Some classes of Banach spaces depending on a parameter

by

J. MUSIELAK and Z. SEMADENI (Poznań)

In this paper we shall consider the following classes of Banach spaces:

$H_p$ — functions satisfying Hölder condition with an exponent $p$,

$C^v_p$ — continuous functions with finite $p$-th variation,

$L_p$ — absolutely continuous functions of order $p$, $S_p$ and $B_p$ — almost periodic functions in the sense of Stepanoff and Besicovitch, respectively,

$M_p$ — strongly $p$-summable sequences.

These classes may be treated as families of Banach spaces $X_p$ depending on a parameter $p$. In each of these classes there are known inclusions between spaces $X_p$, $X_{p'}$ and inequalities between norms $\|\cdot\|_p$, $\|\cdot\|_{p'}$ for $p < p'$. We shall consider the following problem: given a sequence $p_n$ convergent to $p_0$, establish connections between the corresponding spaces $X_{p_n}$ and $X_{p_0}$. This problem is closely related to the problem of the continuity (suitably defined) of the spaces $X_p$ with respect to the parameter $p$.

These problems are considered from a general point of view in paper [8], where, in the following definition, the limit $\overline{\langle X_n \rangle}$ of a sequence $X_n$ of linear metric spaces is introduced. $\langle X_n, \|\cdot\|_n \rangle$ is termed $\overline{\langle X_n, \|\cdot\|_n \rangle}$ (written $\langle X_n, \|\cdot\|_n \rangle = \overline{\langle X_n, \|\cdot\|_n \rangle}$) if the following conditions are satisfied:

1° $X_n$ and almost all $X_n$ are subspaces of a linear space,

2° $X_n$ converges to $X_0$ in the sense of the theory of sets (i.e. $X_0 = \cap_{n=1}^{\infty} X_n = \cap_{n=1}^{\infty} \cap_{k=1}^{\infty} X_n$, $X_0 = \cap_{n=1}^{\infty} \cap_{k=1}^{\infty} X_n$),

3° $\|x\|_n = \lim_{n \to \infty} \|x\|_n$ for all $x \in X_n$.

Next, we write $\langle X_n, \|\cdot\|_n \rangle = \overline{\langle X_n, \|\cdot\|_n \rangle}$ if $\overline{\langle X_n \rangle}$ is dense in $\langle X_n, \|\cdot\|_n \rangle$.

Let $\{X_p\}_{p \in \mathbb{P}}$ be a family of Banach spaces $\langle X_p, \|\cdot\|_p \rangle$ such that
$X_p \subset X_{p'}$, and $[x]_p \gg [x]_{p'}$ for $p > p'$. We say that this family is continuous with respect to $p$, if the following conditions are satisfied (1):

1° If $x \in X_{p_1}$ and if $p_n \searrow p$, then $[x]_p = \lim_{n \to \infty} [x]_{p_n}$,

2° if $x \in \bigcap_{p_1} X_{p}$ and if $p_n \searrow p$, then $[x]_p = \lim_{n \to \infty} [x]_{p_n}$,

3° $\bigcap_{p_1} X_{p}$ is dense in $\langle X_p, \| \cdot \|_p \rangle$,

4° $X_p$ is dense in $\bigcap_{p_1} X_{p}$ with respect to the $F$-norm

$$\| x \|_p^* = \sum_{n=1}^{\infty} 2^{-\kappa} \| x \|_{p_n}$$

where $\kappa(u) = u(1+u)^{-1}$ and $p_n$ is a fixed sequence such that $a < p_1 < p_2 < \cdots$ and $p_a \searrow p$,

5° $[x]_p = \sup_{n \to \infty} [x]_{p_n} < \infty$ implies $X_p$ for all $p$ ($a < p < \beta$).

Moreover, we consider two definitions of semicontinuity. If conditions 1°, 2°, 3° are satisfied, then the family $\langle X_p, \| \cdot \|_p \rangle$ is said to be semicontinuous from above; if 1°, 2°, 3° are satisfied, $\langle X_p, \| \cdot \|_p \rangle$ is said to be semicontinuous from below.

Conditions 1° and 3° mean that $\langle X_p, \| \cdot \|_p \rangle = \langle X_{p_1}, \| \cdot \|_{p_1} \rangle$. At the same time, the space $\langle \bigcap_{p_1} X_{p} \rangle$ is a $B_r$-space and conditions 2° and 4° mean that

$$\langle \bigcap_{p_1} X_{p} \rangle = \langle X_{p_1}, \| \cdot \|_{p_1} \rangle$$

where $\| x \|_{p_1} = \sum_{n=1}^{\infty} 2^{-\kappa(1)} \| x \|_{p_n}$,

$$\langle X_p, \| \cdot \|_p \rangle = \langle X_{p_1}, \| \cdot \|_{p_1} \rangle \cap \langle \bigcap_{p_1} X_{p} \rangle$$

Auxiliaries. The following well-known lemmas are very useful in further considerations.

0.1. Given sets $X$ and $T$, let us suppose that to every $p \in (a, \beta)$ there corresponds a family $\{f_{p_1}(\cdot)\}_{p_1}$ of functions defined in $X$, such that

$$f_{p_1}(x) \leq f_{p_2}(x)$$

for all $x \in X$, $p \leq p'$ and such that

$$p_n \to p \implies f_{p_n}(x) \to f_p(x)$$

(1) if the inclusions for $X_p$ and the inequalities for $\| \cdot \|_p$ are opposite, the definition is analogous.

0.2. Let $T$ be a compact topological space and $f_{p_1}(x)$ are continuous on $T$ for every fixed $p \in (a, \beta)$ and $x \in X$. Then, assuming (1) and (2), the function $\varphi(p)$ defined by (3) is continuous.

Proof. We have to prove that for each $p_n \to p$ the limit $\varphi(p_n)$ exists for every fixed $x \in X$. Since $f_{p_1}(x) \to f_p(x)$ for $p_n \to p$ uniformly on $T$ and $f_{p_1}(x)$ are continuous on $T$ for every fixed $x \in X$, the limit $\varphi(p_n)$ exists for every fixed $x \in X$, and $\varphi(p) = \sup_{x \in X} f_p(x)$.

0.3. Let $X$ be a linear class of bounded functions $u(x)$ defined on an arbitrary set $T$, containing constant functions and such that if $x \in X$ and $p > 0$, then $[x]_p \in X$. Next, let $\widetilde{X}(x)$ be a functional over $X$, satisfying the following conditions:

$$\widetilde{X}(x + y) = \widetilde{X}(x) + \widetilde{X}(y), \quad \widetilde{X}(-x) = -\widetilde{X}(x) \quad \text{for} \lambda \geq 0,$$

$$0 \leq x(t) \leq y(t) \implies \widetilde{X}(x) \leq \widetilde{X}(y),$$

$$\widetilde{X}(1) = 1, \quad \text{where 1 denotes the constant function } x(t) = 1.$$

Then, for any fixed $x \in X$, the functions

$$\varphi_1(p) = \widetilde{X}([x]_p) \quad \text{and} \quad \varphi_2(p) = \left[ \varphi_1(p) \right]^\alpha$$

are continuous for $p > 0$. 

![Image]
Proof. It suffices to show that \( \varphi(t) \) is continuous whenever \( |x(t)| \leq 1 \) and \( p \) runs any interval \((a, \beta)\) with \( \beta > a > 0 \). Let \( u \) be chosen such that \( x(u) = \frac{1}{3} \) and \( p = q \). Then \( |x(u)|^p - |x(u)|^q < \varepsilon \) for all \( t \in T \), whence \( 0 < M(|x|^p) - M(|x|^q) \leq M(|x|^p - |x|^q) \leq M(x) = \varepsilon \) for \( a < p < q < p + \delta < \beta \).

1. Spaces of functions satisfying Hölder conditions.

1.1. Let \( H_\rho \) be the class of all real functions \( x(t) \) defined in \((0, 1)\) vanishing at \( t = 0 \) and satisfying the Hölder condition with the exponent \( \rho \), i.e., the condition

\[
|a(t+h) - a(t)| \leq K|h|^\rho
\]

for all \( t, t + h \in (0, 1) \), where \( K \) is a constant depending on \( x \), and \( 0 < \rho \leq 1 \).

Next, let \( H_\rho^p \) be the subclass of \( H_\rho \) consisting of all functions \( x(t) \) satisfying the condition \( a(t+h) - a(t) = o(h^\rho) \), i.e., such that

\[
\lim_{h \to 0} h^{-\rho} x(h) = 0 \quad \text{where} \quad x(h) = \sup_{t \in (t+h) - (t)} |a(t+h) - a(t)|.
\]

The following inclusions are well-known:

\[
H_\rho \subset H_\rho^p \subset H_\rho \quad \text{for} \quad p < q.
\]

\( H_\rho \) is a non-separable Banach space with respect to the norm

\[
\|a\|_H^p = \sup_{t \in (t+h) - (t)} \sup_{o \in [c, d]} H \|a(t+h) - a(t)\| - \sup_{t \in (t+h) - (t)} \sup_{o \in [c, d]} H \|a(t+h) - a(t)\|.
\]

and \( H_\rho^p \) is closed in \( H_\rho \) in \( |||H^p||| \).

1.2. The set of all polynomial functions vanishing at 0 is dense in every space \( <H_\rho, |||H^p|||> \) where \( p < 1 \).

Proof. The map (4) generates in the space \( L_\rho \) the norms

\[
|y|_p = \sup_{t \in (t+h) - (t)} \sup_{o \in [c, d]} H \|U(y(t+h) - y(t))\| = \sup_{t \in (t+h) - (t)} \sup_{o \in [c, d]} H \|U(y(t+h) - y(t))\|
\]

for \( 0 < p < 1 \). We have to prove that the step functions are dense in \( L_\rho \) in the norm

\[
\|a\|_H^p = \sup_{t \in (t+h) - (t)} \sup_{o \in [c, d]} H \|a(t+h) - a(t)\| - \sup_{t \in (t+h) - (t)} \sup_{o \in [c, d]} H \|a(t+h) - a(t)\|.
\]

holds for every \( y \in L_\rho \) and \( 0 < \delta < 1 \). Let us consider the characteristic function \( y(t) \) of a measurable subset of \((0, 1)\). Choose \( \varepsilon \) with \( 0 < \varepsilon < 1 \) and an integer \( n > 2/\delta \) where \( \delta = \varepsilon^{1-n} \). Write \( J_n = (t-3/3n, t/n) \) and

\[
z(t) = \begin{cases} 1/h & \text{for} \quad t \in J_n \quad \text{and} \quad h = 1, 2, \ldots, n. \end{cases}
\]

Then \( |z - y|_p \leq \varepsilon \sup |z(t) - y(t)| \leq 1 \). Since

\[
\int_{J_n} |z(t) - y(t)|_p \, dt = 0,
\]

we have

\[
\int_{J_n} |z(t) - y(t)|_p \, dt = \int_{J_n} \frac{1}{h} |z(t) - y(t)|_p \, dt = \left( \frac{1}{h} \right)^{-p} \left( \frac{1}{h} \right)^{1-p} \int_{J_n} |z(t) - y(t)|_p \, dt = \varepsilon^{1-p} \leq \frac{\varepsilon}{n} \leq \delta,
\]

where \( k_1 = E(n) + 1 \), \( k_2 = E(n(t+h)) \). Then

\[
\sup_{t \in (t+h) - (t)} \sup_{o \in [c, d]} H \|U(z(t) - y(t))\| \leq \delta^{1-p}.
\]

Finally, by (5) and (6), \( |z - y|_p \leq \delta^{1-p} = \varepsilon \).
Since linear combinations of characteristic functions are dense in \( \langle \mathcal{L}^1 \rangle \), and \( \| y \|_{p'} \geq \| y \|_p \), the step-functions are dense in \( \langle \mathcal{L}^1 \rangle \) for every \( 0 < p' < 1 \).

1.3. The set \( H_p \) is dense in \( \langle H^p, \| \cdot \|_p \rangle \) for every \( 0 < p < 1 \).

Proof. Given \( \varphi \in H_p \), let us write

\[
\varphi_n(t) = n \left[ \begin{array}{l}
\sum_{i=1}^{n} \text{a}(i) \delta_{i}(t) - n \int_{0}^{t} \text{a}(r) \, dr,
\end{array} \right]
\]

where \( \text{a}(t) = \text{a}(1) \) for \( t \geq 1 \). Obviously, \( \varphi_n \in H_p \), and

\[
\| \varphi_n - \varphi \|_p = n \sup \left[ \int_{0}^{t} \right] + \int_{0}^{t} \| \varphi(t+h) - \varphi(t) \|_p \, dr.
\]

Given \( \varepsilon > 0 \), let us choose an \( h_0 > 0 \) such that \( \| \varphi(t+h) - \varphi(t) \|_p < \varepsilon/2 \) for \( 0 < h \leq h_0 \) and \( t \geq 1 \). Then

\[
\| \varphi_n - \varphi \|_p \leq n \sup \left[ \int_{0}^{t} \right] + \int_{0}^{t} \| \varphi(t+h) - \varphi(t) \|_p \, dr \leq \varepsilon
\]

for \( n = 1, 2, \ldots \). Now, let us consider the case \( h_0 \leq h \leq 1 \). Then

\[
\| \varphi_n - \varphi \|_p \leq n \sup \left[ \int_{0}^{t} \right] + \int_{0}^{t} \| \varphi(t+h) - \varphi(t) \|_p \, dr \leq \varepsilon
\]

for \( n \) sufficiently large. Hence \( \| \varphi_n - \varphi \|_p \to 0 \) for \( 0 < p < 1 \).

1.4. The set \( \bigcup_{p \in \mathcal{L}^1} \) is dense in \( \langle H^p, \| \cdot \|_p \rangle \).

1.5. The spaces \( \langle H^p, \| \cdot \|_p \rangle \) and \( \langle H^{p'}, \| \cdot \|_p \rangle \) are separable for \( 0 < p' < p \).

More precisely, the set of all rational polynomial functions(*) is dense in every space \( \langle H^p, \| \cdot \|_p \rangle \) and in every space \( \langle H^p, \| \cdot \|_p \rangle \) for \( 0 < p' < p \).

Proof. By 1.2 and 1.3 the polynomial functions are dense in the space \( \langle H^p, \| \cdot \|_p \rangle \) for every \( 0 < p < 1 \). Thus, we have to prove that any polynomial function may be approximated (in the norm \( \| \cdot \|_p \), \( p < 1 \)) by rational polynomial functions; this follows trivially by the following lemma.

Let \( y(t) \) be a continuous function defined in \( (0, 1) \), being linear in either interval \( (a, b) \) and \( (b, c) \), where \( 0 \leq a < b < c \leq 1 \); next, let \( a < w \leq b \) and let

\[
y(t) = \begin{cases} 
y(t) & \text{for } 0 \leq t \leq w \text{ and for } c \leq t \leq 1, 
\text{linear in } (w, c).
\end{cases}
\]

Then \( \lim_{w \to a} y(t) = 0 \).

Indeed, we have

\[
y(t) = \begin{cases} 
y(t) & \text{for } 0 \leq t \leq w \text{ and for } c \leq t \leq 1, 
\text{linear in } (w, b) \text{ and in } (b, c).
\end{cases}
\]

Obviously,

\[
|A| \leq 2(b-w) \max \left( \frac{|y(b)-y(a)|}{b-a}, \frac{|y(c)-y(b)|}{c-b} \right) = B(b-w).
\]

Next, we have \( y(a, h) \leq |A| \leq B(b-w) \), whence

\[
\sup_{0 < h < c} h^{-p} |y(a, h)| = B(b-w)^{1-p}.
\]

Finally

\[
\sup_{0 < h < c} h^{-p} |y(a, h)| \leq \sup_{0 < h < c} \| y \|_{H^1} \leq \| y \|_{H^1}(b-w)^{-p}
\]

\[
\leq \max \left( \frac{|A|}{b-w}, \frac{|A|}{c-b} \right)(b-w)^{-p} \leq \max \left( B(b-w)^{1-p}, B(b-w)^{-p} \right).
\]

Thus, \( |y|_{H^p} = |z - y|_{H^p} \leq B(b-w)^{-p} \) for sufficiently small \( b-w \).

\[ 
\text{Now, let } 0 < p' < p \leq 1 \text{. Then } H^p \subseteq H_p \text{ and every subset of } H_p \text{ dense in } \langle H^p, \| \cdot \|_p \rangle \text{ is dense in } \langle H^p, \| \cdot \|_p \rangle \text{, too.}
\]

1.6. Let \( 0 \leq p < 1 \) and let \( \varphi \in \bigcup_{p \in \mathcal{L}^1} \). Then

\[
\| \varphi \|_{H^p} = \lim_{p \to 0} \| \varphi \|_{H^p}.
\]

In particular, for \( p = 0 \),

\[
\lim_{p \to 0} \| \varphi \|_p = \sup_{0 \leq h \leq 1} |\varphi(t+h) - \varphi(t)|: 0 \leq t + h \leq 1.
\]

(*) By rational polynomial functions we understand polynomial functions with both coordinates of single-points rational.
At the same time,
\[ \|x\|_p^p = \lim_{t\to+} \|x\|_{p,1}^p \quad \text{for } x \in H_{p,1}, \quad 0 < p \leq 1. \]

Proof. First, we give the proof for \( p > 0 \). Writing
\[ f_{\delta, b}(x) = \frac{\|x(t+h) - x(t)\|^p}{h^p} \quad \text{for } 0 < h \leq 1 \text{ and } 0 \leq t \leq 1 - h, \]
and \( T = [(t, h): 0 \leq h \leq 1, \quad 0 \leq t \leq 1 - h] \) we observe that \( f_{\delta, b}(x) \) are continuous on \( T \) for fixed \( q \) and \( x \in H_{p,1}, \quad q > q \). It suffices to prove this at the points \( (t, 0) \) for \( 0 \leq t \leq 1 \). Since
\[ f_{\delta, b}(x) = \frac{\|x(t+h) - x(t)\|^p}{h^p} \leq \|x\|_{p,1}^p \quad (0 < h \leq 1), \]
so, for any \( x \in \bigcup H_{p,1} \), there exists a \( b_0 > 0 \) (dependent on \( x \)) such that
\[ f_{\delta, b}(x) \text{ are continuous on } T \text{ for } 0 \leq \delta \leq b_0. \]
Thus, 0.2 may be applied to obtain (T). Similarly, (9) follows from 0.1. Now, we proceed to the case \( p = 0 \). Let \( x = 0 \) be a fixed element of \( \bigcup H_{p,1} \); then there exists a \( b_0 \) such that \( x \in H_{p,1}^{b_0} \). Now, choose \( \delta > 0 \) so that
\[ \|x(t+h) - x(t)\| \leq \delta \|x\|^p \quad \text{for all } 0 < h \leq \delta \text{ and for all } t. \]
Then
\[ \|x(t+h) - x(t)\| \leq \delta \|x\|_{p,1}^p \quad \text{for } 0 < h \leq \delta, \quad 0 < p \leq p_0 \text{ and for all } t. \]
Consequently, for \( 0 < p < p_0 \),
\[ \|x\|_{p,1}^p \leq \sup_{h \leq \delta} \|x(t+h) - x(t)\| \leq \sup_{s < t} \|x(t+h) - x(t)\| \delta^p = \|x\|_{p,1}^p \delta^p, \]
where \( p_0 \) and \( \delta \) depend only on \( x \). Thus we have proved
\[ \|x\|_{p,1}^p \leq \|x\|_{p,1}^p \leq \|x\|_{p,1}^p \delta^p \]
which implies (8).

1.7. Let \( C_0 \) be the space of all continuous functions in \( (0, 1) \) vanishing at \( 0 \). \( C_0 \) may be identified with the space \( H_{p,1} \) which is defined analogously to \( H_{p,1}^2 \), as well as \( H_p \) may be identified with the space of bounded functions in \( (0, 1) \), vanishing at \( 0 \). The norm \( \|x\|_{p,1}^p = \max \{ |x(t)|: 0 \leq t \leq 1 \} \) is defined in \( C_0 \) and equivalent to the usual norm \( \|x\| = \max \{ |x(t)|: 0 \leq t \leq 1 \} \); indeed, \( \|x\| \leq \|x\|_{p,1} \leq 2|x| \). Thus
\[ \langle C_0, p \rangle^p = \langle H_p, p \rangle^p = \langle H_{p,1}, p \rangle^p. \]

Moreover, by the preceding considerations,
\[ \langle H_{p,1}, p \rangle^p = \langle H_{p,1}^2, p \rangle^p = \langle H_{p,1}, p \rangle^p \quad \text{for } 0 < p \leq 1. \]

Finally, we conclude that the spaces \( \langle H_{p,1}, p \rangle^p \) form a family of separable Banach spaces, semicontinuous from below with respect to \( p \). At the same time, the spaces \( \langle H_{p,1}, p \rangle^p \) are semicontinuous from above, neither family being continuous (1).

2. Spaces of functions of finite \( p \)-th variation.

2.0. We shall consider the classes \( OV_p \) and \( AC_p \), defined for \( p > 1 \) as follows (1). Given a fixed closed interval \( (a, b) \) and a partition \( \pi: a = t_0 < t_1 < \ldots < t_n = b \), we write
\[ SV_p(\pi, \pi) = \sum_{i=1}^n \|x(t_i) - x(t_{i-1})\|^p \]
for any function \( x(t) \) defined in \( (a, b) \). The value
\[ V_p(x) = \sup_{\pi} SV_p(\pi, \pi) \]
is called the \( p \)-th variation of \( x(t) \) in \( (a, b) \). Let
\[ V_p(x) = [x; V_p(x) = \infty, x(a) = 0] \]
and let \( OV_p \) be the class of all continuous functions belonging to \( V_p \). \( AC_p \) will denote the class of all functions \( x(t) \) vanishing at \( a \) and \( p \)-absolutely continuous, i.e. satisfying the following condition: for every \( \varepsilon > 0 \) a number \( \delta > 0 \) may be chosen so that, for every finite system of non-overlapping subintervals \( (a_i, b_i) \) of the interval \( (a, b) \), the inequality
\[ \sum |x(b_i) - x(a_i)| \leq \delta \text{ implies } \sum |x(b_i) - x(a_i)| \leq \delta \]
All the spaces \( V_p, OV_p \), and \( AC_p \) are Banach spaces with respect to the norm \( \|x\|_{p,1} = V_p(x) (p > 1) \). Moreover, the following inclusions and inequalities hold for all \( x \) and \( 1 < p < p' \):
\[ AC_p \subset OV_p \subset AC_p, \quad V_p(x) \leq V_{p'}(x). \]

The set of all rational polynomial functions is dense in \( AC_{1,1} \) \( \| \|_{1} \)
(1) recently, Ciesielski (2) proved that every function \( x \in H_{p,1} \) may be developed in a series with respect to the Schauder polynomial functions (consisting of the known basis in \( C(0, 1) \), convergent with respect to the norm \( \| \|_{p} \). So he gave a new proof of 1.2, 1.3 and 1.5.

(1) For the definitions and basic properties, see [9], [5] and [6].
2.1. Let \( \pi : a = u_0 < u_1 < \ldots < u_n = b \) be a partition of \((a, b)\), and let \( \sigma(t) \) be a function defined in \((a, b)\), monotone in each interval \( \langle u_i, u_{i+1} \rangle \) and continuous at each point \( u_i \). Then
\[
V_p(\sigma) = \sup_{\pi \in \Pi_n} S_p(\pi, \sigma) \quad \text{for} \quad p \geq 1. \tag{11}
\]

Proof. It suffices to prove that, for any partition \( \pi' : a = v_0 < v_1 < \ldots < v_m = b \) of the interval \((a, b)\), there exists a subpartition \( \pi : a = u_0 < u_1 < \ldots < u_n = b \) of \( \pi_0 \) such that
\[
S_p(\pi', \sigma) < S_p(\pi, \sigma). \tag{12}
\]

Let \( j \) be the least index such that \( u_j \) does not belong to \( \pi_0 \) \((j \geq 1)\). We distinguish two cases.

1° Let \( [\sigma(v_j) - \sigma(v_{j-1})][\sigma(v_{j+1}) - \sigma(v_j)] \geq 0 \). Then
\[
|\sigma(v_{j+1}) - \sigma(v_j)|^p + |\sigma(v_j) - \sigma(v_{j-1})|^p \leq |\sigma(v_{j+1}) - \sigma(v_{j-1})|^p
\]
and \( S_p(\pi', \sigma) \leq S_p(\pi, \sigma) \) where \( \pi_1 : a = v_0 < v_1 < \ldots < v_{j-1} < v_{j+1} < v_{j+2} < \ldots < v_m = b \).

2° Let \( [\sigma(v_0) - \sigma(v_{j-1})][\sigma(v_{j+1}) - \sigma(v_j)] < 0 \). Then there exists an index \( \delta \) such that \( u_{j-1} < v_\delta < u_j \),
\[
|\sigma(u_\delta) - \sigma(v_{j-1})| > |\sigma(v_j) - \sigma(v_{j-1})|
\]
and
\[
|\sigma(v_{j+1}) - \sigma(u_\delta)| > |\sigma(v_j) - \sigma(u_\delta)|.
\]

Obviously, \( S_p(\pi', \sigma) \leq S_p(\pi, \sigma) \) where
\[
\pi_1 : a = v_0 < v_1 < \ldots < v_{j-1} < v_\delta < v_{j+1} < v_{j+2} < \ldots < v_m = b.
\]

Thus, after a finite number of such steps we obtain a subpartition \( \pi \) of \( \pi_0 \) satisfying (12).

The formula (11) is valid in two important cases: for polynomial functions and for step functions. In the second case we assume \( u_i \) \((i = 1, 2, \ldots, m)\) to be the middle points of the intervals in which \( \sigma(t) \) is constant. So, given two arbitrary partitions \( \pi_1 \) and \( \pi_2 \) with the same number of points, the space \( \langle X, \| \sigma \| \rangle \) of polynomial functions with angle points at points of \( \pi_1 \) is isometric with the space \( \langle X, \| \cdot \|_p \rangle \) of step functions with middle points at points of \( \pi_2 \).

2.2. Given a fixed polynomial function \( \sigma(t) \), the function \( \varphi(p) = V_p(\sigma) \) is continuous for \( p \geq 1 \), \( \varphi(1) = 0 \).
\[
\| \sigma \|_p = \lim_{p \to 0} \| \sigma \|_{p+\epsilon} = \lim_{\epsilon \to 0} \| \sigma \|_{p+\epsilon}.
\]
Moreover,
\[
\lim_{p \to 0} \| \sigma \|_p = \| \sigma \|_\infty = \sup \{ |\sigma(t+h) - \sigma(t)| : 0 < t < t+h \leq 1 \}.
\]

Proof. Denoting by \( T \) the set of all subpartitions of the partition \( \pi_0 \) of \( a \) to \( b \) as \( \pi = \tau, f_{\pi}(\sigma) = \sigma_\tau(\pi, \sigma) \) and applying 0.1 and 0.2 we obtain (13).

Since
\[
\left( \frac{1}{m} \sum_{i=0}^{m} |\sigma(t_i) - \sigma(t_{i-1})|^p \right)^{1/p} \leq \left( \frac{1}{m} \sum_{i=1}^{m} |\sigma(t_i) - \sigma(t_{i-1})|^p \right)^{1/p'}
\]
and since \( \| \sigma \|_{p'} = \sup_{x \in [a, b]} |\sigma(x)| \), so \( S_p(\pi, \sigma) \geq S_{p'}(\pi, \sigma) \) and, by 0.1,
\[
S_p(\pi, \sigma) \geq \sup_{x \in [a, b]} \sup_{t \in [0, 1]} |\sigma(t) - \sigma(t_{i-1})| = \| \sigma \|_{p'}^p.
\]

2.3. Since the convergence of a monotone sequence of norms in a dense set implies convergence everywhere (cf. [3], Th. 9.1), the preceding considerations yield
\[
\langle \sigma_{p}, \| \cdot \| \rangle = \langle \sigma_{p}, \| \cdot \| \rangle = \sup_{x \in [a, b]} \langle \sigma \| \sigma \|_p \rangle \quad \text{for} \quad 1 < p < \infty,
\]
\[
\langle \sigma_{p}, \| \cdot \|_p \rangle = \langle \sigma_{p}, \| \cdot \|_p \rangle = \sup_{x \in [a, b]} \langle \sigma \| \sigma \|_p \rangle.
\]

Similarly as in the case of spaces \( \langle H_p, \| \cdot \|_p \rangle \) and \( \langle H_p, \| \cdot \|_p \rangle \), the family \( \langle \sigma_{p}, \| \cdot \|_p \rangle \) depends on \( p \) semantically from below and the family \( \langle \sigma_{p}, \| \cdot \|_p \rangle \) semantically from above (4).

2.4. According to Riesz [7], we may consider another definition of the p-th variation of a function \( \sigma(t) \) defined in \((a, b)\),
\[
\Phi_p(\sigma) = \sup_{x \in [a, b]} \left( \sum_{i=1}^{n} |\sigma(t_i) - \sigma(t_{i-1})|^p \right)^{1/p}.
\]

F. Riesz proved that in order that \( \Phi_p(\sigma) < \infty \) for a function \( \sigma(t) \) and for \( p > 1 \), it is necessary and sufficient that \( \sigma(t) \) be the indefinite integral of a function belonging to \( H_p \), and
\[
\Phi_p(\sigma) = \left( \int_a^b |\sigma'(t)|^p dt \right)^{1/p}.
\]
(cf. [4], p. 224, and [10]). Thus the space
\[
I_{H_p} = \{ \sigma : \Phi_p(\sigma) < \infty, \sigma(a) = 0 \}
\]

(4) Let us note that families \( CT_{H_p} \) and \( CT_{D_p} \), as well as families \( H_p \) and \( H_p \), resemble topologically the lexicographic product of an interval \((a, b)\) of ordered set \( \lambda \) and of a two-point set, provided with the order topology, i.e., the so-called topological (non-metrizable) space obtained by "splitting of the points of an interval into halves".
is identical with
\[ \{ x : x(t) = \int_0^t x(u) \, du, \forall u \in I \} \],
both spaces being provided with the norm \( \| x \|_p = \Phi_p(x) \). Obviously, the family \( \{ ll_p \} \) depends on \( p \) continuously. Moreover, \( \langle ll_p, \| \cdot \|_1 \rangle = \langle AG_1, \| \cdot \|_1 \rangle \) and \( \langle ll_p, \| \cdot \|_2 \rangle = \langle H_1, \| \cdot \|_2 \rangle \) (if \( a = 0, b = 1 \)).

Thus, let us assume, for simplicity, that \( (a, b) = (0, 1) \). Connections between families considered so far may be presented by the following scheme:

\[ \text{C}_0 \quad \text{H}_0^0, \text{H}_p \quad \text{AC}_p, \text{CV}_p \quad \text{AC}_1 \]

2.5. In many considerations (e.g. in the theory of Fourier series) functions satisfying the Hölder condition and being of finite \( p \)-th variation, simultaneously, are very useful. Spaces of such functions \( CV_p \cap H_p \), \( AC_p \cap H_p^1 \), etc. provided with the norms \( \| x \|_p = \| x \|_p^H + \| x \|_p^{CV} \) are Banach spaces, moreover, the space \( \langle AC_p \cap H_p^1, \| \cdot \|_p \rangle \) is separable, rational polygonal functions being dense in it. These spaces may be treated as depending on a double parameter \( (p, q) \), where \( p \geq 1, 0 < q < 1 \).

3. Spaces of almost periodic functions. Let \( S_p, W_p \) and \( B_p \) (\( 1 \leq p < \infty \)) denote the normed spaces of almost periodic functions in the sense of Stepanoff, Weyl and Besicovitch, respectively (\(^2\)). The means

\[ \overline{M}^a_p(x) = \sup_{a < c < b} \int_a^b x(u) \, du, \quad \overline{M}^c_p(x) = \lim_{b \to \infty} \sup_{a < c < b} \int_a^b x(u) \, du, \]
\[ \overline{M}^{a,b}_p(x) = \lim_{b \to \infty} \frac{1}{b-a} \int_a^b x(t) \, dt \]
are defined for any bounded measurable function and, by 0.3, the norms
\[ \| x \|_{\overline{M}^a_p} = \| \overline{M}^a_p(\| x \|_p) \|^{1/p}, \quad \| x \|_{\overline{M}^c_p} = \| \overline{M}^c_p(\| x \|_p) \|^{1/p}, \quad \| x \|_{\overline{M}^{a,b}_p} = \| \overline{M}^{a,b}_p(\| x \|_p) \|^{1/p} \]
depend on \( p \) continuously for any fixed bounded \( x \). Hence, the class \( \langle S_p, \| \cdot \|_{\overline{M}^a_p} \rangle \) depends on \( p \) semicontinuously from below (\(^3\)).

\(^1\) An exposition of these spaces is given in the monograph of Besicovitch and in the paper (1) of Bohr and Fejer.
\(^2\) Professor S. Hartman has remarked that, by some results of Bohr and Fejer (11), condition 5a is not satisfied for the class \( \{ l_p \} \).

\[ \langle B_p, \| \cdot \|_p \rangle \]
depend on \( p \) continuously, since they are equivalent to the spaces \( L_p(G, \mu) \), where \( G \) denotes the Bohr compactification of the additive group of real numbers and \( \mu \) denotes the Haar measure on \( G \). At the same time, this equivalence maps the uniformly almost periodic functions of Bohr on the continuous functions on \( G \) and maps the functions \( e^{it} \) onto the characters on \( G \); \( \mu \) being regular, continuous functions are dense in each space \( L_p(G, \mu) \), \( 1 \leq p < \infty \) (Fejer [3]) (\(^4\)).

We do not consider the spaces \( W_p \), for they are not complete.

4. Spaces of strongly \( p \)-summable sequences.

\[ \langle M_p, \| \cdot \|_{\overline{M}^c_p} \rangle \]
is a non-separable Banach space. Further, let us denote by \( M_p^a \) the closure in \( \langle M_p, \| \cdot \|_{\overline{M}^c_p} \rangle \) of the set of sequences which are constant for almost all \( n \). Obviously, \( M_p^a \) consists exactly of all strongly \( p \)-summable sequences, i.e. of sequences such that

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n |x_k| = 0 \]
for a number \( x \) being a generalized limit of \( \{ x_n \} \).

Finally, let us denote by \( M_p^b \) the closure in \( \langle M_p, \| \cdot \|_{\overline{M}^c_p} \rangle \) of the set of all bounded sequences. Evidently the following inclusion is satisfied

\[ M_p^a \subset M_p^b \subset M_p, \quad M_p \subset M_p^b, \quad M_p^c \subset M_p^d \]
for \( p > p' \geq 1 \).

\(^1\) We are indebted to Professors S. Hartman and C. Byl-Nardzewski who have shown us this method.
where $a_n$ is a sequence tending to infinity and $0 = m_0 < m_1 < m_2 < \ldots$ is a sequence of integers such that $m_i/k \to \infty$, 

$$\frac{2}{3} < \frac{1}{m_k} \sum_{i=m_k}^{m_{k+1}-1} |a_i|^p < 1$$

and

$$\frac{1}{m_k} \sum_{i=m_k}^{m_{k+1}-1} |a_i|^p < \frac{2}{3}$$

for $k = 1, 2, \ldots$ Then $x = \{a_n\}$ belongs to $M_p$ and does not belong to $M^p$, although $x \in M^p$ for every $p'$ with $1 \leq p' < p$. Indeed, let us denote $k = m_{i+1} - m_i$, then

$$|x|^p = \sup_n \frac{|a_1|^p + \cdots + |a_n|^p}{l_1 + \cdots + l_n} \leq 1$$

and

$$\lim_{n \to \infty} |a_1|^p + \cdots + |a_n|^p = \lim_{n \to \infty} |a_n|^p = 0$$

for $p' < p$. Hence

$$0 \leq \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |a_i|^p = \lim_{n \to \infty} \frac{|a_1|^p + \cdots + |a_n|^p}{l_1 + \cdots + l_n} = 0.$$ 

References


Hors par la Rédaãion le 6. 9. 1960

---

On very strong Riesz-summability of orthogonal series

by

J. MEDEI (Sánchez)

1. Let $\{a_n\}$ be a positive, strictly increasing, numerical sequence, with $a_0 = 0$ and $a_n \to \infty$.

A series

$$\sum_{n=0}^{\infty} \frac{a_n + a_{n+1} + \cdots + a_{n+m}}{k}$$

with $k$-th partial sums $s_k$, is said to be summable $(R, \lambda_0, 1)$ to the sum $s$, if

$$s_k = \sum_{n=1}^{k-1} (a_{n+1} - a_n) s_{n} \to s, \quad k \to \infty.$$ 

Obviously, the Riesz-method of summation is a generalization of $(C, 1)$-method, which is obtained by putting $\lambda_0 = n$.

Series (1.1) is said to be very strongly summable $(R, \lambda_0, 1)$ to the sum $s$, if

$$\sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) s_n = o(\lambda_{n+1}), \quad n \to \infty,$$

for every strictly increasing sequence of indices $\{a_n\}$.

In particular, if $a_n = k$ ($k = 0, 1, 2, \ldots$), we shall say that series (1.1) is strongly summable $(R, \lambda_0, 1)$ to the sum $s$.

Series (1.1) is said to be strongly (very strongly) summable $(C, 1)$, if it is strongly (very strongly) summable $(R, \lambda_0, 1)$ with $\lambda_0 = n$.

2. Further, we shall consider the strong and the very strong Riesz-summability of orthogonal series.

Let $\{e_n(x)\}$ denote an orthonormal system defined in the interval $(0, 1)$ and $\{e_n\} \in p$, i.e.

$$\sum_{n=1}^{N} e_n^2 < \infty.$$