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Linear spaces with mixed topology

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This paper (¹) contains a systematic investigation of a topology called *mixed topology* and determined in a natural way by two given topologies defined in the same linear space X. The theory of mixed topologies here presented is closely related to Alexiewicz's investigation of two-norm spaces ([1], [2]) and Orlicz's investigation of Saks spaces ([9], [10]). Roughly speaking, the mixed topology is the unique natural neighbourhood topology corresponding to the sequential topology in a two-norm space. It is also the unique natural extension (to the whole space) of the topology in the unit sphere considered as a Saks space. The connection between spaces with mixed topology and Saks spaces or two-norm spaces makes it possible to apply the theory of linear topological spaces (in particular locally convex spaces) to the investigation of Saks spaces and two-norm spaces. As an example of such applications a theorem on the extension of γ-linear functionals will be proved (see 2.6.4)

1. Preliminaries. By a *linear space* we understand any linear space over the field of reals $\mathcal{R}$. The restriction to real spaces is not essential and the passage to complex spaces presents no difficulty. If X is a linear space and $a \in X$, $A \subset X$, $B \subset X$, $a \in \mathcal{R}$, then we shall use the following notation:

$$a+A = [a+x: x \in A],$$

$$A \pm B = [x \pm y: x \in A \text{ and } y \in B],$$

$$aA = [ax: x \in A].$$

If $y = f(x)$ is a mapping from a set X into another set Y and $Z \subset X$, then the restriction of f to Z is denoted by $f|Z$.

Let X be a linear space. If a topology τ is defined in X in such a way that addition and multiplication by scalars are continuous in both

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variables, then τ is called a *linear topology*. A linear space X with a linear topology τ is called a *linear topological space* and is denoted by $\langle X, \tau\rangle$. If τ is a linear topology, then $\mathfrak{U}(\tau)$ denotes a basis of neighbourhoods for 0 in the τ-topology. For each $a \in X$ the family of sets $a+U$, $U \in \mathfrak{U}(\tau)$, is then a basis of neighbourhoods for the element a.

If τ is a linear topology, then there exists a basis $\mathfrak{U}(\tau)$ satisfying the following conditions:

(l_1) if $U \in \mathfrak{U}(\tau)$ and $\lambda \in \mathcal{R}$, $\lambda \neq 0$, then $\lambda U \in \mathfrak{U}(\tau)$,

(l_2) if $U \in \mathfrak{U}(\tau)$ and $\lambda \in \mathcal{R}$, $|\lambda| \leqslant 1$, then $\lambda U \subset U$,

(l_3) if $U \in \mathfrak{U}(\tau)$, then for every $x \in X$ there exists $\lambda \in \mathcal{R}$, $\lambda \neq 0$, such that $\lambda x \in U$,

(l_4) if $U \in \mathfrak{U}(\tau)$ and $V \in \mathfrak{U}(\tau)$, then there exists $W \in \mathfrak{U}(\tau)$ such that $W \subset U \cap V$,

(l_5) if $U \in \mathfrak{U}(\tau)$, then there exists $V \in \mathfrak{U}(\tau)$ such that $V+V \subset U$.

In the sequel we shall always suppose that any basis of neighbourhoods under consideration satisfies all the conditions (l_1)-(l_5).

If, conversely, $\mathfrak{U}$ is a family of subsets of X satisfying conditions (l_1)-(l_5), then $\mathfrak{U}$ is a basis of neighbourhoods of 0 for some linear topology. A linear topology τ is called a *linear Hausdorff topology* if and only if the basis $\mathfrak{U}(\tau)$ satisfies the condition

(l_6) for every $x \in X$, $x \neq 0$ there exists $U \in \mathfrak{U}(\tau)$ such that $x \notin U$.

A linear topology is called *locally convex* if and only if there exists a basis of convex neighbourhoods of 0. Let τ_1 and τ_2 be two linear topologies defined on X. If for every $U \in \mathfrak{U}(\tau_1)$ there exists a $V \in \mathfrak{U}(\tau_2)$ such that $V \subset U$, then we say that the topology τ_2 is finer than τ_1 (or that τ_1 is coarser than τ_2) and we write $\tau_1 \leqslant \tau_2$. If τ is a linear topology on X and $Z \subset X$, then for each element $a \in Z$ we can take the family of sets $(a+U) \cap Z$, where $U \in \mathfrak{U}(\tau)$, as the basis of neighbourhoods of a. The topology thus defined in Z is denoted by $\tau|Z$ and called *topology induced on Z by τ*.

If $\langle X, \tau_1\rangle$ and $\langle Y, \tau_2\rangle$ are (not necessarily linear) topological spaces, and if $y = u(x)$ is a continuous operation from X to Y, then we say that the operation u is (τ_1, τ_2)-continuous. If in particular $\langle X, \tau_1\rangle$ and $\langle Y, \tau_2\rangle$ are linear topological spaces, then instead of "u is distributive and (τ_1, τ_2)-continuous" we say "u is (τ_1, τ_2)-linear".

Let X be a linear space. We say that a set $A \subset X$ *absorbs* a set $B \subset X$ if there exists a $\lambda > 0$ such that $\lambda B \subset A$. A subset B of a linear topological space $\langle X, \tau\rangle$ is said to be *bounded* (or *τ-bounded*) if B is absorbed by every neighbourhood of 0. The class of all τ-bounded sets is denoted by $\mathrm{Bd}(\tau)$. A set B is τ-bounded if and only if for every sequence $\{x_n\}$ of elements of B the conditions $\lambda_n \geqslant 0$, $\lambda_n \to 0$ imply $\lambda_n x_n \to 0$ in the τ-topology.

Let $\langle X, \tau\rangle$ be a locally convex linear Hausdorff topological space. A set $A \subset X$ is called *symmetric* if $x \in A$ implies $-x \in A$. A set $A \subset X$ is called *absorbing* if, for every $x \in X$, there is a $\lambda > 0$ such that $\lambda x \in A$. A set A is called a *barrel* if A is convex, symmetric, absorbing and closed. The space $\langle X, \tau\rangle$ is called a *t-space* (*espace tonnelé*) if all barrels are neighbourhoods of 0 ([5], [6]). The space $\langle X, \tau\rangle$ is called *bornological* if any convex symmetric set in X which absorbs all bounded subsets of X is a neighbourhood of 0 (see [6]). It is known that

(*) *A locally convex linear Hausdorff topological space $\langle X, \tau\rangle$ is bornological if and only if, for every locally convex topology τ_1 defined on X, the condition $\mathrm{Bd}(\tau) = \mathrm{Bd}(\tau_1)$ implies $\tau_1 \leqslant \tau$.*

2.1. Suppose that in a linear space X two linear Hausdorff topologies τ and τ^* are defined. Let $\mathfrak{U}(\tau)$ and $\mathfrak{U}(\tau^*)$ be bases of neighbourhoods for 0 in topologies τ and τ^* respectively. Neighbourhoods in $\mathfrak{U}(\tau)$ will be denoted by $U, V, \ldots$, and neighbourhoods in $\mathfrak{U}(\tau^*)$ will be denoted by $U^*, V^*, \ldots$ In the sequel we shall sometimes postulate (but only when explicitly stated) the following conditions:

(n) $\tau^* \leqslant \tau$;

(o) the neighbourhoods belonging to $\mathfrak{U}(\tau)$ are τ-bounded;

(d) the neighbourhoods belonging to $\mathfrak{U}(\tau)$ are convex and τ^*-closed.

Condition (o) implies the metrisability of space $\langle X, \tau\rangle$. The space $\langle X, \tau\rangle$ is then a Fréchet space or an incomplete Fréchet space. If the topology τ is locally convex (and, in particular, if condition (d) is satisfied), then condition (o) is equivalent to the statement that the space $\langle X, \tau\rangle$ is a normed space (i. e. a B^*-space).

For each sequence $U_n^* \in \mathfrak{U}(\tau^*)$ and for each $U \in \mathfrak{U}(\tau)$, we shall denote by $\gamma(U_1^*, U_2^*, \ldots; U)$, or shortly by U^γ, the set

$$\bigcup_{n=1}^{\infty} (U_1^* \cap U + U_2^* \cap 2U + \ldots + U_n^* \cap nU), \tag{1}$$

i. e. the set of all sums $x_1+x_2+\ldots+x_n$ ($n = 1, 2, \ldots$), where $x_k \in U_k^*$ and $\frac{1}{k} x_k \in U$.

It is easy to verify that the family $\mathfrak{R}$ of all the sets (1) is a basis of neighbourhoods for 0 in a linear Hausdorff topology. In fact, if $\gamma(U_1^*, U_2^*, \ldots; U) \in \mathfrak{R}$ and $\lambda \in \mathcal{R}$, then $\lambda \cdot \gamma(U_1^*, U_2^*, \ldots; U) = \gamma(\lambda U_1^*, \lambda U_2^*, \ldots; \lambda U)$. Therefore conditions ($l_1$) and ($l_2$) are satisfied. For each $x \in X$ there exists a $\lambda \in \mathcal{R}$, $\lambda \neq 0$, such that $\lambda x \in U_1^*$ and $\lambda x \in U$. Then $\lambda x \in \gamma(U_1^*,$

$U_2^*, \ldots; U)$, which proves (l_3). If $W_k^* \subset U_k^* \cap V_k^*$ $(k = 1, 2, \ldots)$ and $W \subset U \cap V$, then $\gamma(W_1^*, W_2^*, \ldots; W) \subset \gamma(U_1^*, U_2^*, \ldots; U) \cap \gamma(V_1^*, V_2^*, \ldots; V)$, which proves (l_4). Condition (l_5) can be verified as follows: Let us choose $V_n^* \in \mathfrak{U}(\tau^*)$ and $V \in \mathfrak{U}(\tau)$ in such a way that $V_n^* + V_n^* \subset U_n^*$ and $V + V \subset U$. Then $V^\gamma + V^\gamma \subset U^\gamma$, where $V^\gamma = \gamma(V_1^*, V_2^*, \ldots; V)$. In fact, if $x \in V^\gamma + V^\gamma$, then $x = y + z$, where $y = y_1 + y_2 + \ldots + y_m$, $y_k \in V_k^*$, $\frac{1}{k} y_k \in V$ $(k = 1, 2, \ldots, m)$ and $z = z_1 + z_2 + \ldots + z_n$, $z_k \in V_k^*$, $\frac{1}{k} z_k \in V$ $(k = 1, 2, \ldots, n)$. Hence (if, for instance, $m \leqslant n$) $x = (y_1 + z_1) + (y_2 + z_2) + \ldots + (y_m + z_m) + z_{m+1} + \ldots + z_n$ and it follows from $y_k + z_k \in U_k^*$, $\frac{1}{k}(y_k + z_k) \in U$ $(k = 1, 2, \ldots, m)$, $z_k \in U_k^*$, $\frac{1}{k} z_k \in U$ $(k = m+1, m+2, \ldots, n)$ that $x \in U^\gamma$.

Hence the family $\mathfrak{R}$ is a basis of neighbourhoods for 0 in a new linear topology. We shall call this topology the *mixed topology* (²) determined by the topologies τ and τ^*.

We denote the mixed topology by $\gamma[\tau, \tau^*]$ or shortly by τ^γ. The mixed topology satisfies condition (l_0), i. e. it is a Hausdorff topology. This follows at once from the following statement:

2.1.1. *For each* $U^* \in \mathfrak{U}(\tau^*)$ *there exists a* $\gamma(U_1^*, U_2^*, \ldots; U) \in \mathfrak{R}$ *such that* $\gamma(U_1^*, U_2^*, \ldots; U) \subset U^*$.

In fact, for every $U^* \in \mathfrak{U}(\tau^*)$ there exists $U_1^* \in \mathfrak{U}(\tau^*)$ such that $U_1^* + U_1^* \subset U^*$. Furthermore, there exists $U_2^* \in \mathfrak{U}(\tau^*)$ such that $U_2^* + U_2^* \subset U_1^*$. By induction, there exists a $U_n^* \in \mathfrak{U}(\tau^*)$ such that $U_n^* + U_n^* \subset U_{n-1}^*$. We have $U_1^* + U_2^* + \ldots + U_n^* \subset U^*$ for each n, and therefore $\gamma(U_1^*, U_2^*, \ldots; U) \subset U^*$ for each U.

Lemma 2.1.1. may be written in the form

$$(2) \qquad \tau^* \leqslant \gamma[\tau, \tau^*].$$

If condition (n) is satisfied, then

$$(3) \qquad \gamma[\tau, \tau^*] \leqslant \tau.$$

In fact, for every U^γ of form (1) there exists a $V \in \mathfrak{U}(\tau)$ such that $V \subset U_1^* \cap U \subset U^\gamma$.

If $\tau^* \geqslant \tau$, then

$$(4) \qquad \gamma[\tau, \tau^*] = \tau^*.$$

In fact, for every U^γ there exists a $U^* \in \mathfrak{U}(\tau^*)$ such that $U^* \subset U_1^* \cap U \subset U^\gamma$. Hence $\tau^* \geqslant \gamma[\tau, \tau^*]$, which, together with (2), proves (4).

(²) The term "the space with mixed topology" was first used by A. Alexiewicz and Z. Semadeni in paper [3].

If the topologies τ and τ^* are locally convex, then the topology τ^γ is also locally convex. In fact, if U_n^* $(n = 1, 2, \ldots)$ and U are convex, then the sets $U_1^* \cap U + U_2^* \cap 2U + \ldots + U_n^* \cap nU$ are convex, as algebraic sums of convex sets. Therefore the set $\gamma(U_1^*, U_2^*, \ldots; U)$ is convex, as the set-theoretical union of an increasing sequence of convex sets.

Remark. We could take as a basis of neighbourhoods of 0 in the mixed topology the class of all sets of the form

$$(5) \qquad \bigcup_{n=1}^{\infty} (U_1^* \cap a_1 U + U_2^* \cap a_2 U + \ldots + U_n^* \cap a_n U),$$

where $\{a_n\}$ is an arbitrary fixed sequence of real numbers tending to infinity. In fact, there exists a subsequence $\{a_{m_n}\}$ such that $|a_{m_n}| \geqslant n$. Then

$$U_{m_n}^* \cap nU \subset U_{m_n}^* \cap a_{m_n} U$$

and

$$\gamma(U_{m_1}^*, U_{m_2}^*, \ldots; U) \subset \bigcup_{n=1}^{\infty} (U_1^* \cap a_1 U + U_2^* \cap a_2 U + \ldots + U_n^* \cap a_n U).$$

Conversely, if $\{k_n\}$ is an increasing sequence of positive integers such that $k_n \geqslant |a_n|$, then $U_{k_n}^* \cap a_n U \subset U_{k_n}^* \cap k_n U$, and

$$\bigcup_{n=1}^{\infty} (U_{k_1}^* \cap a_1 U + U_{k_2}^* \cap a_2 U + \ldots + U_{k_n}^* \cap a_n U) \subset \gamma(U_1^*, U_2^*, \ldots; U).$$

Therefore, the bases of neighbourhoods of the form (1) and (5) are equivalent.

2.2. Let τ and τ^* be two linear Hausdorff topologies defined on X. Let τ' be an arbitrary linear topology defined on X. We say that the topology τ' satisfies *condition* (P_1) (with respect to the pair (τ, τ^*)) if

$$(P_1) \qquad \tau' | Z = \tau^* | Z \quad \text{for each} \quad Z \in \mathrm{Bd}(\tau).$$

2.2.1. *The mixed topology* $\gamma[\tau, \tau^*]$ *satisfies the condition* (P_1).

2.2.2. *If the topology* τ *satisfies condition* (o), *then for every linear topology* τ' *defined on* X, *the condition*

$$\tau' | Z \leqslant \tau^* | Z \quad \textit{for each} \quad Z \in \mathrm{Bd}(\tau)$$

implies the inequality

$$\tau' \leqslant \gamma[\tau, \tau^*].$$

In particular, the mixed topology is the finest of all linear topologies which satisfy (P_1).

The proofs of lemmas 2.2.1. and 2.2.2. were given in my previous paper [12].

The topology τ' is said to satisfy *condition* (P_2) (with respect to the pair (τ, τ^*)) provided

(P_2) If $y = u(x)$ is a distributive operation defined on X with values belonging to another topological linear space $\langle Y, \tau_1\rangle$, and if for each $Z \in \mathrm{Bd}(\tau)$ the operation $u|Z$ is $(\tau^*|Z, \tau_1)$-continuous, then the operation u is (τ', τ_1)-linear.

2.2.3. *If the topology τ satisfies condition* (o), *then the topology $\gamma[\tau, \tau^*]$ satisfies the condition* (P_2).

2.2.4. COROLLARY. *If the topology τ satisfies condition* (o), *then a distributive operation $y = u(x)$ defined on X with values belonging to another topological linear space $\langle Y, \tau_1\rangle$ is (τ^γ, τ_1)-continuous if and only if the operation $u|Z$ is $(\tau^*|Z, \tau_1)$-continuous for each $Z \in \mathrm{Bd}(\tau)$.*

2.2.5. *If the topology τ satisfies condition* (o) *and a linear topology τ' satisfies condition* (P_2), *then $\tau' \geqslant \tau^\gamma$. In other words, the topology τ^γ is the coarsest of all the linear topologies which satisfy* (P_2).

The proofs of 2.2.3-2.2.5 were given in [12]. The following theorem is an immediate consequence of 2.2.2 and 2.2.5:

2.2.6. THEOREM. *If the topology τ satisfies condition* (o) *and if a linear topology τ' satisfies conditions* (P_1) *and* (P_2) *simultaneously, then $\tau' = \tau^\gamma$.*

Henceforth we shall assume that the topology τ is locally convex and that $\mathfrak{U}(\tau)$ is a basis of convex neighbourhoods of 0.

2.3. For each sequence $U_n^* \in \mathfrak{U}(\tau^*)$ $(n = 0, 1, 2, \ldots)$ and for each $U \in \mathfrak{U}(\tau)$ let us write

(6) $$U^{\gamma_1} = \gamma_1(U_0^*, U_1^*, \ldots; U) = U_0^* \cap \bigcap_{n=1}^{\infty} (nU + U_n^*).$$

We shall prove that the class of all the sets (6) is a basis of neighbourhoods for 0 in the mixed topology $\gamma[\tau, \tau^*]$. First we shall show, however, that if $\{a_n\}$ is an arbitrary sequence of positive numbers tending to infinity, then every set of the form

(7) $$V_0^* \cap \bigcap_{n=1}^{\infty} (a_n V + V_n^*), \quad \text{where} \quad V \in \mathfrak{U}(\tau),\ V_n^* \in \mathfrak{U}(\tau^*),$$

contains a set of form (6), and conversely. In fact, suppose that U^{γ_1} is an arbitrary set of form (6). Let $\{k_n\}$ be an increasing sequence of positive integers such that $k_n \geqslant a_n$ for $n = 1, 2, \ldots$ Let $V_n^* \in \mathfrak{U}(\tau^*)$ be such a sequence that $V_0^* \subset \bigcap_{p=0}^{k_1-1} U_p^*$, $V_n^* \subset \bigcap_{p=k_n}^{k_{n+1}-1} U_p^*$. Then

$$V_0^* \cap (a_n U + V_n^*) \subset U_0^* \cap \bigcap_{p=k_n}^{k_{n+1}-1} (pU + U_p^*)$$

and

$$V_0^* \cap \bigcap_{n=1}^{\infty} (a_n U + V_n^*) \subset U_0^* \cap \bigcap_{p=1}^{\infty} (pU + U_p^*).$$

It can be shown by a similar argument that, for every set of form (7), there are neighbourhoods $U^* \in \mathfrak{U}(\tau^*)$, $n = 0, 1, 2, \ldots$, such that set (7) contains set (6) for $U = V$.

Now we shall show that every set (7) is a neighbourhood of 0 in the mixed topology. On account of the preceding remark it suffices to show that every set

(8) $$U_0^* \cap \bigcap_{n=1}^{\infty} \left(\tfrac{1}{2}n(n+1)\, U + U_n^*\right)$$

is a neighbourhood of 0 in the mixed topology.

Let V_1^* be an arbitrary member of $\mathfrak{U}(\tau^*)$, satisfying the condition $V_1^* + V_1^* \subset U_0^*$. Let us take, by induction, $V_n^* \in \mathfrak{U}(\tau^*)$ $(n > 1)$ such that

$$V_n^* + V_n^* \subset U_{n-1}^* \cap V_{n-1}^*.$$

We have

(9) $$V_1^* + V_2^* + \ldots + V_n^* \subset U_0^*,$$

and for every p

(10) $$V_n^* + V_{n+1}^* + \ldots + V_{n+p}^* \subset V_n^* + V_n^* \subset U_{n-1}^*.$$

By (10), we obtain

$$\gamma(V_1^*, V_2^*, \ldots; U) = \bigcup_{p=1}^{\infty} \big(V_1^* \cap U + V_2^* \cap 2U + \ldots + V_{n-1}^* \cap (n-1)\,U +$$
$$+ V_n^* \cap nU + \ldots + V_{n+p}^* \cap (n+p)\,U\big) \subset \bigcup_{p=1}^{\infty} \big(U + 2U + \ldots + (n-1)\,U +$$
$$+ V_n^* + \ldots + V_{n+p}^*\big) \subset \tfrac{1}{2}n(n-1)\,U + U_{n-1}^*.$$

The last inclusion being valid for each $n > 1$, it follows from (9) that set (8) contains the set $\gamma(V_1^*, V_2^*, \ldots; U)$. Therefore set (8) is a neighbourhood of 0 in the mixed topology.

Now we shall show that every neighbourhood of 0 in the mixed topology contains a set of form (6). Let $U^\gamma = \gamma(U_1^*, U_2^*, \ldots; U)$ be a neighbourhood of 0 in the mixed topology. Let us write, for brevity, $m_n = 2n-1$ $(n = 1, 2, \ldots)$. There exists a sequence $V_0^*, V_1^*, \ldots$, such that $V_0^* + V_0^* \subset U_{m_1}^*$, $V_{p-1}^* + V_{p-1}^* \subset U_{m_p}^*$, $V_{p-1}^* \supset V_p^*$ $(p = 1, 2, \ldots)$. We shall prove that $V^{\gamma_1} = \gamma_1(V_0^*, V_1^*, \ldots; U) \subset U^\gamma$. Let $x \in V^{\gamma_1}$. Then $x \in V_0^*$ and, for each $n = 1, 2, \ldots$, there exists a decomposition $x = y_n + z_n$

where $y_n \in nU$, $z_n \in V_n^*$. Let $x_1 = y_1$ and $x_n = y_n - y_{n-1}$ for $n > 1$. We have, for every n, the following obvious identity:

(11) $$x_1 + x_2 + \ldots + x_n + z_n = y_1 + (y_2 - y_1) + \ldots + (y_n - y_{n-1}) + z_n = y_n + z_n = x.$$

Furthermore

$$z_{n-1} = x_n + z_n,$$

and therefore $x_n = z_{n-1} - z_n \in V_{n-1}^* + V_n^*$. On the other hand, $x_n = y_n - y_{n-1} \in nU + (n-1)U = (2n-1)U = m_n U$. Hence

$$x_n \in (V_{n-1}^* + V_n^*) \cap (2n-1)U.$$

It follows immediately from the definition of V_n^* that

$$V_{n-1}^* + V_n^* \subset V_{n-1}^* + V_{n-1}^* \subset U_{m_n}^*.$$

Hence

(12) $$x_n \in U_{m_n}^* \cap m_n U.$$

It follows from the equality $z_n = x - y_n$ that $z_n \in (k_0 + n)U$ where k_0 is a positive number such that $x \in k_0 U$. If $n_0 > k_0 - 1$, then $2n_0 + 1 = m_{n_0+1} > k_0 + n_0$ and $z_{n_0} \in (k_0 + n_0)U \subset m_{n_0+1}U$. On the other hand,

$$z_{n_0} \in V_{n_0}^* \subset V_{n_0}^* + V_{n_0}^* \subset U_{m_{n_0+1}}^*.$$

Therefore

(13) $$z_{n_0} \in U_{m_{n_0+1}}^* \cap m_{n_0+1} U.$$

It follows from (11), (12) and (13) that

$$x = x_1 + x_2 + \ldots + x_{n_0} + z_{n_0} \in U_{m_1}^* \cap m_1 U + \ldots + U_{m_{n_0}}^* \cap m_{n_0} U + U_{m_{n_0+1}}^* \cap m_{n_0+1} U \subset U^\gamma.$$

This proves that $V^{\gamma_1} \subset U^\gamma$.

Therefore all sets (6) (or all sets (7)) form a basis of neighbourhoods of 0 in the mixed topology.

If, in particular, τ is a normed topology defined by the norm $\|\,\|$ and $S_n = [x\colon \|x\| \leqslant n]$, then the sets

(14) $$U_0^* \cap \bigcap_{n=1}^{\infty} (rS_n + U_n^*), \quad U_n^* \in \mathfrak{U}(\tau^*), \quad r > 0,$$

compose a basis of neighbourhoods of 0 in the mixed topology.

Remark. If the topologies τ and τ^* are locally convex and if condition (o) is satisfied, then the class of all the sets

(15) $$\operatorname{conv} \bigcup_{n=1}^{\infty} (U_n^* \cap nU)$$

is also a basis of neighbourhoods of 0 in the mixed topology.

In fact, it is clear that sets (15) form a basis of neighbourhoods for a locally convex linear topology τ_1. The inequality $\tau_1 \geqslant \tau^\gamma$ is obvious. The inverse inequality follows from 2.2.2, because the topology τ_1 has the property (P_1).

2.3.1. Theorem. *Suppose that the topologies τ and τ^* satisfy condition* (d). *Then $x_n \to x_0$ in the mixed topology if and only if simultaneously*

a) *$x_n \to x_0$ in the τ^*-topology,*

b) *the sequence $\{x_n\}$ is bounded in the τ-topology.*

Proof(3). Let us suppose that $x_n \to x_0$ in the τ^*-topology and the sequence $\{x_n\}$ is τ-bounded. Let Z be set of all elements x_n, $n = 0, 1, \ldots$ We have

$$x_n \to x_0 \quad \text{in} \quad \tau^*|Z\text{-topology}.$$

The τ-boundedness of Z implies the equality $\tau^*|Z = \tau^\gamma|Z$, on account of the proposition 2.2.1. Hence $x_n \to x_0$ in $\tau^\gamma|Z$-topology, i. e. $x_n \to x_0$ in the mixed topology τ^γ.

Suppose now that $x_n \to x_0$ in the mixed topology. It follows from the inequality $\tau^* \leqslant \tau^\gamma$ that $x_n \to x_0$ in the τ^*-topology. It suffices to prove that $\{x_n\}$ is τ-bounded. Suppose the contrary. We may assume that $x_0 = 0$, i. e. that $x_n \to 0$ in the mixed topology. If the sequence $\{x_n\}$ is not τ-bounded, then there exists a neighbourhood $U \in \mathfrak{U}(\tau)$ and an increasing sequence of indices $\{k_n\}$ such that $x_{k_n} \notin nU$ for $n = 1, 2, \ldots$ It follows from condition (d) that all the sets $n\overline{U}$ are τ^*-closed. Hence, for each n, there exists $U_n^* \in \mathfrak{U}(\tau^*)$ such that $x_{k_n} \notin nU + U_n^*$. Therefore the set $U^{\gamma_1} = \bigcap_{n=1}^{\infty} (nU + U_n^*)$ contains no of the elements x_{k_n}. On the other hand, the set U^{γ_1} is a neighbourhood of 0 in the mixed topology. This contradicts the hypothesis that $\{x_n\}$ converges to 0 in the mixed topology. Therefore the sequence $\{x_n\}$ is τ-bounded.

2.3.2. Corollary. *Under the hypotheses of theorem 2.3.1, a sequence $\{x_n\}$ is a Cauchy sequence in the mixed topology if and only if simultaneously*

a) *$\{x_n\}$ is a Cauchy sequence in the τ^*-topology,*

b) *$\{x_n\}$ is τ-bounded.*

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(3) I have proved this theorem in paper [12]. The proof given here is simpler than that in [12].

2.4. We suppose in this section that the topologies τ and τ^* satisfy conditions (o) and (d).

2.4.1. *A set $Z \subset X$ is bounded in the mixed topology if and only if it is bounded in topologies τ and τ^* simultaneously. In symbols*

$$\mathrm{Bd}(\tau^\gamma) = \mathrm{Bd}(\tau) \cap \mathrm{Bd}(\tau^*).$$

Proof. Suppose that $A \in \mathrm{Bd}(\tau) \cap \mathrm{Bd}(\tau^*)$. If $x_n \in A$ and $\lambda_n \geqslant 0$, $\lambda_n \to 0$, then $\lambda_n x_n \to 0$ in both topologies τ and τ^*. Hence the sequence $\{\lambda_n x_n\}$ is τ-bounded. In virtue of theorem 2.3.1 we infer that $\lambda_n x_n \to 0$ in the mixed topology, and consequently $A \in \mathrm{Bd}(\tau^\gamma)$. Therefore $\mathrm{Bd}(\tau) \cap \mathrm{Bd}(\tau^*) \subset \mathrm{Bd}(\tau^\gamma)$. Inequality (2) implies the inclusion $\mathrm{Bd}(\tau^\gamma) \subset \mathrm{Bd}(\tau^*)$. Suppose that $A \in \mathrm{Bd}(\tau^\gamma)$. Let $\{x_n\}$ be a sequence of elements of the set A, and let $\lambda_n \geqslant 0$, $\lambda_n \to 0$. Since $\sqrt{\lambda_n} \to 0$, we have $\sqrt{\lambda_n} x_n \to 0$ in the mixed topology. In virtue of theorem 2.3.1 the sequence $\{\sqrt{\lambda_n} x_n\}$ is τ-bounded. Consequently $\sqrt{\lambda_n} \cdot \sqrt{\lambda_n} x_n = \lambda_n x_n \to 0$ in the τ-topology and therefore the set A is τ-bounded. Hence $\mathrm{Bd}(\tau^\gamma) \subset \mathrm{Bd}(\tau)$.

Corollary. *If the topologies τ and τ^* satisfy condition* (n), *then a set $A \subset X$ is bounded in the mixed topology if and only if it is τ-bounded.*

2.4.2. *If $\gamma[\tau, \tau^*] = \tau$, then $\tau^* \geqslant \tau$. If, in particular, $\tau^* \leqslant \tau$, then the equality $\gamma[\tau, \tau^*] = \tau$ implies $\tau = \tau^*$.*

Proof. Let V be any τ-bounded neighbourhood of 0 in the τ-topology. By hypothesis, V contains a neighbourhood $\gamma(U_1^*, U_2^*, \ldots; U)$. It follows from the τ-boundedness of the set V, that $V \subset n_0 U$ for some $n_0 \geqslant 1$. We shall show that $U_{n_0+1}^* \subset V$. Suppose this inclusion is false. Then there exists $x_0 \in U_{n_0+1}^*$, $x_0 \notin V$. It is clear that $\lambda x_0 \notin V$, $\lambda x_0 \in (n_0+1) U$ for a number λ, $0 < \lambda \leqslant 1$. We have $\lambda x_0 \in U_{n_0+1}^*$ (see condition (l_2)). Hence

$$\lambda x_0 \in U_{n_0+1}^* \cap (n_0+1) U \subset \gamma(U_1^*, U_2^*, \ldots; U) \subset V.$$

This contradicts the assumption $\lambda x_0 \notin V$. Therefore $U_{n_0+1}^* \subset V$, and consequently $\tau^* \geqslant \tau$.

2.4.3. *If $\tau_1^* | Z = \tau_2^* | Z$ for each $Z \in \mathrm{Bd}(\tau)$, then*

$$\gamma[\tau, \tau_1^*] = \gamma[\tau, \tau_2^*]. \tag{16}$$

In particular:

$$\gamma[\tau, \tau^*] = \gamma[\tau, \gamma[\tau, \tau^*]]. \tag{17}$$

Proof. By 2.2.1,

$$\gamma[\tau, \tau_2^*] | Z = \tau_2^* | Z = \tau_1^* | Z \quad \text{for} \quad Z \in \mathrm{Bd}(\tau).$$

Therefore the topology $\gamma[\tau, \tau_2^*]$ has property (P_1) (for $\tau^* = \tau_1^*$). Consequently $\gamma[\tau, \tau_1^*] \geqslant \gamma[\tau, \tau_2^*]$, by 2.2.2. Replacing τ_1^* by τ_2^* and conversely

we obtain the inverse inequality. Hence equality (16) is true. Setting in (16) $\tau_1^* = \tau^*$, $\tau_2^* = \gamma[\tau, \tau^*]$ we obtain equality (17).

It follows from 2.4.3 that, contrary to the case considered in 2.4.2, the equality $\gamma[\tau, \tau^*] = \tau^*$ does not imply the equality $\tau = \tau^*$. In fact, if $\tau_1^* \leqslant \tau$, $\tau_1^* \neq \tau$ and $\tau^* = \gamma[\tau, \tau_1^*]$, then $\tau^* \leqslant \tau$, $\tau^* \neq \tau$ and $\tau^* = \gamma[\tau, \tau^*]$.

2.4.4. *If condition* (n) *is satisfied and if $\langle X, \gamma[\tau, \tau^*]\rangle$ is a bornological space, then $\tau = \tau^*$.*

Proof. By 2.4.1 (corollary) we have $\mathrm{Bd}(\gamma[\tau, \tau^*]) = \mathrm{Bd}(\tau)$. Therefore $\gamma[\tau, \tau^*] \geqslant \tau$, by (*). Hence $\gamma[\tau, \tau^*] = \tau$, on account of inequality (3). This equality implies $\tau = \tau^*$, by 2.4.2.

2.4.5. *If condition* (n) *is satisfied and if $\langle X, \gamma[\tau, \tau^*]\rangle$ is a t-space, then $\tau = \tau^*$.*

Proof. Let $U \in \mathfrak{U}(\tau)$. By condition (d), the neighbourhood U is τ^*-closed. It follows from the inequality $\tau^* \leqslant \gamma[\tau, \tau^*]$ that U is closed in the mixed topology. Therefore the set U is a barrel in the space $\langle X, \gamma[\tau, \tau^*]\rangle$ and consequently U is a neighbourhood of 0 in the mixed topology. Hence $\gamma[\tau, \tau^*] \geqslant \tau$, and, by 2.4.2, $\tau = \tau^*$.

Theorems 2.4.4 and 2.4.5 show that, in non-trivial cases, spaces with mixed topology fail to be bornological or tonnelé.

2.5. Let X_0 be a linear subspace of the space X. We shall consider on the space X_0 the following two topologies:

a) the topology $\gamma[\tau, \tau^*] | X_0$, i. e. the topology induced on X_0 by the mixed topology $\gamma[\tau, \tau^*]$.

b) the topology $\gamma[\tau | X_0, \tau^* | X_0]$, i. e. the mixed topology constructed from the topologies induced on X_0 by τ and τ^*.

It is easy to verify that

$$\gamma[\tau, \tau^*] | X_0 \leqslant \gamma[\tau | X_0, \tau^* | X_0].$$

In fact, the sets

$$X_0 \cap U_0^* \cap \bigcap_{n=1}^{\infty} (U_n^* + nU)$$

are neighbourhoods of 0 in the topology $\gamma[\tau, \tau^*] | X_0$, and the sets

$$X_0 \cap U_0^* \cap \bigcap_{n=1}^{\infty} (U_n^* \cap X_0 + nU \cap X_0)$$

are neighbourhoods of 0 in the topology $\gamma[\tau | X_0, \tau^* | X_0]$. It is clear that the second set is contained in the first. The inverse inclusion is, in general, false.

Suppose now that the subspace X_0 and the topologies τ and τ^* satisfy the following condition:

(c_1) if $A \subset X$, $A \in \mathrm{Bd}(\tau)$, A is τ^*-closed and $A \cap X_0 = \emptyset$, then there exists $V^* \in \mathfrak{U}(\tau^*)$ such that $(A+V^*) \cap X_0 = \emptyset$.

Condition (c_1) is satisfied, in particular, if X_0 is τ^*-closed and if every τ-bounded τ^*-closed set is compact (= bicompact) in the τ^*-topology.

2.5.1. THEOREM. *Suppose that conditions* (d) *and* (o) *are satisfied. Then condition* (c_1) *implies the equality*

$$\gamma[\tau \mid X_0, \tau^* \mid X_0] = \gamma[\tau, \tau^*] \mid X_0. \tag{18}$$

Proof. Let $U_0^* \cap \bigcap_{n=1}^{\infty} (U_n^* \cap X_0 + nU \cap X_0)$ be a neighbourhood of 0 in the topology $\gamma[\tau \mid X_0, \tau^* \mid X_0]$. On account of conditions (d) and (o) we may assume that the sets nU are τ^*-closed and τ-bounded. Let W_n^* be a τ^*-open τ^*-neighbourhood of 0, such that $W_n^* + W_n^* \subset U_n^*$. The set $nU \cap X_0 + W_n^*$ is τ^*-open. Hence the set

$$A_n = nU \setminus (nU \cap X_0 + W_n^*)$$

is τ^*-closed. Since the set A_n is τ-bounded and $A_n \cap X_0 = \emptyset$, then, by condition (c_1), there exists $V_n^* \in \mathfrak{U}(\tau^*)$ such that $(A_n + V_n^*) \cap X_0 = \emptyset$. We may suppose that $V_n^* \subset W_n^*$ and therefore $W_n^* + V_n^* \subset U_n^*$.

We have

$$\begin{aligned} X_0 \cap (nU + V_n^*) &\subset X_0 \cap [(A_n + V_n^*) \cup (nU \cap X_0 + W_n^* + V_n^*)] \\ &\subset X_0 \cap [(A_n + V_n^*) \cup (nU \cap X_0 + U_n^*)] = X_0 \cap (nU \cap X_0 + U_n^*) \\ &= nU \cap X_0 + U_n^* \cap X_0. \end{aligned}$$

The last equality follows from the fact that the set X_0 is linear. Therefore

$$X_0 \cap U_0^* \cap \bigcap_{n=1}^{\infty} (nU + V_n^*) \subset X_0 \cap U_0^* \cap \bigcap_{n=1}^{\infty} (nU \cap X_0 + U_n^* \cap X_0),$$

which implies $\gamma[\tau, \tau^*] \mid X_0 \geqslant \gamma[\tau \mid X_0, \tau^* \mid X_0]$. The inverse inequality being always valid, we get equality (18).

2.6. Let X be a linear space with a homogeneous norm $\| \, \|$, and let $\| \, \|^*$ be another F-norm defined on the space X. A sequence $\{x_n\}$ of elements of the space X is said to be *γ-convergent to* x_0, in symbols $x_n \underset{\gamma}{\rightarrow} x_0$, if $\|x_n - x_0\|^* \rightarrow 0$ and $\sup_n \|x_n\| < \infty$. The γ-convergence is also called *two-norm convergence*. The space X with γ-convergence is denoted by $\langle X, \| \, \|, \| \, \|^* \rangle$ and called a *two-norm space*. The theory of two-norm spaces has been developed by Alexiewicz ([1], [2]). Two-norm convergence in some concrete spaces was examined earlier by G. Fichtenholz [7]. The following conditions are important in the theory of two-norm spaces:

(n_1) $\|x_n\| \rightarrow 0$ implies $\|x_n\|^* \rightarrow 0$,

(n_2) $\|x_n - x_0\|^* \rightarrow 0$ implies $\liminf_{n \to \infty} \|x_n\| \geqslant \|x_0\|$,

(n_3) if $\|x_n\| \leqslant K$, $\lim_{\substack{p \to \infty \\ q \to \infty}} \|x_p - x_q\|^* = 0$, then there exists $x_0 \in X$ such that $\|x_0\| \leqslant K$ and $\|x_n - x_0\|^* \rightarrow 0$.

Let τ be the linear topology defined by the norm $\| \, \|$, and let τ^* be the linear topology defined by the norm $\| \, \|^*$. Let $\mathfrak{U}(\tau)$ be the class of all solid spheres $S_r = [x : \|x\| \leqslant r]$. Condition (o) from 2.1. is obviously satisfied. Condition (n) from 2.1 is identical with (n_1), and condition (d) is identical with (n_2).

If $\langle Y, \tau_1 \rangle$ is a topological linear space and if $y = u(x)$ is a distributive operation defined on X with values in Y, then the operation u is said to be *(γ, τ_1)-linear* provided $x_n \underset{\gamma}{\rightarrow} x_0$ implies $u(x_n) \underset{\tau_1}{\rightarrow} u(x_0)$. In particular, a distributive functional $\xi(x)$ defined on X is said to be *γ-linear* provided $x_n \underset{\gamma}{\rightarrow} x_0$ implies $\xi(x_n) \rightarrow \xi(x_0)$.

The connection between the two-norm spaces and the spaces with mixed topology is stated by the following theorem:

2.6.1. THEOREM. A) *If* $\langle X, \| \, \|, \| \, \|^* \rangle$ *is a two-norm space satisfying condition* (n_2), *then the mixed topology* $\tau^\gamma = \gamma[\tau, \tau^*]$ *has the following properties*:

(i) $\tau^\gamma \mid S = \tau^* \mid S$, *where* $S = [x : \|x\| \leqslant 1]$;

(ii) $x_n \underset{\gamma}{\rightarrow} x_0$ *if and only if* $x_n \rightarrow x_0$ *in the* τ^γ*-topology*;

(iii) *for every linear topological space* $\langle Y, \tau_1 \rangle$, *and for every operation* $y = u(x)$ *from* X *to* Y, u *is* (γ, τ_1)*-linear if and only if it is* (τ^γ, τ_1)*-linear.*

B) *The mixed topology* $\gamma[\tau, \tau^*]$ *is a unique linear topology possessing properties* (i) *and* (iii).

Proof of A). Property (i) follows from 2.2.1. Property (ii) is an immediate consequence of theorem 2.3.1. A distributive operation u is (γ, τ_1)-linear if and only if the operation $u \mid S$ is $(\tau^* \mid S, \tau_1)$-continuous, or, which is the same, if and only if, for each $Z \in \mathrm{Bd}(\tau)$ the operation $u \mid Z$ is $(\tau^* \mid Z, \tau_1)$-continuous. Therefore property (iii) follows at once from 2.2.4.

Proof of B). Suppose that a topology τ' has properties (i) and (iii). It follows from (i) that the topology τ' satisfies condition (P_1) from 2.2. Hence $\tau' \leqslant \tau^\gamma$, by 2.2.2. If the operation u from X to Y is distributive and $u \mid S$ is $(\tau^* \mid S, \tau_1)$-continuous, then, by condition (iii), u is (τ', τ_1)-linear. Consequently, the topology τ' satisfies condition (P_2) from 2.2, and $\tau' \geqslant \tau^\gamma$ by 2.2.5. Consequently $\tau' = \tau^\gamma$.

There exists a close relation between two-norm spaces and Saks spaces. The theory of Saks spaces has been developed by W. Orlicz ([9], [10]). His definitions are as follows. Let $X_s = S$ be the unit solid sphere in a normed space $\langle X, \| \, \| \rangle$, and let $\| \, \|^*$ be another F-norm defined on X. If the set X_s with the metric $d(x_1, x_2) = \|x_1 - x_2\|^*$ $(x_1, x_2 \in X_s)$ is a complete metric space (i. e. if condition (n_3) is satisfied), then the space $\langle X_s, \tau^*|X_s\rangle$ is called a *Saks space*. Let $\langle Y, \tau_1\rangle$ be any topological linear space and let $y = u(x)$ be an operation defined on the space X_s with values in Y. The operation u is called *distributive* if the conditions $\lambda_1, \lambda_2 \in \mathcal{R}$, $x_1, x_2 \in X_s$, $\lambda_1 x_1 + \lambda_2 x_2 \in X_s$ imply $u(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 u(x_1) + \lambda_2 u(x_2)$. If the operation u is distributive and $(\tau^*|X_s, \tau_1)$-continuous, then u is called (X_s, Y)-*linear*.

As long as we deal with convergence and linear operations, it makes no difference whether we consider Saks spaces or two-norm spaces. In fact, if $x_n \to x_0$ in the Saks space $\langle X_s, \tau^*|X_s\rangle$, then $x_n \xrightarrow[\gamma]{} x_0$ in the two-norm space $\langle X, \|\;\|, \|\;\|^*\rangle$. Conversely, if $x_n \xrightarrow[\gamma]{} x_0$ in the space $\langle X, \|\;\|, \|\;\|^*\rangle$, then there exists $\lambda \in \mathcal{R}$, $\lambda \neq 0$ such that $\lambda x_n \in X_s$, $\lambda x_0 \in X_s$ and $\lambda x_n \to \lambda x_0$ in the space $\langle X_s, \tau^*|X_s\rangle$. If $y = u(x)$ is a (X_s, Y)-linear operation from X_s to $\langle Y, \tau_1\rangle$, then u may be extended in a unique manner to a (γ, τ_1)-linear operation defined on the whole space X. Conversely, if u is a (γ, τ_1)-linear operation from X to Y, then the operation $u|X_s$ is (X_s, Y)-linear.

The following theorem states the relation between condition (n_3) and the properties of mixed topology:

2.6.2. *Condition* (n_3) *is satisfied if and only if simultaneously*

a) *the topologies* τ *and* τ^* *satisfy condition* (d) *from* 2.1,

b) *the space* $\langle X, \tau^\gamma\rangle$ *is sequentially complete.*

This theorem easily follows from theorem 2.3.1 and corollary 2.3.2.

It follows from a theorem due to D. A. Raikov ([11], p. 223) that 2.6.2. may be strengthened as follows:

2.6.3. *If condition* (n_3) *is satisfied, then the space* $\langle X, \tau^\gamma\rangle$ *is complete, i. e. every Cauchy filter on* $\langle X, \tau^\gamma\rangle$ *converges to a point of the space.*

It follows from theorem 2.6.1 that a distributive operation from $\langle X, \tau^\gamma\rangle$ into another topological linear space is continuous if and only if it is sequentially continuous. Thus we see that the space $\langle X, \tau^\gamma\rangle$, although it fails to be bornological (see 2.4.4), has in this case the following important property of bornological spaces: the notions of continuity and sequential continuity of operations defined on this space coincide. A. Alexiewicz and Z. Semadeni have shown [3] that if τ^* is nonmetrizable, then the space $\langle X, \tau^\gamma\rangle$ does not always possess this property. The example due to Alexiewicz and Semadeni is as follows. Let X be the space of all bounded measurable functions $x = x(t)$ defined on $\langle 0, 1\rangle$. The τ-topology is defined by the norm $\|x\| = \sup_{0\leqslant t\leqslant 1} |x(t)|$, and the τ^*-topology is defined by the set of pseudonorms $\|x\|_t^* = |x(t)|$ $(0 \leqslant t \leqslant 1)$. The functional $\xi(x) = \int_0^1 x(t)\,dt$, defined on the space $\langle X, \tau^\gamma\rangle$, is sequentially continuous, but not continuous.

Alexiewicz and Semadeni [4] have constructed another linear topology generating γ-convergence. The procedure applied by Alexiewicz and Semadeni requires the hypothesis that the space $\langle X, \|\;\|^*\rangle$ is B_0^*-space. Their definition is as follows: Suppose that condition (n_1) is satisfied. Let Ξ be the conjugate space to $\langle X, \|\;\|\rangle$ with the usual norm $\|\xi\| = \sup_{\|x\|\leqslant 1} |\xi(x)|$. Let Ξ_γ be the space of all γ-linear functionals defined on $\langle X, \|\;\|, \|\;\|^*\rangle$. Since $\Xi_\gamma \subset \Xi$, the norm $\|\xi\|$ is defined for each $\xi \in \Xi_\gamma$. Let $U^* \in \mathfrak{U}(\tau^*)$, $\xi_n \in \Xi_\gamma$, $\|\xi_n\| \leqslant 1$, $0 < a_n \to \infty$. We write

$$V(U^*, \{\xi_n\}, \{a_n\}) = U^* \cap \bigcap_{n=1}^{\infty} [x: |\xi_n(x)| \leqslant a_n].$$

The sets $V(U^*, \{\xi_n\}, \{a_n\})$ constitute a basis of neighbourhoods for 0 in a locally convex linear topology on X. This is the topology μ of Alexiewicz and Semadeni. The topology μ has the following properties ([4], p. 127):

(α) $x_n \xrightarrow[\gamma]{} x_0$ if and only if $x_n \xrightarrow[\mu]{} x_0$,

(β) the γ-linear functionals are identical with the functionals linear with respect to the topology μ.

Following Alexiewicz and Semadeni a linear locally convex Hausdorff topology τ' is called *appropriate* if it satisfies conditions (α) and (β), i. e. if for sequences γ-convergence is equivalent to τ'-convergence, and if the class of γ-linear functionals is identical with the class of functionals linear in the topology τ'. Alexiewicz and Semadeni have shown that there exist different appropriate topologies for the space $\langle X, \|\;\|, \|\cdot\|^*\rangle$ ([4], p. 134). It follows from theorem 2.6.1 that, if we modify the definition of "appropriate" topology taking conditions (i) and (iii) in the place of (α) and (β), then the "appropriate" topology is determined uniquely.

Theorem 2.6.1 enables us to apply the mixed topology to the study of two-norm spaces. For example, the problem of extension of γ-linear functionals is closely connected with the problem of relativization of the mixed topology to a linear subspace. This is shown by the following theorem:

2.6.4. THEOREM. *Let* $\langle X, \|\;\|, \|\;\|^*\rangle$ *be a two-norm space such that* $\langle X, \|\;\|^*\rangle$ *is a* B_0^*-*space. Let* X_0 *be a linear subspace of the space* X *such that*

$$\gamma[\tau, \tau^*]|X_0 = \gamma[\tau|X_0, \tau^*|X_0].$$

Let ξ_0 *be some* γ-*linear functional on* X_0. *Then there exists a* γ-*linear extension* ξ *of* ξ_0 *on the whole space* X.

Theorem 2.6.4 is an immediate consequence of theorem 2.6.1 and of the well-known theorem on extension of linear functionals in locally convex spaces.

Alexiewicz and Semadeni [4] have shown that γ-linear functionals do not have, in general, the extension property. Hence, equality (18) is not true in general. However, we considered in 2.5 a case where equality (18) was true. From theorems 2.5.1 and 2.6.3 we obtain

2.6.5. THEOREM. *Let $\langle X, \|\ \|, \|\ \|^*\rangle$ be a two-norm space such that $\langle X, \|\ \|^*\rangle$ is a B_0^*-space and such that the following condition is satisfied:*

(c) *the sphere $S = [x: \|x\| \leqslant 1]$ is τ^*-compact.*

If a linear subspace $X_0 \subset X$ is τ^*-closed and if ξ_0 is a γ-linear functional on X_0, then there exists a γ-linear extension ξ of ξ_0 on the whole space X.

Another theorem on extension of γ-linear functionals for two-norm spaces which are simultaneously vector lattices, has been proved by Alexiewicz and Semadeni [3].

3. We shall now give some examples of spaces with mixed topology. These examples will be preceded by theorem 3.1.1 which enables us to establish the form of τ^γ-neighbourhoods in many concrete spaces.

3.1. Suppose that in a linear space X the topology τ is defined by a homogeneous norm $\|\ \|$, and the topology τ^* is defined by a set (uncountable, in general) of homogeneous pseudonorms $\|\ \|_\beta^*$ $(\beta \in B)$. Suppose, moreover, that the norm $\|\ \|$ and the pseudonorms $\|\ \|_\beta^*$ satisfy the condition

(19) $$\|x\| = \sup_{\beta \in B} \|x\|_\beta^* \quad \text{for each} \quad x \in X.$$

It is obvious that the topologies τ and τ^* satisfy conditions (o), (n) and (d) from 2.1.

We shall need in the next theorem the following property:

(r) If $\beta_n \in B$, $x \in X$ and $\varepsilon > 0$, then for every positive integer p there are elements y and z in X, such that $x = y+z$, $\|z\|_{\beta_i}^* = 0$ for $i = 1, 2, \ldots, p$, and $\|y\| \leqslant \max(\|x\|_{\beta_1}^*, \|x\|_{\beta_2}^*, \ldots, \|x\|_{\beta_p}^*) + \varepsilon$.

3.1.1. THEOREM. *Suppose that conditions* (19) *and* (c) (see theorem 2.6.5), *or* (19) *and* (r) *are satisfied. Then the sets*

(20) $$\bigcap_{i=1}^{\infty} [x: \|x\|_{\beta_i}^* \leqslant a_i],$$

where $\beta_i \in B$ and $0 < a_i \to \infty$, constitute the basis of neighbourhoods of 0 in the mixed topology $\gamma[\tau, \tau^*]$. The mixed topology is also determined by the pseudonorms

$$[x]_{(\beta_i),(a_i)} = \sup_i \frac{\|x\|_{\beta_i}^*}{a_i}, \quad \beta_i \in B,\ 0 < a_i \to \infty.$$

Proof. It is easy to verify that the class of all sets (20) is a basis of neighbourhoods for 0 in a locally convex linear topology τ_1. Let $Z \in \mathrm{Bd}(\tau)$, i. e.

$$Z \subset [x: \|x\| \leqslant r]$$

for some $r > 0$. Take any neighbourhood of an element $x_0 \in Z$ in the $\tau_1|Z$-topology. We can suppose that this neighbourhood is of the form

$$Z \cap \bigcap_{i=1}^{\infty} [x: \|x-x_0\|_{\beta_i}^* \leqslant a_i], \quad 0 < a_i \to \infty, \quad \beta_i \in B.$$

If $x, x_0 \in Z$, then, in virtue of (19), $\|x-x_0\|_{\beta_i}^* \leqslant \|x-x_0\| \leqslant 2r$. It follows from the condition $a_i \to \infty$ that there exists an integer i_0 such that $a_i > 2r$ for $i \geqslant i_0$. We have

$$Z \cap \bigcap_{i=1}^{i_0} [x: \|x-x_0\|_{\beta_i}^* \leqslant a_i] = Z \cap \bigcap_{i=1}^{\infty} [x: \|x-x_0\|_{\beta_i}^* \leqslant a_i].$$

Hence $\tau^*|Z \geqslant \tau_1|Z$, and consequently $\tau_1 \leqslant \gamma[\tau, \tau^*]$, by 2.2.2.

The proof of the inverse inequality requires the hypothesis that either condition (r) or condition (c) is satisfied.

Suppose first that condition (r) holds. Every neighbourhood of 0 in the mixed topology $\gamma[\tau, \tau^*]$ contains a set

(21) $$U_0^* \cap \bigcap_{n=1}^{\infty} (U_n^* + nU),$$

where $U = [x: \|x\| \leqslant r]$, $r > 0$, $U_n^* = [x: \max_{1 \leqslant i \leqslant k_n} \|x\|_{\beta_i}^* \leqslant \varepsilon_n]$, $\beta_i \in B$, $\varepsilon_n > 0$, $k_n < k_{n+1}$ for $n = 0, 1, \ldots$ Let $a_i = \min(\varepsilon_0, r/2)$ for $1 \leqslant i \leqslant k_0$ and $a_i = \frac{1}{2}nr$ for $k_{n-1} < i \leqslant k_n$. Let x be an arbitrary element of set (20). We have $\|x\|_{\beta_i}^* \leqslant a_i \leqslant \varepsilon_0$ for $1 \leqslant i \leqslant k_0$ and therefore $x \in U_0^*$. Let m be a positive integer. It follows from condition (r) (for $p = k_m$, $\varepsilon = \frac{1}{2}mr$) that there are elements $y \in X$ and $z \in X$ such that $y+z = x$, $\|y\| \leqslant \max_{1 \leqslant i \leqslant k_m} \|x\|_{\beta_i}^* + \frac{1}{2}mr$, $\|z\|_{\beta_i}^* = 0$ for $1 \leqslant i \leqslant k_m$. We have $z \in U_m^*$ and $\|y\| \leqslant \frac{1}{2}mr + \frac{1}{2}mr = mr$, i. e. $y \in mU$. Consequently $x \in mU + U_m^*$. The number m being arbitrary, we infer that x belongs to set (21). Therefore set (21) contains set (20) and $\tau_1 \geqslant \gamma[\tau, \tau^*]$, which, together with the preceding inequality, gives the equality $\tau_1 = \gamma[\tau, \tau^*]$.

Suppose now that condition (c) is satisfied. We shall use in this case arguments similar to those used in the proof of lemma 1 in [5] (p. 73). Let U^γ be an open neighbourhood of 0 in the mixed topology. It follows form the equality $\tau^*|S = \gamma[\tau, \tau^*]|S$ that there are $\beta_1, \beta_2, \ldots, \beta_{k_1} \in B$, such that

$$\bigcap_{i=1}^{k_1} [x: \|x\|_{\beta_i}^* \leqslant \varepsilon] \cap S \subset U^\gamma \cap S.$$

Suppose that there are indices $\beta_1, \beta_2, \ldots, \beta_{k_1}, \beta_{k_1+1}, \ldots, \beta_{k_n}$ such that

$$(22) \qquad \bigcap_{p=1}^{n} \bigcap_{i=k_{p-1}+1}^{k_p} [x: \|x\|_{\beta_i}^* \leqslant a_i] \cap nS \subset U^\gamma \cap nS,$$

where $a_i = \varepsilon$ for $1 \leqslant i \leqslant k_1$, $a_i = p-1$ for $k_{p-1} < i \leqslant k_p$ $(p = 2, 3, \ldots)$, $k_0 = 0$. We shall prove that there are $\beta_{k_n+1}, \ldots, \beta_{k_{n+1}}$ such that, setting $a_i = n$ for $k_n < i \leqslant k_{n+1}$, we shall have

$$\bigcap_{p=1}^{n+1} \bigcap_{i=k_{p-1}+1}^{k_p} [x: \|x\|_{\beta_i}^* \leqslant a_i] \cap (n+1)S \subset U^\gamma \cap (n+1)S.$$

Suppose this be false. Then the set

$$(23) \qquad C_{\gamma_1, \gamma_2, \ldots, \gamma_l} = \bigcap_{p=1}^{n} \bigcap_{i=k_{p-1}+1}^{k_p} [x: \|x\|_{\beta_i}^* \leqslant a_i] \cap \bigcap_{j=1}^{l} [x: \|x\|_{\gamma_j}^* \leqslant n]$$

has, for each finite sequence of indices $\gamma_1, \gamma_2, \ldots, \gamma_l \in B$, a non-void intersection with the set $(n+1)S \setminus U^\gamma$. But $\tau^* | (n+1)S = \gamma[\tau, \tau^*] | (n+1)S$ and the set U^γ is $\gamma[\tau, \tau^*]$-open. Therefore, by condition (c), the set $(n+1)S \setminus U^\gamma$ is τ^*-compact. Sets (23) are τ^*-closed, and it follows from the equality

$$C_{\gamma_1, \gamma_2, \ldots, \gamma_l} \cap C_{\gamma_1', \gamma_2', \ldots, \gamma_{l'}'} = C_{\gamma_1, \gamma_2, \ldots, \gamma_l, \gamma_1', \gamma_2', \ldots, \gamma_{l'}'}$$

that the intersection of the members of each finite family of sets (23) has common elements with the τ^*-compact set $(n+1)S \setminus U^\gamma$. Hence there exists an element

$$(24) \qquad x_0 \in [(n+1)S \setminus U^\gamma] \cap \bigcap_{\gamma_1, \ldots, \gamma_l} C_{\gamma_1, \ldots, \gamma_l}.$$

We have $\|x_0\|_\gamma^* \leqslant n$ for each $\gamma \in B$, and, by condition (19), $\|x_0\| \leqslant n$, i. e. $x_0 \in nS$. Hence, by (22),

$$x_0 \in U^\gamma \cap nS \subset U^\gamma \cap (n+1)S,$$

in contradiction to (24). Consequently, there exist a sequence of indices $\beta_i \in B$ $(i = 1, 2, \ldots)$, an increasing sequence of positive integers k_n $(n = 1, 2, \ldots)$ and a sequence $0 < a_i \to \infty$ $(i = 1, 2, \ldots)$ such that the inclusion (22) is true for each n. We shall have

$$\bigcap_{i=1}^{\infty} [x: \|x\|_{\beta_i}^* \leqslant a_i] \cap nS \subset U^\gamma,$$

and, since n is arbitrary,

$$\bigcap_{i=1}^{\infty} [x: \|x\|_{\beta_i}^* \leqslant a_i] \subset U^\gamma.$$

Consequently $\tau_1 \geqslant \gamma[\tau, \tau^*]$. This completes the proof of theorem 3.1.1.

We now give some examples of spaces with mixed topology satisfying the conditions of theorem 3.1.1.

A) Let X be the space $\boldsymbol{m}$ of bounded sequences $x = \{t_i\}$ of real numbers. The topology τ is defined by the norm $\|x\| = \sup_i |t_i|$, and the topology τ^* is defined by the pseudonorms $\|x\|_i^* = |t_i|$ $(i = 1, 2, \ldots)$. It is obvious that conditions (19), (r) and (c) are satisfied. Consequently, the sets $\bigcap_{i=1}^{\infty} [x: |t_i| \leqslant a_i]$, where $0 < a_i \to \infty$, constitute, by theorem 3.1.1, a basis of neighbourhoods for 0 in the mixed topology.

B) Let X be the space $\boldsymbol{l}$ of sequences $x = \{t_i\}$ of real numbers such that $\sum_{i=1}^{\infty} |t_i| < \infty$. The topology τ is defined by the norm $\|x\| = \sum_{i=1}^{\infty} |t_i|$, and the topology τ^* is defined by the pseudonorms $\|x\|_i^* = |t_i|$ $(i = 1, 2, \ldots)$. The pseudonorms $\| \|_i^*$ do not satisfy condition (19). We easily observe, however, that the pseudonorms $\| \|_i^*$ are equivalent to the pseudonorms $[x]_i = \sum_{k=1}^{i} |t_k|$ $(i = 1, 2, \ldots)$, and the pseudonorms $[\ ]_i$ satisfy condition (19). Conditions (c) and (r) are also satisfied and we conclude, by theorem 3.1.1, that the sets $\bigcap_{i=1}^{\infty} [x: \sum_{k=1}^{i} |t_k| \leqslant a_i]$, where $0 < a_i \to \infty$, constitute a basis of neighbourhoods for 0 in the mixed topology.

C) Let T be an abstract set, and let X be the space of all bounded, real-valued functions $x = x(t)$ defined on T. The topology τ is defined by the norm $\|x\| = \sup_{t \in T} |x(t)|$, and the topology τ^* is defined by the pseudonorms $\|x\|_t^* = |x(t)|$ $(t \in T)$. Conditions (19), (r) and (c) are satisfied. The sets $\bigcap_{i=1}^{\infty} [x: |x(t_i)| \leqslant a_i]$, where $t_i \in T$, $0 < a_i \to \infty$, constitute a basis of neighbourhoods for 0 in the mixed topology.

D) Let T be a completely regular Hausdorff space. Let X be the space $C^*(T)$ of bounded, real-valued, continuous functions $x = x(t)$ on T. Let $\{T_\beta\}_{\beta \in B}$ be a family of (non-necessarily all) compact subsets of T such that $\bigcup_{\beta \in B} T_\beta = T$. The topology τ is defined by the norm $\|x\| = \sup_{t \in T} |x(t)|$, and the topology τ^* is defined by the pseudonorms $\|x\|_\beta^* = \sup_{t \in T_\beta} |x(t)|$. Condition (19) is obviously satisfied. We shall prove that condition (r) is also satisfied. Let $\beta_n \in B$, $x \in X$, $\varepsilon > 0$, and let p be a positive integer. It is obvious that there exists an open set $G_p \subset T$ such that $\bigcup_{i=1}^{p} T_{\beta_i} \subset G_p$, and

$$(25) \qquad \sup_{t \in G_p} |x(t)| \leqslant \sup_{t \in \bigcup_{i=1}^{p} T_{\beta_i}} |x(t)| + \varepsilon = \max(\|x\|_{\beta_1}^*, \ldots, \|x\|_{\beta_p}^*) + \varepsilon.$$

The set $T \setminus G_p$ is closed and disjoint with the compact set $F_p = \bigcup_{i=1}^{p} T_{\beta_i}$.

The space T being completely regular, there exists a bounded, real-valued, continuous function $f(t)$ on X such that $0 \leqslant f(t) \leqslant 1$ for each $t \in T$, $f(t) = 0$ for $t \in F_p$, $f(t) = 1$ for $t \in T \setminus G_p$. Let $y(t) = [1-f(t)]\cdot x(t)$ and $z(t) = f(t)\cdot x(t)$. We have $y \in X$, $z \in X$ and $x = y+z$. Furthermore, $\|y\|_\beta^* \leqslant \|x\|_\beta^*$ and $\|z\|_\beta^* \leqslant \|x\|_\beta^*$ for each $\beta \in B$. Since $z(t) = 0$ for $t \in F_p$, we have $\|z\|_{\beta_i}^* = 0$ for $i = 1, 2, \ldots, p$. Since $y(t) = 0$ for $t \in T \setminus G_p$, we have in view of formula (25) $\|y\| = \sup_{t \in T} |y(t)| = \sup_{t \in G_p} |y(t)| \leqslant \sup_{t \in G_p} |x(t)| \leqslant \max(\|x\|_{\beta_1}^*, \ldots, \|x\|_{\beta_p}^*) + \varepsilon$. Consequently, the functions y and z have required properties, and condition (r) has been proved.

Applying theorem 3.1.1 we see that the sets

$$\bigcap_{i=1}^{\infty} [x: \sup_{t \in T_{\beta_i}} |x(t)| \leqslant a_i],$$

where $\beta_i \in B$ and $0 < a_i \to \infty$, constitute a basis of neighbourhoods for 0 in the mixed topology τ^γ. In this case the mixed topology is identical with the topology introduced by J. Mařík [8].

E) Let X be the space conjugate to a normed space Z. Let τ be the strong topology on X, defined by the usual norm $\| \ \|$ of elements of X as functionals, and let τ^* be the weak topology $\sigma(X, Z)$. The topology τ^* may be defined by the pseudonorms $\|x\|_z^* = |x(z)|$, where $z \in Z$, $\|z\| \leqslant 1$. The pseudonorms $\| \ \|_z^*$ satisfy condition (19). It is well known that condition (c) is also satisfied. By theorem 3.1.1 the sets $\bigcap_{i=1}^{\infty} [x: |x(z_i)| \leqslant a_i]$, where $z_i \in Z$, $\|z_i\| \leqslant 1$, $0 < a_i \to \infty$, constitute a basis of neighbourhoods for 0 in the topology τ^γ. We can also say that the sets

$$[x: \sup_i |x(z_i)| \leqslant 1], \tag{26}$$

where $z_i \in Z$, $\|z_i\| \to 0$, constitute a basis of neighbourhoods for 0 in the mixed topology. Consequently, the mixed topology is identical in this case with the topology τ_c of uniform convergence of functionals on the compact subsets of Z ([5], p. 74). In fact, the inequality $\tau_c \geqslant \gamma[\tau, \tau^*]$ follows at once from (26). On the other hand, the topology τ_c has property (P_1) from 2.2, and therefore $\gamma[\tau, \tau^*] \geqslant \tau_c$, by 2.2.2.

3.2. We now give two other examples of spaces with mixed topology. The spaces mentioned in F) and G) do not satisfy the conditions of theorem 3.1.1.

F) Let X be the space $\boldsymbol{M}$ of measurable, real-valued functions $x(t)$ equivalent to bounded functions on $\langle 0, 1\rangle$. The topology τ is defined by the norm $\|x\| = \operatorname{sup\,ess}_{0 \leqslant t \leqslant 1} |x(t)|$, and the topology τ^* is defined by the norm $\|x\|^* = \int_0^1 |x(t)|\,dt$. Conditions (o), (n) and (d) from 2.1 are satisfied.

For each function $x \in X$ and for each $p \geqslant 0$ we write

$$x_{(p)}(t) = \begin{cases} x(t) - p & \text{if} \quad x(t) \geqslant p, \\ 0 & \text{if} \quad -p \leqslant x(t) \leqslant p, \\ x(t) + p & \text{if} \quad x(t) \leqslant -p. \end{cases}$$

The homothetic images (with centre 0) of sets

$$\bigcap_{n=0}^{\infty} \Big[x: \int_0^1 |x_{(n)}(t)|\,dt \leqslant \varepsilon_n\Big], \tag{27}$$

where $\{\varepsilon_n\}$ are arbitrary sequences of positive numbers, constitute a basis of neighbourhoods for 0 in the mixed topology. In fact, set (27) is identical with the set $U_0^* \cap \bigcap_{n=1}^{\infty} (U_n^* + nU)$, where $U_n^* = [x: \int_0^1 |x(t)|\,dt \leqslant \varepsilon_n]$ and $U = [x: \|x\| \leqslant 1]$.

G) Let X be the space $\boldsymbol{L}$ of integrable functions on $\langle 0, 1\rangle$. The topology τ is defined by the norm $\|x\| = \int_0^1 |x(t)|\,dt$ and the topology τ^* is defined by the norm

$$\|x\|^* = \int_0^1 \frac{|x(t)|}{1+|x(t)|}\,dt.$$

Conditions (o), (n) and (d) from 2.1 are satisfied. In this case the mixed topology τ^γ is not locally convex. Alexiewicz ([2], p. 54) has shown that there are no non-trivial linear functionals on the space $\langle X, \tau^\gamma\rangle$.

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Extinguishing a class of functions

by

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Let E be a set of real positive numbers. By $L(E)$ we shall denote the family of all intervals of the form

$$I = \{(x, y): ax+y = t,\ x \geqslant 0,\ y \geqslant 0\},$$

where $a \in E$ and $0 < t < \infty$. A complex-valued continuous function φ of two variables defined on the first quadrant is said to be *extinguished* by the set E if $\int_I \varphi(x, y)\,ds = 0$ for any interval $I \in L(E)$. It is well known ([2], p. 63) that

(*) *The unique function extinguished by the right half-line is the function identically equal to* 0.

Let $\mathscr{A}_n$ denote the class of all complex-valued functions φ of two variables defined on the first quadrant and having the representation

$$\varphi(x, y) = \sum_{j=1}^{n} f_j(x) g_j(y),$$

where all the functions $f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_n$ are continuous on the right half-line. By $\mathfrak{E}_n$ we shall denote the class of all sets E of positive numbers such that the unique function belonging to $\mathscr{A}_n$ and extinguished by E is the function identically equal to 0. From Titchmarsh's Theorem on convolution ([3], p. 327) it follows that all one-point sets belong to $\mathfrak{E}_1$. Indeed, if a function φ is extinguished by a set $\{a\}$ and $\varphi(x, y) = f(x)g(y)$, then we have the equality

$$\int_{ax+y=t} f(x)g(y)\,ds = 0 \qquad (t > 0).$$

Hence for any positive t we get the equality

$$\int_0^t f(x)g\big(a(t-x)\big)\,dx = 0,$$