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# Some classes of Banach spaces depending on a parameter

by

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In this paper we shall consider the following classes of Banach spaces:

$H_p$ —functions satisfying Hölder condition with an exponent $p$,
$CV_p$ —continuous functions with finite $p$-th variation,
$AC_p$ —absolutely continuous functions of order $p$,
$S_p$ and $B_p$ —almost periodic functions in the sense of Stepanoff and Besicovitch, respectively,
$M_p$ —strongly $p$-summable sequences.

These classes may be treated as families of Banach spaces $X_p$ depending on a parameter $p$. In each of these classes there are known inclusions between spaces $X_p$, $X_{p'}$ and inequalities between norms $\| \|_p$, $\| \|_{p'}$ for $p<p'$. We shall consider the following problem: given a sequence $p_n$ convergent to $p_0$, establish connections between the corresponding spaces $X_{p_n}$ and $X_{p_0}$. This problem is closely related to the problem of the continuity (suitably defined) of the spaces $X_p$ with respect to the parameter $p$.

These problems are considered from a general point of view in paper [8], where, in the following definition, the limit $\underset{n\to\infty}{\mathfrak{S}}(X_n)$ of a sequence $X_n$ of linear metric spaces is introduced. $\langle X_0, \| \|_0\rangle$ is termed $\mathfrak{S}$-*limit of* $\langle X_n, \| \|_n\rangle$ (written $\langle X_0, \| \|_0\rangle = \underset{n\to\infty}{\mathfrak{S}}\langle X_n, \| \|_n\rangle$) if the following conditions are satisfied:

1° $X_0$ and almost all $X_n$ are subspaces of a linear space,

2° $X_n$ converges to $X_0$ in the sense of the theory of sets (i. e. $X_0=\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}X_n=\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}X_n$),

3° $\|x\|_0=\lim_{n\to\infty}\|x\|_n$ for all $x\in X_0$.

Next, we write $\langle X_0, \| \|_0\rangle = \underset{n\to\infty}{\overline{\mathfrak{S}}}\langle X_n, \| \|_n\rangle$ if $\underset{n\to\infty}{\mathfrak{S}}(X_n)$ is dense in $\langle X_0, \| \|_0\rangle$.

Let $\{X_p\}_{a<p<\beta}$ be a family of Banach spaces $\langle X_p, \| \|_p\rangle$ such that

$X_p \subset X_{p'}$ and $\|x\|_p \geqslant \|x\|_{p'}$ for $p > p'$. We say that this family is *continuous with respect to* $p$, if the following conditions are satisfied [1]:

1° If $x \in \bigcup_{\varepsilon>0} X_{p+\varepsilon}$ and if $p_n \searrow p$, then $\|x\|_p = \lim_{n\to\infty} \|x\|_{p_n}$,

2° if $x \in \bigcap_{\varepsilon>0} X_{p-\varepsilon}$ and if $p_p \nearrow p$, then $\|x\|_p = \lim_{n\to\infty} \|x\|_{p_n}$,

3° $\bigcup_{\varepsilon>0} X_{p+\varepsilon}$ is dense in $\langle X_p, \| \|_p\rangle$,

4° $X_p$ is dense in $\bigcap_{\varepsilon>0} X_{p-\varepsilon}$ with respect to the $F$-norm

$$\|x\|_p^* = \sum_{n=1}^{\infty} 2^{-n}\varphi(\|x\|_{p_n})$$

where $\varphi(u) = u(1+u)^{-1}$ and $p_n$ is a fixed sequence such that $a < p_1 < p_2 < \ldots$ and $p_n \to p$,

5° $\{x: x \in \bigcap_{\varepsilon>0} X_{p-\varepsilon},\ \sup_{\varepsilon>0} \|x\|_{p-\varepsilon} < \infty\} = X_p$ for all $p$ $(\alpha < p < \beta)$.

Moreover, we consider two definitions of semicontinuity. If conditions 1°, 2°, 4°, 5° are satisfied, then the family $\langle X_p, \| \|_p\rangle$ is said to be *semicontinuous from above*; if 1°, 2°, 3° are satisfied, $\langle X_p, \| \|_p\rangle$ is said to be *semicontinuous from below*.

Conditions 1° and 3° mean that $\langle X_p, \| \|_p\rangle = \underset{\varepsilon\to 0+}{\mathfrak{S}} \langle X_{p+\varepsilon}, \| \|_{p+\varepsilon}\rangle$. At the same time, the space $\langle \bigcap_{\varepsilon>0} X_{p-\varepsilon}, \| \|_p^*\rangle$ is a $B_0$-space and conditions 2° and 4° mean that

$$\langle \bigcap_{\varepsilon>0} X_{p-\varepsilon}, \| \|_p^*\rangle = \underset{n\to\infty}{\mathfrak{S}} \langle X_{p_n}, \| \|_n^0\rangle \quad \text{where} \quad \|x\|_n^0 = \sum_{k=1}^{n} 2^{-k}\varphi(\|x\|_{p_k}),$$

$$\langle X_p, \| \|_p\rangle = \underset{\varepsilon\to 0+}{\mathfrak{S}} \langle X_p, \| \|_{p-\varepsilon}\rangle, \qquad \langle \bigcap_{\varepsilon>0} X_{p-\varepsilon}, \| \|_p^*\rangle = \underset{n\to\infty}{\mathfrak{S}} \langle X_p, \| \|_n^0\rangle.$$

**Auxiliaries.** The following well-known lemmas are very useful in further considerations.

**0.1.** *Given sets $X$ and $T$, let us suppose that to every $p \in (\alpha, \beta)$ there corresponds a family $\{f_{p,\tau}(x)\}_{\tau\in T}$ of functions defined in $X$, such that*

(1) $$f_{p,\tau}(x) \leqslant f_{p',\tau}(x)$$

*for all $x \in X$, $\tau \in T$, $p \leqslant p'$ and such that*

(2) $$p_n \to p \quad \textit{implies} \quad f_{p_n,\tau}(x) \to f_{p,\tau}(x)$$

[1] If the inclusions for $X_p$ and the inequalities for $\| \|_p$ are opposite, the definition is analogous.



*for all $x \in X$ and $\tau \in T$. Then, for any fixed $x \in X$, the function*

(3) $$\varphi(p) = \sup\{f_{p,\tau}(x): \tau \in T\}$$

*is non-decreasing and left-side continuous with respect to $p$.*

*More generally, $\mathfrak{R}$ being a $\sigma$-ideal of subsets of $T$ with $T \notin \mathfrak{R}$, the function*

$$\varphi_{\mathfrak{R}}(p) = \underset{T}{\sup\nolimits_{\mathfrak{R}}} f_{p,\tau}(x)$$

*is also non-decreasing and left-side continuous in $(\alpha, \beta)$.*

In this case (1) and (2) are assumed to be satisfied for $\mathfrak{R}$-almost every $\tau \in T$ and $\underset{T}{\sup\nolimits_{\mathfrak{R}}} g(\tau)$ denotes the $\mathfrak{R}$-essential supremum, i. e. the least upper bound of numbers $a$ such that the set $\{\tau \in T: g(\tau) > a\}$ belongs to $\mathfrak{R}$.

Proof. The monotony of $\varphi_{\mathfrak{R}}(p)$ being trivial, let us suppose that $p_n \nearrow p$ and $\alpha < p < \beta$. Obviously, the limit $A = \lim_{n\to\infty} \varphi_{\mathfrak{R}}(p_n)$ exists and $A \leqslant \varphi_{\mathfrak{R}}(p)$. Since $A = \sup_n \varphi_{\mathfrak{R}}(p_n)$, for each $n$ there exists $R_n \in \mathfrak{R}$ such that $f_{p_n,\tau}(x) \leqslant A$ for $\tau \in T \setminus R_n$. Next, there exists $R_0 \in \mathfrak{R}$ such that $f_{p_n,\tau}(x) \to f_{p,\tau}(x)$ for $\tau \in T \setminus R_0$, hence $f_{p,\tau}(x) \leqslant A$ for $\tau \in T \setminus \bigcup_{n=0}^{\infty} R_n$. Since $\bigcup_{n=0}^{\infty} R_n \in \mathfrak{R}$, we have $\varphi_{\mathfrak{R}}(p) \leqslant A$.

**0.2.** *Let us suppose that $T$ is a compact topological space and $f_{p,\tau}(x)$ are continuous on $T$ for any fixed $p \in (\alpha, \beta)$ and $x \in X$. Then, assuming* (1) *and* (2), *the function $\varphi(p)$ defined by* (3) *is continuous.*

Proof. We have to prove that $p_n \searrow p_0$ implies $\varphi(p_n) \to \varphi(p_0)$ for any fixed $x \in X$. Since $f_{p_n,\tau}(x) \searrow f_{p_0,\tau}(x)$ and $f_{p_n,\tau}(x)$ are continuous on $T$ $(n = 0, 1, 2, \ldots)$, by the theorem of Dini $f_{p_n,\tau}(x)$ converge uniformly on $T$; hence $\sup_\tau f_{p_n,\tau}(x)$ tends to $\sup_\tau f_{p_0,\tau}(x)$.

**0.3.** *Let $X$ be a linear class of bounded functions $x(t)$ defined on an arbitrary set $T$, containing constant functions and such that if $x \in X$ and $p > 0$, then $|x|^p \in X$. Next, let $\overline{\mathfrak{M}}(x)$ be a functional over $X$, satisfying the following conditions:*

$$\overline{\mathfrak{M}}(x+y) \leqslant \overline{\mathfrak{M}}(x) + \overline{\mathfrak{M}}(y), \qquad \overline{\mathfrak{M}}(\lambda x) = \lambda\overline{\mathfrak{M}}(x) \quad \text{for} \quad \lambda \geqslant 0,$$

$$0 \leqslant x(t) \leqslant y(t) \quad \textit{implies} \quad \overline{\mathfrak{M}}(x) \leqslant \overline{\mathfrak{M}}(y),$$

$\overline{\mathfrak{M}}(1) = 1$, *where* 1 *denotes the constant function* $x(t) = 1$.

*Then, for any fixed $x \in X$, the functions*

$$\varphi_1(p) = \overline{\mathfrak{M}}(|x|^p) \quad \textit{and} \quad \varphi_2(p) = |\varphi_1(p)|^{1/p}$$

*are continuous for $p > 0$.*

Proof. It suffices to show that $\varphi_1(p)$ is continuous whenever $|x(t)| \leqslant 1$ and $p$ runs any interval $(\alpha, \beta)$ with $\beta > \alpha > 0$. Let us choose $\varepsilon > 0$ and then $\delta > 0$ so that $\alpha < p < \beta$ and $|p-q| < \delta$ imply $|u^p - u^q| < \varepsilon$ for all $0 \leqslant u \leqslant 1$. Then $\big||x(t)|^p - |x(t)|^q\big| < \varepsilon$ for all $t \in T$, whence $0 \leqslant \overline{\mathfrak{M}}(|x|^p) - \overline{\mathfrak{M}}(|x|^q) \leqslant \overline{\mathfrak{M}}(|x|^p - |x|^q) \leqslant \overline{\mathfrak{M}}(\varepsilon) = \varepsilon$ for $\alpha < p < q < p+\delta < \beta$.

**1. Spaces of functions satisfying Hölder conditions.**

**1.0.** Let $H_p$ be the class of all real functions $x(t)$ defined in $\langle 0, 1\rangle$ vanishing at $t = 0$ and satisfying the Hölder condition with the exponent $p$, i. e. the condition

$$|x(t+h) - x(t)| \leqslant K|h|^p \quad \text{for all} \quad t, t+h \in \langle 0, 1\rangle,$$

$K$ being a constant depending on $x$, and $0 < p \leqslant 1$.

Next, let $H_p^0$ be the subclass of $H_p$ consisting of all functions $x(t)$ satisfying the condition $x(t+h) - x(t) = o(h^p)$, i. e. such that

$$\lim_{h\to 0+} h^{-p}\psi(x, h) = 0 \quad \text{where} \quad \psi(x, h) = \sup_{0\leqslant t\leqslant 1-h} |x(t+h) - x(t)|.$$

The following inclusions are well-known:

$$H_{p'} \subset H_p^0 \subset H_p \quad \text{for} \quad p < p'.$$

$H_p$ is a non-separable Banach space with respect to the norm

$$\|x\|_p^H = \sup_{0<h\leqslant 1} \sup_{0\leqslant t\leqslant 1-h} |x(t+h) - x(t)|\, h^{-p} = \sup_{0<h\leqslant 1} h^{-p}\psi(x, h),$$

and $H_p^0$ is closed in $\langle H_p, \|\ \|_p^H\rangle$.

**1.1.** *The space* $H_1$ (i. e. the space of functions satisfying the usual Lipschitz condition, vanishing at 0) *is isometric with the space* $L_\infty$ *of all essentially bounded measurable functions in* $\langle 0, 1\rangle$. Indeed, every $x \in H_1$ is absolutely continuous, whence

$$x(t) = \int_0^t x'(\tau)\,d\tau \quad \text{and} \quad \|x\|_1^H = \sup_{t,h} \frac{|x(t+h) - x(t)|}{h} = \operatorname*{ess\,sup}_{0\leqslant t\leqslant 1} |x'(t)|.$$

Thus

$$U(y) = \int_0^t y(\tau)\,d\tau \tag{4}$$

establishes a one-to-one linear and norm-preserving map of $L_\infty$ onto $H_1$. It is easily seen that $U$ transforms the step functions onto the polygonal functions. Hence no function $x = U(y)$ with $y$ non-equivalent to any Riemann-integrable function cannot be approximated by polygonal functions (in the norm $\|\ \|_1^H$).



Now, given two functions $x, y \in H_1$ and a point $t_0$, if $x(t)$ is differentiable at $t_0$ and if $y(t)$ has an angle point at $t_0$, i. e. if

$$y'_+(t_0) - y'_-(t_0) = \lim_{h\to 0+} \frac{y(t_0+h) - y(t_0)}{h} - \lim_{h\to 0+} \frac{y(t_0) - y(t_0-h)}{h} = 2b \neq 0,$$

then $\|x-y\|_1^H \geqslant |b|$. In particular, a polygonal function with an angle point at a non-rational value of $t$ cannot be approximated (in the norm $\|\ \|_1^H$) by polygonal functions possessing angle points at rational values of $t$ only.

**1.2.** *The set of all polygonal functions vanishing at* 0 *is dense in every space* $\langle H_1, \|\ \|_p^H\rangle$ *where* $p < 1$.

Proof. The map (4) generates in the space $L_\infty$ the norms

$$\|y\|_p^U = \|U(y)\|_p^H = \sup_{0<h\leqslant 1} h^{-p} \sup_{0\leqslant t\leqslant 1-h} |Uy(t+h) - Uy(t)|$$
$$= \sup_{0<h\leqslant 1} h^{-p} \sup_{0\leqslant t\leqslant 1-h} \Big|\int_t^{t+h} y(\tau)\,d\tau\Big|, \quad \text{where} \quad 0 < p \leqslant 1.$$

We have to prove that the step functions are dense in $\langle L_\infty, \|\ \|_p^U\rangle$ for $0 < p < 1$. The inequality

$$\sup_{0<h\leqslant\delta} h^{-p} \sup_{0\leqslant t\leqslant 1-h} \Big|\int_t^{t+h} y(\tau)\,d\tau\Big| \leqslant \sup_{0<h\leqslant\delta} h^{1-p} \operatorname*{ess\,sup}_{0\leqslant t\leqslant 1} |y(t)| = \delta^{1-p}\|y\|_1^U \tag{5}$$

holds for every $y \in L_\infty$ and $0 < \delta \leqslant 1$. Let us consider the characteristic function $y(t)$ of a measurable subset of $\langle 0, 1\rangle$. Choose $\varepsilon$ with $0 < \varepsilon < 1$ and an integer $n > 2/\delta$ where $\delta = \varepsilon^{1/(1-p)}$. Write $\Delta_k = \langle (k-1)/n, k/n\rangle$ and

$$z(t) = \frac{1}{|\Delta_k|} \int_{\Delta_k} y(\tau)\,d\tau \quad \text{for} \quad t \in \Delta_k \text{ and } k = 1, 2, \ldots, n.$$

Then $\|z-y\|_1^U = \operatorname*{ess\,sup}_{0\leqslant t\leqslant 1} |z(t) - y(t)| \leqslant 1$. Since

$$\int_{\Delta_k} [z(\tau) - y(\tau)]\,d\tau = 0,$$

we have

$$\Big|\int_t^{t+h} [z(\tau) - y(\tau)]\,d\tau\Big| = \Big|\int_t^{k_1/n} + \int_{k_2/n}^{t+h}\Big| \leqslant \frac{2}{n}\|z-y\|_1^U \leqslant \frac{2}{n} < \delta,$$

where $k_1 = E(nt)+1$, $k_2 = E[n(t+h)]$. Then

$$\sup_{\delta\leqslant h\leqslant 1} h^{-p} \sup_{0\leqslant t\leqslant 1-h} \Big|\int_t^{t+h} [z(\tau) - y(\tau)]\,d\tau\Big| \leqslant \delta^{1-p}. \tag{6}$$

Finally, by (5) and (6), $\|z-y\|_p^U \leqslant \delta^{1-p} = \varepsilon$.

Since linear combinations of characteristic functions are dense in $\langle L_\infty, \| \|_1^U\rangle$ and $\|y\|_1^U \geqslant \|y\|_p^U$, the step-functions are dense in $\langle L_\infty, \| \|_p^U\rangle$ for every $0 < p < 1$.

**1.3.** *The set $H_1$ is dense in $\langle H_p^0, \| \|_p^H\rangle$ for every $0 < p < 1$.*

Proof. Given $x \epsilon H_p^0$, let us write

$$x_n(t) = n\int_t^{t+1/n} x(\tau)d\tau - n\int_0^{1/n} x(\tau)d\tau,$$

where $x(t) = x(1)$ for $t \geqslant 1$. Obviously, $x_n \epsilon H_1$ and

$$x_n(t) - x(t) = n\int_0^{1/n}[x(t+\tau) - x(t) - x(\tau)]d\tau,$$

$$\psi(x_n - x, h) = n\sup_{0\leqslant t\leqslant 1-h}\left|\int_0^{1/n}[x(t+h+\tau) - x(t+h) - x(t+\tau) + x(t)]d\tau\right|.$$

Given $\varepsilon > 0$, let us choose an $h_0 > 0$ such that $|x(t+h) - x(t)|h^{-p} < \varepsilon/2$ for $0 < h \leqslant h_0$ and $t \geqslant 0$. Then

$$\begin{aligned} &h^{-p}\psi(x_n - x, h)\\ &\leqslant n\sup_{0\leqslant t\leqslant 1-h}\int_0^{1/n}\left[\left|\frac{x(t+h+\tau) - x(t+\tau)}{h^p}\right| + \left|\frac{x(t+h) - x(t)}{h^p}\right|\right]d\tau < \varepsilon\end{aligned}$$

for $n = 1, 2, \ldots$ Now, let us consider the case $h_0 \leqslant h \leqslant 1$. Then

$$\begin{aligned} &h^{-p}\psi(x_n - x, h)\\ &\leqslant n\sup_{0\leqslant t\leqslant 1-h}\int_0^{1/n}\left[\left|\frac{x(t+h+\tau) - x(t+h)}{\tau^p}\right| + \left|\frac{x(t+\tau) - x(t)}{\tau^p}\right|\right]\frac{\tau^p}{h^p}d\tau\\ &\leqslant \frac{2\|x\|_p^H}{n^p h_0^p} < \varepsilon\end{aligned}$$

for $n$ sufficiently large. Hence $\|x_n - x\|_p^H \to 0$ for $0 < p < 1$.

**1.4.** *The set $\bigcup_{\varepsilon>0} H_{p+\varepsilon}$ is dense in $\langle H_p^0, \| \|_p^H\rangle$.*

**1.5.** *The spaces $\langle H_p^0, \| \|_p^H\rangle$ and $\langle H_p, \| \|_{p'}^H\rangle$ are separable for $0 < p' < p \leqslant 1$.*

More precisely, *the set of all rational polygonal functions* (²) *is dense in every space $\langle H_p^0, \| \|_p^H\rangle$ and in every space $\langle H_p, \| \|_{p'}^H\rangle$ for $0 < p' < p \leqslant 1$.*

Proof. By 1.2 and 1.3 the polygonal functions are dense in the

(²) By *rational polygonal functions* we understand polygonal functions with both coordinates of angle-points rational.



space $\langle H_p^0, \| \|_p^H\rangle$ for every $0 < p < 1$. Thus, we have to prove that any polygonal function may be approximated (in the norm $\| \|_p$, $p < 1$) by rational polygonal functions; this follows trivially by the following lemma.

Let $y(t)$ be a continuous function defined in $\langle 0, 1\rangle$, being linear in either interval $\langle a, b\rangle$ and $\langle b, c\rangle$, where $0 \leqslant a < b < c \leqslant 1$; next, let $a < w < b$ and let

$$z(t) = \begin{cases} y(t) \text{ for } 0 \leqslant t \leqslant w \text{ and for } c \leqslant t \leqslant 1,\\ \text{linear in } \langle w, c\rangle.\end{cases}$$

Then $\lim_{w\to b}\|z - y\|_p^H = 0$.

Indeed, we have

$$x(t) = z(t) - y(t) = \begin{cases} 0 \text{ for } 0 \leqslant t \leqslant w \text{ and for } c \leqslant t \leqslant 1,\\ A = y(w) - y(b) + \dfrac{b-w}{c-w}[y(c) - y(w)] \text{ for } t = b,\\ \text{linear in } \langle w, b\rangle \text{ and in } \langle b, c\rangle.\end{cases}$$

Obviously,

$$|A| \leqslant 2(b-w)\max\left(\frac{|y(b) - y(a)|}{b-a}, \frac{|y(c) - y(b)|}{c-b}\right) = B(b-w).$$

Next, we have $\psi(x, h) \leqslant |A| \leqslant B(b-w)$, whence

$$\sup_{b-w\leqslant h\leqslant 1} h^{-p}\psi(x, h) \leqslant B(b-w)^{1-p};$$

finally

$$\begin{aligned}\sup_{0<h\leqslant b-w} h^{-p}\psi(x, h) &\leqslant \sup_{0<h\leqslant b-w}\|x\|_1^H h^{1-p} \leqslant \|x\|_1^H(b-w)^{1-p}\\ &\leqslant \max\left(\frac{|A|}{b-w}, \frac{|A|}{c-b}\right)(b-w)^{1-p} \leqslant \max\left(B, B\frac{b-w}{c-b}\right)(b-w)^{1-p}.\end{aligned}$$

Thus, $\|x\|_p^H = \|z - y\|_p^H \leqslant B(b-w)^{1-p}$ for sufficiently small $b - w$.

Now, let $0 < p' < p \leqslant 1$. Then $H_p \subset H_{p'}^0$ and every subset of $H_p$ dense in $\langle H_{p'}^0, \| \|_{p'}^H\rangle$ is dense in $\langle H_p, \| \|_{p'}^H\rangle$, too.

**1.6.** *Let $0 \leqslant p < 1$ and let $x \epsilon \bigcup_{\varepsilon>0} H_{p+\varepsilon}$. Then*

$$\|x\|_p^H = \lim_{\varepsilon\to 0+}\|x\|_{p+\varepsilon}^H. \tag{7}$$

*In particular, for $p = 0$,*

$$\lim_{\varepsilon\to 0+}\|x\|_\varepsilon^H = \|x\|_0^H = \sup\{|x(t+h) - x(t)|: 0 \leqslant t < t+h \leqslant 1\}. \tag{8}$$

*At the same time,*

(9) $$\|x\|_p^H = \lim_{\varepsilon\to 0+} \|x\|_{p-\varepsilon}^H \quad \text{for} \quad x\in H_p,\ 0<p\leqslant 1.$$

Proof. First, we give the proof for $p>0$. Writing

$$f_{q,(t,h)}(x) = \begin{cases} |x(t+h)-x(t)|\,h^{-q} & \text{for } 0<h\leqslant 1 \text{ and } 0\leqslant t\leqslant 1-h, \\ 0 & \text{for } h=0 \text{ and all } t\in\langle 0,1\rangle \end{cases}$$

and $T=\{(t,h)\colon 0\leqslant h\leqslant 1,\ 0\leqslant t\leqslant 1-h\}$ we observe that $f_{q,(t,h)}(x)$ are continuous on $T$ for fixed $q$ and $x\in H_{q'}$, $q'>q$. It suffices to prove this at the points $(t,0)$ for $0\leqslant t\leqslant 1$. Since

$$f_{q,(t,h)}(x) = \frac{|x(t+h)-x(t)|}{h^{q'}}\,h^{q'-q}\leqslant \|x\|_{q'}^H h^{q'-q} \qquad (0<h\leqslant 1),$$

so, for any $x\in\bigcup_{\varepsilon>0} H_{p+\varepsilon}$, there exists a $\delta_0>0$ (dependent on $x$) such that $_{p+\delta,(t,h)}(x)$ are continuous on $T$ for $0\leqslant\delta\leqslant\delta_0$. Thus, 0.2 may be applied to obtain (7). Similarly, (9) follows from 0.1.

Now, we proceed to the case $p=0$. Let $x\neq 0$ be a fixed element of $\bigcup_{p>0} H_p$; then there exists a $p_0$ such that $x\in H_{p_0}^0$.

Now, choose $\delta>0$ so that

$$|x(t+h)-x(t)|\,h^{-p_0} < \tfrac{1}{2}\|x\|_0^H$$

for all $0<h\leqslant\delta$ and for all $t$. Then

$$|x(t+h)-x(t)|\,h^{-p} < \tfrac{1}{2}\|x\|_0^H \quad \text{for} \quad 0<h\leqslant\delta,\ 0<p\leqslant p_0 \text{ and for all } t.$$

Consequently, for $0<p<p_0$,

$$\|x\|_p^H = \sup_{h\geqslant\delta}\ \sup_{0\leqslant t\leqslant 1-h} |x(t+h)-x(t)|\,h^{-p} \leqslant \sup_{0\leqslant t<t+h\leqslant 1} |x(t+h)-x(t)|\,\delta^{-p} = \|x\|_0^H\delta^{-p},$$

where $p_0$ and $\delta$ depend only on $x$. Thus we have proved

(10) $$\|x\|_0^H\leqslant \|x\|_p^H\leqslant \|x\|_0^H\delta^{-p}$$

which implies (8).

**1.7.** Let $C_0$ be the space of all continuous functions in $\langle 0,1\rangle$ vanishing at 0. $C_0$ may be identified with the space $H_0^0$ which is defined analogously to $H_p^0$, as well as $H_0$ may be identified with the space of bounded functions in $\langle 0,1\rangle$, vanishing at 0. The norm $\|x\|_0^H = \max\{|x(t_1)-x(t_2)|\colon 0\leqslant t_1<t_2\leqslant 1\}$ is defined in $C_0$ and equivalent to the usual norm $\|x\| = \max\{|x(t)|\colon 0\leqslant t\leqslant 1\}$; indeed, $\|x\|\leqslant\|x\|_0^H\leqslant 2\|x\|$. Thus

$$\langle C_0, \|\,\|_0^H\rangle = \overline{\underset{p\to 0}{\mathfrak{S}}}\langle H_p^0, \|\,\|_p^H\rangle = \overline{\underset{p\to 0}{\mathfrak{S}}}\langle H_p, \|\,\|_p^H\rangle.$$



Moreover, by the preceding considerations,

$$\langle H_p^0, \|\,\|_0^H\rangle = \overline{\underset{\varepsilon\to 0}{\mathfrak{S}}}\langle H_{p+\varepsilon}^0, \|\,\|_{p+\varepsilon}^H\rangle = \overline{\underset{\varepsilon\to 0}{\mathfrak{S}}}\langle H_{p+\varepsilon}, \|\,\|_{p+\varepsilon}^H\rangle \quad \text{for} \quad 0<p<1.$$

Finally, we conclude that the spaces $\langle H_p^0, \|\,\|_p^H\rangle$ form a family of separable Banach spaces, semicontinuous from below with respect to $p$. At the same time, the spaces $\langle H_p, \|\,\|_p^H\rangle$ are semicontinuous from above, neither family being continuous [3].

## 2. Spaces of functions of finite $p$-th variation.

**2.0.** We shall consider the classes $CV_p$ and $AC_p$, defined for $p\geqslant 1$ as follows [4]. Given a fixed closed interval $\langle a,b\rangle$ and a partition $\pi$: $a=t_0<t_1<\ldots<t_m=b$, we write

$$S_p(\pi,x) = \Big(\sum_{i=1}^{m} |x(t_i)-x(t_{i-1})|^p\Big)^{1/p}$$

for any function $x(t)$ defined in $\langle a,b\rangle$. The value

$$V_p(x) = \sup_\pi S_p(\pi,x)$$

is called the *$p$-th variation of $x(t)$ in $\langle a,b\rangle$*. Let

$$V_p = \{x\colon V_p(x)<\infty,\ x(a)=0\}$$

and let $CV_p$ be the class of all continuous functions belonging to $V_p$. $AC_p$ will denote the class of all functions $x(t)$ vanishing at $a$ and $p$-absolutely continuous, i. e. satisfying the following condition: for every $\varepsilon>0$ a number $\delta>0$ may be chosen so that, for every finite system of non-overlapping subintervals $(a_i,\beta_i)$ of the interval $\langle a,b\rangle$, the inequality $\sum(\beta_i-a_i)^p<\delta$ implies $\sum|x(\beta_i)-x(a_i)|^p<\varepsilon$.

All the spaces $V_p$, $CV_p$ and $AC_p$ are Banach spaces with respect to the norm $\|x\|_p^V = V_p(x)$ $(p\geqslant 1)$. Moreover, the following inclusions and inequalities hold for all $x$ and $1\leqslant p<p'$:

$$AC_p\subset CV_p\subset AC_{p'}, \qquad V_{p'}(x)\leqslant V_p(x).$$

The set of all rational polygonal functions is dense in $\langle AC_1, \|\,\|_1^V\rangle$ (indeed, the map (4) transforms isometrically the space $L_1$ onto $AC_1$, and rational step functions are dense in $L_1$); hence, $\langle AC_1, \|\,\|_p^V\rangle$ is separable for all $p\geqslant 1$. Since the set $AC_1$ is dense in $\langle AC_p, \|\,\|_p^V\rangle$ (see [5] and [6]), all the spaces $\langle AC_p, \|\,\|_p^V\rangle$ have a common separable dense subset.

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(3) Recently, Ciesielski [2] proved that every function $x\in H_p^0$ may be developed in a series with respect to the Schauder polygonal functions (consisting of the known basis in $C\langle 0,1\rangle$), convergent with respect to the norm $\|\,\|_p$. So he gave a new proof of 1.2, 1.3 and 1.5.

(4) For the definitions and basic properties, see [9], [5] and [6].

**2.1.** *Let $\pi_0$: $a = u_0 < u_1 < \ldots < u_n = b$ be a partition of $\langle a, b\rangle$, and let $x(t)$ be a function defined in $\langle a, b\rangle$, monotone in each interval $\langle u_i, u_{i+1}\rangle$ and continuous at each point $u_i$. Then*

(11) $$V_p(x) = \sup_{\pi\subset\pi_0} S_p(\pi, x) \quad \text{for} \quad p \geqslant 1.$$

Proof. It suffices to prove that, for any partition $\pi'$: $a = v_0 < v_1 < \ldots < v_m = b$ of the interval $\langle a, b\rangle$, there exists a subpartition $\pi$: $a = u_0 < u_{n_1} < \ldots < u_{n_k} = b$ of $\pi_0$ such that

(12) $$S_p(\pi', x) \leqslant S_p(\pi, x).$$

Let $j$ be the least index such that $v_j$ does not belong to $\pi_0$ ($j \geqslant 1$). We distinguish two cases.

1° Let $[x(v_j)-x(v_{j-1})][x(v_{j+1})-x(v_j)] \geqslant 0$. Then

$$|x(v_{j+1})-x(v_j)|^p + |x(v_j)-x(v_{j-1})|^p \leqslant |x(v_{j+1})-x(v_{j-1})|^p$$

and $S(\pi', x) \leqslant S(\pi_1, x)$ where $\pi_1$: $a = v_0 < v_1 < \ldots < v_{j-1} < v_{j+1} < \ldots < v_m = b$.

2° Let $[x(v_j)-x(v_{j-1})][x(v_{j+1})-x(v_j)] < 0$. Then there exists an index $i_0$ such that $v_{j-1} < u_{i_0} < v_{j+1}$,

$$|x(u_{i_0})-x(v_{j-1})| \geqslant |x(v_j)-x(v_{j-1})|$$
$$\text{and} \quad |x(v_{j+1})-x(u_{i_0})| \geqslant |x(v_{j+1})-x(v_j)|.$$

Obviously, $S(\pi', x) \leqslant S(\pi_2, x)$, where

$$\pi_2: \ a = v_0 < v_1 < \ldots < v_{j-1} < u_{i_0} < v_{j+1} < \ldots < v_n = b.$$

Thus, after a finite number of such steps we obtain a subpartition $\pi$ of $\pi_0$ satisfying (12).

The formula (11) is valid in two important cases: for polygonal functions and for step functions. In the second case we assume $u_i$ ($i = 1, 2, \ldots, n$) to be the middle points of the intervals in which $x(t)$ is constant. So, given two arbitrary partitions $\pi_1$ and $\pi_2$ with the same number of points, the space $\langle X_1, \| \|_p^V\rangle$ of polygonal functions with angle points at points of $\pi_1$ is isometric with the space $\langle X_2, \| \|_p^V\rangle$ of step functions with middle points at points of $\pi_2$.

**2.2.** *Given a fixed polygonal function $x(t)$, the function $\varphi(p) = V_p(x)$ is continuous for $p \geqslant 1$, i. e.*

(13) $$\|x\|_p^V = \lim_{\varepsilon\to 0+} \|x\|_{p+\varepsilon}^V = \lim_{\varepsilon\to 0+} \|x\|_{p-\varepsilon}^V.$$

*Moreover,*

(14) $$\lim_{p\to\infty} \|x\|_p^V = \|x\|_0^H = \sup\{|x(t+h)-x(t)|:\ 0 \leqslant t < t+h \leqslant 1\}.$$



Proof. Denoting by $T$ the set of all subpartitions of the partition $\pi_0$ of angle points of $x(t)$, $\pi = \tau$, $f_{p,\tau}(x) = S_p(\pi, x)$ and applying 0.1 and 0.2 we obtain (13).

Since

$$\left(\frac{1}{m}\sum_{i=1}^{m} |x(t_i)-x(t_{i-1})|^p\right)^{1/p} \leqslant \left(\frac{1}{m}\sum_{i=1}^{m} |x(t_i)-x(t_{i-1})|^{p'}\right)^{1/p'} \quad \text{for} \quad 1 \leqslant p \leqslant p'$$

and since $\|x\|_p^V = \sup_{\pi\subset\pi_0} S_p(\pi, x)$, so $S_p(\pi, x) \nearrow \sup_i |x(t_i)-x(t_{i-1})|$ and, by 0.1,

$$\sup_{\pi\subset\pi_0} S_p(\pi, x) \to \sup_{\pi\subset\pi_0} \sup_i |x(t_i)-x(t_{i-1})| = \|x\|_0^H.$$

**2.3.** Since the convergence of a monotone sequence of norms in a dense set implies convergence everywhere (cf. [8], Th. 9.1), the preceding considerations yield

$$\langle AC_p, \| \|_p^V\rangle = \overline{\mathfrak{S}}_{\varepsilon\to 0+} \langle AC_{p-\varepsilon}, \| \|_{p-\varepsilon}^V\rangle = \overline{\mathfrak{S}}_{\varepsilon\to 0+} \langle CV_{p-\varepsilon}, \| \|_{p-\varepsilon}^V\rangle \quad \text{for} \quad 1 < p < \infty,$$

$$\langle C_0, \| \|_0^H\rangle = \overline{\mathfrak{S}}_{p\to\infty} \langle AC_p, \| \|_p^V\rangle = \overline{\mathfrak{S}}_{p\to\infty} \langle CV_p, \| \|_p^V\rangle.$$

Similarly as in the case of spaces $\langle H_p^0, \| \|_p^H\rangle$ and $\langle H_p, \| \|_p^H\rangle$, the family $\langle AC_p, \| \|_p^V\rangle$ depends on $p$ semicontinuously from below and the family $\langle CV_p, \| \|_p^V\rangle$ — semicontinuously from above ([5]).

**2.4.** According to Riesz [7], we may consider another definition of the $p$-th variation of a function $x(t)$ defined in $\langle a, b\rangle$:

$$\Phi_p(x) = \sup_{\pi}\left(\sum_{i=1}^{m} \frac{|x(t_i)-x(t_{i-1})|^p}{|t_i - t_{i-1}|^{p-1}}\right)^{1/p}, \qquad p \geqslant 1.$$

F. Riesz proved that in order that $\Phi_p(x) < \infty$ for a function $x(t)$ and for $p > 1$, it is necessary and sufficient that $x(t)$ be the indefinite integral of a function belonging to $L_p$, and

$$\Phi_p(x) = \left(\int_a^b |x'(t)|^p dt\right)^{1/p}$$

(cf. [4], p. 224, and [10]). Thus the space

$$IL_p = \{x: \Phi_p(x) < \infty,\ x(a) = 0\}$$

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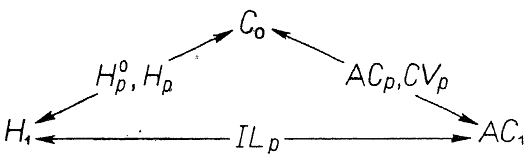
([5]) Let us note that families $CV_p$ and $AC_p$, as well as families $H_p$ and $H_p^0$, resemble topologically the lexigraphical product of an interval (i. e. of an ordered set of type $\lambda$) and of a two-point set, provided with the order topology, i. e. the so called topological (non-metrisable) space obtained by "spliting of the points of an interval into halves".

is identical with

$$\left\{x : x(t) = \int_a^t z(u)\,du,\ z \in L_p\right\},$$

both spaces being provided with the norm $\|x\|_p^\Phi = \Phi_p(x)$. Obviously, the family $\langle IL_p, \|\ \|_p^\Phi\rangle$ depends on $p$ continuously. Moreover, $\langle IL_1, \|\ \|_1^\Phi\rangle = \langle AC_1, \|\ \|_1^V\rangle$ and $\langle IL_\infty, \|\ \|_\infty^\Phi\rangle = \langle H_1, \|\ \|_1^H\rangle$ (if $a = 0$, $b = 1$).

Thus, let us assume, for simplicity, that $\langle a, b\rangle = \langle 0, 1\rangle$. Connections between families considered so far may be presented by the following scheme:




**2.5.** In many considerations (e. g. in the theory of Fourier series) functions satisfying the Hölder condition and being of finite $p$-th variation, simultaneously, are very useful. Spaces of such functions $CV_p \cap H_q$, $AC_p \cap H_q^0$ etc. provided with the norms $\|x\|_{p,q}^{VH} = \|x\|_p^V + \|x\|_q^H$ are Banach spaces, moreover, the space $\langle AC_p \cap H_q^0, \|\ \|_{p,q}^{VH}\rangle$ is separable, rational polygonal functions being dense in it. These spaces may be treated as depending on a double parameter $(p, q)$, where $p \geqslant 1$, $0 < q < 1$.

**3. Spaces of almost periodic functions.** Let $S_p$, $W_p$ and $B_p$ $(1 \leqslant p < \infty)$ denote the normed spaces of almost periodic functions in the sense of Stepanoff, Weyl and Besicovitsch, respectively ([6]). The means

$$\overline{\mathfrak{M}}^S(x) = \sup_{-\infty<t<\infty} \int_t^{t+1} x(u)\,du, \qquad \overline{\mathfrak{M}}^W(x) = \overline{\lim_{l\to\infty}} \sup_{-\infty<t<\infty} \frac{1}{l}\int_t^{t+l} x(u)\,du,$$

$$\overline{\mathfrak{M}}^B(x) = \overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$$

are defined for any bounded measurable function and, by 0.3, the norms

$$\|x\|_p^S = [\overline{\mathfrak{M}}^S(|x|^p)]^{1/p}, \qquad \|x\|_p^W = [\overline{\mathfrak{M}}^W(|x|^p)]^{1/p}, \qquad \|x\|_p^B = [\overline{\mathfrak{M}}^B(|x|^p)]^{1/p}$$

depend on $p$ continuously for any fixed bounded $x$. Hence, the class $\langle S_p, \|\ \|_p^S\rangle$ depends on $p$ semicontinuously from below ([7]).

([6]) An exposition of these spaces is given in the monography of Besicovitch and in the paper [1] of Bohr and Følner.

([7]) Professor S. Hartman has remarked that, by some results of Bohr and Følner [1], condition 5° is not satisfied for the class $\{S_p\}$.



The spaces $\langle B_p, \|\ \|_p^B\rangle$ depend on $p$ continuously, since they are equivalent to the spaces $L_p(G, \mu)$, where $G$ denotes the Bohr compactification of the additive group of real numbers and $\mu$ denotes the Haar measure on $G$. At the same time, this equivalence maps the uniformly almost periodic functions of Bohr on the continuous functions on $G$ and maps the functions $e^{i\lambda t}$ onto the characters on $G$; $\mu$ being regular, continuous functions are dense in each space $L_p(G, \mu)$, $1 \leqslant p < \infty$ (Følner [3]) ([8]).

We do not consider the spaces $W_p$, for they are not complete.

**4. Spaces of strongly $p$-summable sequences.**

**4.0.** Let us denote, for $p \geqslant 1$, by $M_p$ the class of all sequences $x = \{x_n\}$ such that

$$\|x\|_p^M = \sup_n \left(\frac{1}{n}\sum_{i=1}^n |x_i|^p\right)^{1/p} < \infty.$$

$\langle M_p, \|\ \|_p^M\rangle$ is a non-separable Banach space. Further, let us denote by $M_p^c$ the closure in $\langle M_p, \|\ \|_p^M\rangle$ of the set of sequences which are constant for almost all $n$. Obviously, $M_p^c$ consists exactly of all strongly $p$-summable sequences, i. e. of sequences such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n |x_i - a|^p = 0$$

for a number $a$ being a generalized limit of $\{x_n\}$.

Finally, let us denote by $M_p^b$ the closure in $\langle M_p, \|\ \|_p^M\rangle$ of the set of all bounded sequences. Evidently the following inclusion are satisfied

$$M_p \subset M_{p'}, \qquad M_p^c \subset M_{p'}^c, \qquad M_p^b \subset M_{p'}^b$$

for $p > p' \geqslant 1$.

**4.1.** Applying 0.3 with $\overline{\mathfrak{M}}(x) = \sup_n \frac{1}{n}\sum_{i=1}^n x_i$ we conclude that the families $\langle M_p^c, \|\ \|_p^M\rangle$ and $\langle M_p^b, \|\ \|_p^M\rangle$ are semicontinuous from below for $p \geqslant 1$.

**4.2.** *The class $M_p^c$ as well as the class $M_p^b$ is not continuous with respect to $p$.*

Proof. We shall prove that condition 5° is not satisfied. Let

$$x_n = \begin{cases} a_k & \text{for } n = m_k,\ k = 1, 2, \ldots, \\ 0 & \text{elsewhere,} \end{cases}$$

([8]) We are indebted to Professors S. Hartman and C. Ryll-Nardzewski who have shown us this method.

where $\alpha_k$ is a sequence tending to infinity and $0 = m_0 < m_1 < m_2 < \ldots$ is a sequence of integers such that $m_k/k \to \infty$,

$$\frac{2}{3} < \frac{1}{m_k}\sum_{i=1}^{m_k}|x_i|^p < 1 \quad \text{and} \quad \frac{1}{m_k-1}\sum_{i=1}^{m_k-1}|x_i|^p < \frac{1}{3}$$

for $k = 1, 2, \ldots$ Then $x = \{x_n\}$ belongs to $M_p$ and does not belong to $M_p^b$, although $x \in M_{p'}^c$ for every $p'$ with $1 \leqslant p' < p$. Indeed, let us denote $l_k = m_{k+1} - m_k$; then

$$(\|x\|_p^M)^p = \sup_k \frac{|\alpha_1|^p + \ldots + |\alpha_k|^p}{l_1 + \ldots + l_k} \leqslant 1$$

and

$$\lim_{k\to\infty} \frac{|\alpha_1|^{p'} + \ldots + |\alpha_k|^{p'}}{l_1 + \ldots + l_k} \leqslant \lim_{k\to\infty} \frac{|\alpha_1|^{p'} + \ldots + |\alpha_k|^{p'}}{|\alpha_1|^p + \ldots + |\alpha_k|^p} = \lim_{k\to\infty} \frac{|\alpha_k|^{p'}}{|\alpha_k|^p} = 0$$

for $p' < p$. Hence

$$0 \leqslant \overline{\lim_{n\to\infty}} \frac{1}{n}\sum_{i=1}^{n}|x_i|^{p'} \leqslant \lim_{k\to\infty} \frac{|\alpha_1|^{p'} + \ldots + |\alpha_k|^{p'}}{l_1 + \ldots + l_k} = 0.$$

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# On very strong Riesz-summability of orthogonal series

by

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**1.** Let $\{\lambda_n\}$ be a positive, strictly increasing, numerical sequence, with $\lambda_0 = 0$ and $\lambda_n \to \infty$.

A series

$$(1.1) \qquad u_0 + u_1 + \ldots + u_n + \ldots,$$

with $n$-th partial sums $s_n$, is said to be *summable* $(R, \lambda_n, 1)$ *to the sum* $s$, if

$$\tau_n = \frac{1}{\lambda_{n+1}}\sum_{k=0}^{n}(\lambda_{k+1} - \lambda_k)s_k \to s, \quad \text{as} \quad n \to \infty.$$

Obviously, the Riesz-method of summation is a generalization of $(C, 1)$-method, which is obtained by putting $\lambda_n = n$.

Series (1.1) is said to be *very strongly summable* $(R, \lambda_n, 1)$ *to the sum* $s$, if

$$\sum_{k=0}^{n}(\lambda_{k+1} - \lambda_k)(s_{v_k} - s)^2 = o(\lambda_{n+1}), \quad \text{as} \quad n \to \infty,$$

for every strictly increasing sequence of indices $\{v_n\}$.

In particular, if $v_k = k$ $(k = 0, 1, 2, \ldots)$, we shall say that series (1.1) is *strongly summable* $(R, \lambda_n, 1)$ *to the sum* $s$.

Series (1.1) is said to be *strongly* (*very strongly*) *summable* $(C, 1)$, if it is strongly (very strongly) summable $(R, \lambda_n, 1)$ with $\lambda_n = n$.

**2.** Further, we shall consider the strong and the very strong Riesz-summability of orthogonal series.

Let ON$\{\varphi_n(x)\}$ denote an orthonormal system defined in the interval $\langle 0, 1\rangle$ and $\{c_n\} \in l^2$, i. e.

$$(2.1) \qquad \sum_{n=0}^{\infty} c_n^2 < \infty.$$