consequently, to prove the inequality
\[
P_n \leq \frac{1}{2}(n^2 - n + 4) \quad (n \geq 2)
\]
it is sufficient to show that for independent \( a \) and \( b \) the relation
\[
E_{a}(a, b) \geq \gamma_{a} \quad (n \geq 1)
\]
holds.

For \( n = 1 \) the last relation is evident. Now let us suppose that \( n \geq 2 \) and \( E_{a}(a, b) \geq \gamma_{a} \) for \( k < n \). Let \( \varphi \) be a function belonging to \( A(a, a_n) \), extinguished by the set \( E_{a}(a, b) \) and having the representation \( \varphi(x, y) = \sum_{j=1}^{n} f_j(x) g_j(y) \). If the functions \( g_1, g_2, \ldots, g_n \) are linearly dependent, then \( \varphi \in A(a, a_n) \) and, by the inclusion \( E_{a-1}(a, a_n) \subset E_{a}(a, b) \), the function \( \varphi \) is extinguished by the set \( E_{a-1}(a, a_n) \). Consequently, \( \varphi(x, y) = 0 \) in the whole first quadrant. Finally let us suppose that the functions \( g_1, g_2, \ldots, g_n \) are linearly independent. Then we have the operational equality
\[
\sum_{j=1}^{n} f_j T^{a_j} g_j = 0 \quad \text{for any} \quad \lambda \in E_{a}(a, b).
\]

Applying Lemma 3 we get \( f_1 = f_2 = \ldots = f_n = 0 \) and, consequently, \( \varphi(x, y) = 0 \) in the whole first quadrant. Thus we have proved relation (14).

Now we shall prove the inequality \( P_n > n \) (\( n \geq 2 \)). Let \( B \) be an arbitrary \( n \)-point set: \( B = \{ \gamma_1, \gamma_2, \ldots, \gamma_n \} \). Put
\[
g_j(x) = \sin^j \left( \frac{x}{2^n} \left[ \log \frac{\gamma_1}{\gamma_2} \right]^{-1} \right) \quad (j = 1, 2, \ldots, n).
\]

It is easy to see that all the functions \( g_1, g_2, \ldots, g_n \) are linearly independent and
\[
T^{a_j} \gamma_j = \frac{\gamma_j}{\gamma_n} \gamma_j + 1 \quad (j = 1, 2, \ldots, n)
\]

Hence
\[
T^{a_j} g_j = \frac{\gamma_j}{\gamma_n} T^{a_j} g_j \quad (j = 1, 2, \ldots, n)
\]
and, consequently, the rank of the matrix \( [T^{a_j} g_j] \) (\( j = 1, 2, \ldots, n; s = 1, 2, \ldots, n \)) is smaller than \( n \). There exists then a system of operators \( a_1, a_2, \ldots, a_n \), satisfying the equalities
\[
\sum_{j=1}^{n} a_j T^{a_j} g_j = 0 \quad (s = 1, 2, \ldots, n),
\]
where at least one operator \( a_j (1 \leq j \leq n) \) is different from 0. Writing the operators \( a_j \) in the form \( a_j = \frac{b_j}{b_n} \) (\( j = 1, 2, \ldots, n \)), where \( f_j, f_1, f_2, \ldots, f_n \)
are continuous functions, we have, according to (15), the following equalities
\[ \sum_{i=1}^{n} f_i T^n g_i = 0 \quad (s = 1, 2, \ldots, n). \]

In other words the function
\[ \psi(x, y) = \sum_{i=1}^{n} f_i(x) g_i(y) \]
is extinguished by the set \( E \). Since not all function \( f_1, f_2, \ldots, f_n \) vanish and \( g_1, g_2, \ldots, g_n \) are linearly independent, \( \psi(x, y) \) is not identically equal to 0 in the first quadrant. Thus \( E \in \mathfrak{E} \) and, consequently, \( P_n \geq n \).

References


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A proof of Schwartz's theorem on kernels

by

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L. Schwartz has shown that every bilinear continuous functional \( \mathcal{B}(\varphi_1, \varphi_2) \) on the space \( D(Q_1) \times D(Q_2) \) (see the definition below) may be represented by a linear continuous functional \( T \) on the space \( D(Q_1) \times D(Q_2) \), i.e.

\[ \mathcal{B}(\varphi_1, \varphi_2) = T(\varphi_1 \times \varphi_2) \quad \text{for} \quad \varphi_1 \in D(Q_1), \quad i = 1, 2, \]

where \( (\varphi_1 \times \varphi_2)(x_1, x_2) = \varphi_1(x_1) \varphi_2(x_2) \) for \( x_i \in Q_i, \quad i = 1, 2. \)

Since every such functional corresponds to a linear continuous map \( L \) of \( D(Q_1) \) into \( D'(Q_2) \) defined by

\[ L(\varphi_1)(\varphi_2) = \mathcal{B}(\varphi_1, \varphi_2), \]
equality (1) may be written symbolically in the form

\[ L(\varphi_1)(\varphi_2) = \int T(x_1, x_2) \varphi_1(x) \varphi_2(x_2) dx_1 dx_2 \quad \text{for any} \quad \varphi_1 \in D(Q_1), \]

and therefore Schwartz's theorem may be interpreted as a theorem concerning representation of linear continuous operations by kernels.

The theorem is a special case of a general theorem of A. Grothendieck on topological tensor products.

The purpose of this paper is to give a simple proof of Grothendieck's theorem for a special case which often occurs in applications. The proof is based only on elementary properties of \((F)\)-spaces (\((E)\)-spaces in the Polish terminology) and \((LF)\)-spaces.

For the convenience of the reader we shall make a short review of the properties to be used in the paper.

1. Let \( X \) be a linear space over the complex field. Given a family of semi-norms \( ||x||_A \) (\( a \in \mathcal{A} \)) on \( X \), we can define a topology on \( X \) taking the family of sets \( \{ x : ||x - x_0||_A < \varepsilon, \quad i = 1, 2, \ldots, n \} \) as a fundamental system of neighbourhoods of the point \( x_0 \).

This topology is a Hausdorff topology if and only if the family of semi-norms is separating, i.e. if, for every \( x \neq 0 \), there is an \( a \in \mathcal{A} \) such