STUDIA MATHEMATICA, T. XX. (1961)

Remarks on Leżański's determinants

by

ROMAN SIKORSKI (Warszawa)

This paper is a supplement to my paper [6].

In [6] I have given some formulas for solutions of a linear equation

$$(1)\qquad (I+T)x = x_0$$

in a Banach space X, and the adjoint equation in a conjugate space Ξ

$$(2)\qquad \xi(I+T) = \xi_0,$$

by means of the Leżański [3, 4] determinants and subdeterminants. These formulas are abstract analogues of the known formulas from classical Algebra for solutions of a system of linear algebraic equations. In the case where X is the space of all continuous functions on an interval, and T is an integral operator with a continuous kernel, the formulas given in [6] do not coincide with the formulas for solutions from Fredholm's determinant theory of integral equations. Also the Leżański subdeterminants do not coincide, in this case, with the original Fredholm subdeterminants. In § 3 I give the definition of some notions which are abstract analogues of the classical Fredholm subdeterminants in any Banach space. I quote also formulas for solutions of (1) and (2) which are abstract analogues of the original Fredholm formulas. Such notions and formulas have been investigated by Grothendieck [1] under more restrictive hypotheses on T. The method applied in this paper is, I think, simpler.

To explain the difference between Leżański's subdeterminants and Fredholm's subdeterminants, I examine in § 4 an integral model of the theory of determinants in Banach spaces, and I quote some integral formulas for the subdeterminants. To write formulas for Leżański's subdeterminants in an integral form, it is necessary to introduce a substitute of the Dirac delta distribution, which enables to write the identity operator I in an integral form. Consequently all formulas in § 4 should be interpreted only formally, as another kind of writing the exact formulas for subdeterminants from §§ 2, 3. Also the convergence of some function series and function-like series should be understood as convergence in norm of corresponding multilinear functionals represented formally by considered kernels. However, in the formulas for Fredholm's subdeterminants the substitute of Dirac delta distribution does not appear, and sometimes the convergence of the function series defining formally the subdeterminants can be also interpreted as a convergence of functions in a suitably defined space of functions. This is e. g. in the case where X is the space of all continuous functions on an interval, and T is an integral operator with a continuous kernel. In this case we get the original formulas of Fredholm.

In my paper [6] I have proved a theorem on the multiplication of determinants in Banach spaces. This theorem was proved under a hypothesis of a commutative character. In § 5 I shall show that this hypothesis is superflous if the definition of multiplication is suitably modified. The result is based on a theorem on trace proved by Grothendieck [2].

§ 6 contains a differential definition of Leżański's determinant.

§ 1. Fundamental notions and formulas We recall the following definitions from [6].

X and Ξ are two fixed (real or complex) Banach spaces whose elements are denoted respectively by $x, y, z, \ldots$ and $\xi, \eta, \zeta, \ldots$ (with indices if necessary). We suppose that there is defined a scalar multiplication ξx of elements of Ξ and X such that ξx is a bilinear functional on $\Xi\times X$ and

$$|\xi| = \sup_{|x|\leqslant 1}|\xi x| \quad \text{and} \quad |x| = \sup_{|\xi|\leqslant 1}|\xi x|.$$

Thus Ξ can be identified with a closed subspace of the space X^* of all linear bounded functionals on X, and analogously X can be identified with a closed subspace of Ξ^*.

If A is a bilinear functional on $\Xi\times X$, then its value at a point $(\xi, x)\in\Xi\times X$ is denoted by ξAx.

By $\mathfrak{D}$ we shall denote the set of all bilinear functionals A on $\Xi\times X$ such that:

(o) for every fixed $x\in X$ there exists a $y\in X$ such that $\xi Ax = \xi y$ for every $\xi\in\Xi$ (this unique y will be denoted by Ax);

(o′) for every fixed $\xi\in\Xi$ there exists an $\eta\in\Xi$ such that $\xi Ax = \eta x$ for every $x\in X$ (this unique η will be denoted by ξA).

Of course, the transformations $y = Ax$ and $\eta = \xi A$ are conjugate bounded endomorphisms in the spaces X and Ξ respectively. Hence it follows that every $A\in\mathfrak{D}$ can be simultaneously interpreted as a bilinear functional on $\Xi\times X$, or as an endomorphism in X, or as an endomorphism in Ξ. The three possible interpretations of $A\in\mathfrak{D}$ will be systematically used in the whole paper. In order to distinguish none of the three interpretations, we shall use the name *operators* for elements of $\mathfrak{D}$. Operators will be sometimes understood as bilinear functionals on $\Xi\times X$, or as endomorphisms in X or in Ξ.

The set $\mathfrak{D}$ of operators is linear with respect to the natural definition of addition and multiplication by scalars. It is a Banach space with respect to the ordinary norm

$$|A| = \sup_{\substack{|x|\leqslant 1\\|\xi|\leqslant 1}}|\xi Ax| = \sup_{|x|\leqslant 1}|Ax| = \sup_{|\xi|\leqslant 1}|\xi A|.$$

It is also a Banach algebra with the following definition of multiplication: the product A_1A_2 of bilinear functionals $A_1, A_2\in\mathfrak{D}$ is the bilinear functional $\xi(A_1A_2)x = (\xi A_1)(A_2x)$. In other words, the product A_1A_2 interpreted as endomorphism in X (in Ξ) is the superposition of the endomorphisms A_1, A_2 in X (of the endomorphisms A_2, A_1 in Ξ). The unit element of the algebra $\mathfrak{D}$ is the fundamental bilinear functional I: $\xi Ix = \xi x$. By definition, $Ix = x$ and $\xi I = \xi$ for every $x\in X$ and $\xi\in\Xi$.

Let x_0, ξ_0 fixed. The operator K_0 defined by the formula

$$\xi K_0 x = \xi x_0\cdot\xi_0 x$$

(i. e. the product of scalars ξx_0 and $\xi_0 x$) is called *one-dimensional* and denoted by $x_0\cdot\xi_0$. By definition, $K_0x = x_0\cdot\xi_0x$ and $\xi K = \xi x_0\cdot\xi_0$ (the dot replaces here parantheses).

For every bounded linear functional F on $\mathfrak{D}$ the symbol T_F denotes the bilinear functional on $\Xi\times X$:

$$\xi T_F x = F(x\cdot\xi).$$

The space $\mathfrak{DN}$ of all F such that $T_F\in\mathfrak{D}$ is a Banach space under the ordinary norm of F. Any elements of $\mathfrak{DN}$ is called *quasi-nucleus*. If, for an operator T, there exists a quasi-nucleus F such that $T = T_F$, then T is said to be *quasi-nuclear* and F is said to be a *quasi-nucleus of* T. Observe that the canonical transformation which maps any $F\in\mathfrak{DN}$ onto $T_F\in\mathfrak{D}$ is linear and bounded. Moreover

$$|T_F|\leqslant|F|.$$

As an example of a functional F in $\mathfrak{DN}$ we quote here the following *one-dimensional* functional

$$F(A) = \xi_0 Ax_0 \quad \text{for} \quad A\in\mathfrak{D}$$

where ξ_0, x_0 are fixed. This functional will be denoted by $\xi_0\otimes x_0$. Another example is given by the *finitely dimensional* functional

$$F = \sum_{i=1}^{m}\xi_i\otimes x_i, \tag{3}$$

i. e. the functional:

(4) $$F(A) = \sum_{i=1}^{m} \xi_i A x_i \quad \text{for} \quad A \in \mathfrak{O}.$$

Observe that if F is defined by (3), then T_F is the operator $\sum_{i=1}^{m} x_i \cdot \xi_i$.

The class of all finitely dimensional functionals F (i .e. functionals (4)) is a linear subspace of $\mathfrak{Q}\mathfrak{N}$. Its closure in $\mathfrak{Q}\mathfrak{N}$ will be denoted by $\mathfrak{N}$. The elements of $\mathfrak{N}$ are called *nucleus*. If, for an operator T, there exists a nucleus F such that $T = T_F$, then T is said to be *nuclear* and F is said to be a *nucleus of* T.

By the *trace* of a quasi-nucleus (or: nucleus) F we understand the number

$$\mathrm{tr}(F) = F(I).$$

For instance,

$$\mathrm{tr}\left(\sum_{i=1}^{m} \xi_i \otimes x_i\right) = \sum_{i=1}^{m} \xi_i x_i .$$

Suppose that $B(\xi_1, \ldots, \xi_m, x_1, \ldots, x_m)$ is a multilinear functional defined on the Cartesian product $\Xi^m \times X^m$. Suppose also that B, considered as a function of two variables ξ_i, x_j only (all remaining variables being constant), belongs to $\mathfrak{O}$, i. e. it is an operator $\xi_i A x_j$. The number $F(A)$, where $F \in \mathfrak{Q}\mathfrak{N}$, will be also denoted by

$$F_{\xi_i, x_j} B(\xi_1, \ldots, \xi_m, x_1, \ldots, x_m).$$

Of course, the last expression is a linear function of each of the variables $\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_m, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m$, but it does not depend on ξ_i and x_j.

§ 2. Determinant and subdeterminants. Let

$$\theta_m\begin{pmatrix}\xi_1, \ldots, \xi_m \\ x_1, \ldots, x_m\end{pmatrix} = \begin{vmatrix}\xi_1 x_1, & \ldots, & \xi_1 x_m \\ \cdot & \cdots & \cdot \\ \xi_m x_1, & \ldots, & \xi_m x_m\end{vmatrix}.$$

For every quasi-nucleus F, let

$$A_{0,0}(F) = 1,$$

$$A_{0,m}(F) = \frac{1}{m!} F_{\eta_1, y_1} \ldots F_{\eta_m, y_m} \theta_m\begin{pmatrix}\eta_1, \ldots, \eta_m \\ y_1, \ldots, y_m\end{pmatrix} \quad (m = 1, 2, \ldots)$$

and, for $n = 1, 2, \ldots,$ and $m = 0, 1, 2, \ldots$

$$A_{n,m}(F) = \frac{1}{m!} A_{n,m}\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix}$$
$$= F_{\eta_1, y_1} \ldots F_{\eta_m, y_m} \theta_{n+m}\begin{pmatrix}\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m \\ x_1, \ldots, x_n, y_1, \ldots, y_m\end{pmatrix}.$$

Leżański's *determinant of* F is the number

$$D_0(F) = \sum_{m=0}^{\infty} A_{0,m}(F).$$

Leżański's *subdeterminant of* F *of order* n is the $2n$-linear functional

$$D_n(F) = D_n\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix} = \sum_{m=0}^{\infty} A_{n,m}(F)$$
$$= \sum_{m=0}^{\infty} A_{n,m}\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix}.$$

The determinant $D(F)$ and subdeterminants $D_n(F)$ are *determinant* and *subdeterminants for the linear equations*

(5) $$(I + T_F)x = x_0,$$

(6) $$\xi(I + T_F) = \xi_0.$$

Viz. there exists an integer r such that $D_r(F) \neq 0$ but all $D_j(F)$ with $j < r$ vanish identically. Let $\xi_1, \ldots, \xi_r, x_1, \ldots, x_r$ be fixed points such that

(7) $$D_r\begin{pmatrix}\xi_1, \ldots, \xi_r \\ x_1, \ldots, x_r\end{pmatrix} \neq 0.$$

Let B be the operator

$$\xi B x = \frac{D_{r+1}\begin{pmatrix}\xi, \xi_1, \ldots, \xi_r \\ x, x_1, \ldots, x_r\end{pmatrix}}{D_r\begin{pmatrix}\xi_1, \ldots, \xi_r \\ x_1, \ldots, x_r\end{pmatrix}}$$

and let z_i, ξ_i $(i = 1, \ldots, r)$ be defined by the identities

$$\zeta_i x = \frac{D_r\begin{pmatrix}\xi_1, \ldots \ldots \ldots \ldots, \xi_r \\ x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_r\end{pmatrix}}{D_r\begin{pmatrix}\xi_1, \ldots, \xi_r \\ x_1, \ldots, x_r\end{pmatrix}},$$

$$\xi z_i = \frac{D_r\begin{pmatrix}\xi_1, \ldots, \xi_{i-1}, \xi, \xi_{i+1}, \ldots, \xi_r \\ x_1, \ldots \ldots \ldots \ldots \ldots x_s\end{pmatrix}}{D_r\begin{pmatrix}\xi_1, \ldots, \xi_r \\ x_1, \ldots, x_r\end{pmatrix}}.$$

Then the equation (5) has a solution x iff $\zeta_i x_0 = 0$ for $i = 1, \ldots, r$. The general form of the solution of (1) is given by the formula

$$x = Bx_0 + c_1 z_1 + \ldots + c_r z_r. \tag{8}$$

Similarly, the equation (6) has a solution ξ iff $\xi_0 z_i = 0$ for $i = 1, \ldots, r$. Then the general form of the solution of (6) is given by the formula

$$\xi = \xi_0 B + c_1 \xi_1 + \ldots + c_r \xi_r. \tag{9}$$

The numbers $c_1, \ldots, c_r$ are here arbitrary.

§ 3. The Fredholm determinant and subdeterminants. We precede the definition by the proof of the following identity where, for brevity, $T = T_F$:

$$D_n\begin{pmatrix}\xi_1 T, \ldots, \xi_n T\\ x_1, \ldots, x_n\end{pmatrix} = D_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ Tx_1, \ldots, Tx_n\end{pmatrix}. \tag{10}$$

The proof is based on the following known equality (see e. g. Sikorski [8])

$$D_n(F) = D_0(F)\cdot \mathcal{D}_n(F) \qquad (n = 0, 1, 2, \ldots) \tag{11}$$

where $\mathcal{D}_0(F) = 1$ and, for $n > 0$,

$$\mathcal{D}_n(F) = \mathcal{D}_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix} = \begin{vmatrix}\xi_1(I+T)^{-1}x_1, & \ldots, & \xi_1(I+T)^{-1}x_n\\ \ldots & \ldots & \ldots\\ \xi_n(I+T)^{-1}x_1, & \ldots, & \xi_n(I+T)^{-1}x_n\end{vmatrix}. \tag{12}$$

$\mathcal{D}_n(F)$ is defined for all $F \in \mathfrak{Q}\mathfrak{N}$ such that $(I+T_F)^{-1}$ exists, i. e. for all $F \in \mathfrak{Q}\mathfrak{N}$ such that $D_0(F) \neq 0$. Observe that the set of all F such that $D_0(F) = 0$ is nowhere dense in $\mathfrak{Q}\mathfrak{N}$ since $D_0(F)$ is an analytic function of F.

It immediately follows from (2) that

$$\mathcal{D}_n\begin{pmatrix}\xi_1 T, \ldots, \xi_n T\\ x_1, \ldots, x_n\end{pmatrix} = \mathcal{D}_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ Tx_1, \ldots, Tx_n\end{pmatrix}.$$

Hence it follows that (10) holds for all $F \in \mathfrak{Q}\mathfrak{N}$ with $D_0(F) \neq 0$. By continuity, (10) holds everywhere.

Observe also that

$$A_{n,m}\begin{pmatrix}\xi_1 T, \ldots, \xi_n T\\ x_1, \ldots, x_n\end{pmatrix} = A_{n,m}\begin{pmatrix}\xi_1, \ldots, \xi_n\\ Tx_1, \ldots, Tx_n\end{pmatrix}. \tag{13}$$

To obtain it, it suffices to develop the left and the right side of (10) into series of homogenous polynomials of F. Since the expansion is unique, the left side of (13) is the polynomial of the degree $n+m$ in the expansion of the left of (10), and the right side of (13) is the polynomial of the degree $n+m$ in the expansion of the right side of (10), we get the equality (13). The both sides of (13) will be denoted by

$$A^*_{n,m}(F) = A^*_{n,m}\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix} = A_{n,m}\begin{pmatrix}\xi_1 T, \ldots, \xi_n T\\ x_1, \ldots, x_n\end{pmatrix} = A_{n,m}\begin{pmatrix}\xi_1, \ldots, \xi_n\\ Tx_1, \ldots, Tx_n\end{pmatrix}.$$

The *Fredholm determinant* $D^*_0(F)$ of a quasi-nucleus F coincides with the determinant of F:

$$D^*_0(F) = D_0(F).$$

The *Fredholm subdeterminant* $D^*_n(F)$ of F of an order $n > 0$ is the $2n$-linear functional

$$D^*_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix} = D_n\begin{pmatrix}\xi_1 T, \ldots, \xi_n T\\ x_1, \ldots, x_n\end{pmatrix} = D_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ Tx_1, \ldots, Tx_n\end{pmatrix} \tag{14}$$

$$= \sum_{m=0}^{\infty} A^*_{n,m}\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix}.$$

Here D^*_n is expressed by D_n. However it is possible also to express D_n by means of D^*_n. For this purpose we assume the following notation: $\theta_k \wedge D^*_m$ will denote the following $2(k+m)$-linear functional on $\Xi^{k+m} \times X^{k+m}$:

$$\sum_{\mathfrak{p},\mathfrak{q}} \operatorname{sgn}\mathfrak{p}\cdot\operatorname{sgn}\mathfrak{q}\cdot\theta_k\begin{pmatrix}\xi_{p_1}, \ldots, \xi_{p_k}\\ x_{q_1}, \ldots, x_{q_k}\end{pmatrix}\cdot D^*_m\begin{pmatrix}\xi_{p_{k+1}}, \ldots, \xi_{p_{k+m}}\\ x_{q_{k+1}}, \ldots, x_{q_{k+m}}\end{pmatrix}$$

where the sum $\sum_{\mathfrak{p},\mathfrak{q}}$ is extended over all the permutations $\mathfrak{p} = (p_1, \ldots, p_{k+m})$, $\mathfrak{q} = (q_1, \ldots, q_{k+m})$ of numbers $1, \ldots, k+m$ such that

$$p_1 < \ldots < p_k, \quad p_{k+1} < \ldots < p_{k+m},$$
$$q_1 < \ldots < q_k, \quad q_{k+1} < \ldots < q_{k+m},$$

and $\operatorname{sgn}\mathfrak{p}$, $\operatorname{sgn}\mathfrak{q}$ denote the numbers $+1$ or -1 iff the permutations $\mathfrak{p}$, $\mathfrak{q}$ are even or odd, respectively. Similarly we define $\theta_k \wedge D_m$.

THEOREM 1. *The following identities hold*:

$$D_n = \theta_n\cdot D^*_0 - \theta_{n-1}\wedge D^*_1 + \ldots + (-1)^{n-1}\theta_1\wedge D^*_{n-1} + (-1)^n D^*_n,$$
$$D^*_n = \theta_n\cdot D_n - \theta_{n-1}\wedge D_1 + \ldots + (-1)^{n-1}\theta_1\wedge D_{n-1} + (-1)^n D_n.$$

Let

$$\mathcal{D}_n^*(F) = \mathcal{D}_n^*\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix} = \begin{vmatrix} \xi_1 T(I+T)^{-1}x_1, \ldots, \xi_1 T(I+T)^{-1}x_n \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \xi_n T(I+T)^{-1}x_1, \ldots, \xi_n T(I+T)^{-1}x_n \end{vmatrix}.$$

Since $(I+T)^{-1} = I - T(I+T)^{-1}$, the square matrix defining $\mathcal{D}_n$ (see (12)) is the difference of the matrix defining θ_n (see § 2) and the matrix defining $\mathcal{D}_n^*$. By a known theorem on the determinant of the difference of two matrices, $\mathcal{D}_n = \theta_n \cdot \mathcal{D}_0^* - \theta_{n-1} \wedge \mathcal{D}_1^* + \ldots + (-1)^{n-1}\theta_1 \wedge \mathcal{D}_{n-1}^* + (-1)^n \mathcal{D}_n^*$. By (11) and (14), $D_n^* = D_0 \cdot \mathcal{D}_n^*$. Hence, multiplying the obtained identity by $D_0(F)$ we get the first identity mentioned in Theorem 1 for all F such that $D_0(F) \neq 0$. By continuity, this identity holds for every $F \epsilon \mathfrak{O}\mathfrak{N}$. In the same way we prove the second identity.

The determinant and subdeterminants of F satisfy the identities (see Sikorski [8]):

$$D_{n+1}\begin{pmatrix}\xi_0 A, \xi_1, \ldots, \xi_n \\ x_0, x_1, \ldots, x_n\end{pmatrix} = \sum_{i=0}^{n} (-1)^i \xi_0 x_i \cdot D_n\begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots, \xi_n \\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\end{pmatrix},$$

$$D_{n+1}\begin{pmatrix}\xi_0, \xi_1, \ldots, \xi_n \\ Ax_0, x_1, \ldots, x_n\end{pmatrix} = \sum_{i=0}^{n} (-1)^i \xi_i x_0 \cdot D_n\begin{pmatrix}\xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n \\ x_1, \ldots\ldots\ldots\ldots, x_n\end{pmatrix}$$

where, for brevity, $A = I+T = I+T_F$. Hence it follows that

(15)
$$D_{n+1}^*\begin{pmatrix}\xi_0 A, \xi_1, \ldots, \xi_n \\ x_0, x_1, \ldots, x_n\end{pmatrix} = \sum_{i=0}^{n} (-1)^i \xi_0 T x_i \cdot D_n^*\begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots, \xi_n \\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\end{pmatrix},$$

(16)
$$D_{n+1}^*\begin{pmatrix}\xi_0, \xi_1, \ldots, \xi_n \\ Ax_0, x_1, \ldots, x_n\end{pmatrix} = \sum_{i=0}^{n} (-1)^i \xi_i T x_0 \cdot D_n^*\begin{pmatrix}\xi_0, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n \\ x_1, \ldots\ldots\ldots\ldots, x_n\end{pmatrix}.$$

THEOREM 2. *The smallest integer r such that $D_r^*(F)$ is not identically equal to zero coincides with the smallest integer r such that $D_r(F) \neq 0$. Then*

(17) $$D_r^*(F) = (-1)^r D_r(F).$$

Let $\xi_1, \ldots, \xi_r, x_1, \ldots, x_r$ be fixed points such that

$$D_r^*\begin{pmatrix}\xi_1, \ldots, \xi_r \\ x_1, \ldots, x_r\end{pmatrix} \neq 0.$$

Let ζ_i, z_i $(i = 1, \ldots, r)$ be defined by the identities

$$\zeta_i x = \frac{D_r^*\begin{pmatrix}\xi_1, \ldots\ldots\ldots\ldots, \xi_r \\ x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_r\end{pmatrix}}{D_r^*\begin{pmatrix}\xi_1, \ldots, \xi_r \\ x_1, \ldots, x_r\end{pmatrix}},$$

$$\xi z_i = \frac{D_r^*\begin{pmatrix}\xi_1, \ldots, \xi_{i-1}, \xi, \xi_{i+1}, \ldots, \xi_r \\ x_1, \ldots\ldots\ldots\ldots, x_r\end{pmatrix}}{D_r^*\begin{pmatrix}\xi_1, \ldots, \xi_r \\ x_1, \ldots, x_r\end{pmatrix}},$$

and let B^* be the operator

$$\xi B^* x = \frac{D_{r+1}^*\begin{pmatrix}\xi, \xi_1, \ldots, \xi_r \\ x, x_1, \ldots, x_r\end{pmatrix}}{D_r^*\begin{pmatrix}\xi_1, \ldots, \xi_r \\ x_1, \ldots, x_r\end{pmatrix}}.$$

Then the equation (5) *has a solution x iff $\zeta_i x_0 = 0$ for $i = 1, \ldots, r$, and the equation* (6) *has a solution iff $\xi_0 z_i = 0$ for $i = 1, \ldots, r$. The general form of the solution of* (5) *is given by the formula*

$$x = x_0 - B^* x_0 + c_1 z_1 + \ldots + c_r z_r,$$

and the general form of the solution of (6) *is given by the formula*

$$\xi = \xi_0 - \xi_0 B^* + c_1 \zeta_1 + \ldots + c_r \zeta_r,$$

where $c_1, \ldots, c_r$ are arbitrary.

The first remark and the equality (17) follows from Theorem 1. By (17), the elements $\zeta_1, \ldots, \zeta_r, z_1, \ldots, z_r$ defined in Theorem 2 coincide with those defined on p. 149. Thus, to complete the proof, it suffices to show that if $\zeta_j x_0 = 0$, $\xi_0 z_j = 0$ for $j = 1, \ldots, r$, then $x = x_0 - B^* x_0$, $\xi = \xi_0 - \xi_0 B^*$ are solutions of (5) and (6) respectively. This follows from identities (15) and (16) which, for $n = r$, can be written in the form

$$\xi(I+T)B^* x = \xi T x - \sum_{i=1}^{r} \xi T x_i \cdot \zeta_i x,$$

$$\xi B^*(I+T)x = \xi T x - \sum_{i=1}^{r} \xi z_i \cdot \eta_i T x$$

or, equivalently, in the form

$$(I+T)(I-B^*) = I + \sum_{i=1}^{r} T x_i \cdot \zeta_i,$$

$$(I-B^*)(I+T) = I + \sum_{i=1}^{r} z_i \cdot \eta_i T.$$

Multiplying the first equation by x_0 from the right side and the second equation by ξ_0 from the left side, we get $(I+T)(I-B^*)x_0 = x_0$ and $\xi_0(I-B^*)(I+T) = \xi_0$, q. e. d.

§ 4. The integral model. Let μ be a measure defined on a σ-field of subsets of a set Γ. The integrals taken over the whole space Γ will be denoted, for brevity, by $\int f(t)\,dt$ instead of $\int\limits_\Gamma f(t)\,d\mu(t)$, and similarly for multiple integrals.

Suppose that X and Ξ are some Banach spaces of measurable functions on Γ, such that all the conditions from § 1 hold. Suppose moreover that ξx is defined by the formula

$$\xi x = \int \xi(t)\,x(t)\,dt.$$

An operator K is said to be an *integral operator* provided it is of the form

$$\xi K x = \iint \xi(s)\,K(s,t)\,x(t)\,ds\,dt \tag{18}$$

where $K(s,t)$ is a measurable function called the *kernel* of K. For instance, every finitely dimensional operator $K = \sum\limits_{i=1}^{m} x_i\cdot\xi_i$ is an integral operator, the kernel $K(s,t)$ being defined by the formula $K(s,t) = \sum\limits_{i=1}^{m} x_i(s)\cdot\xi_i(t)$.

A quasi-nucleus F is said to be an *integral quasi-nucleus* provided there exists a measurable function $T(t,s)$ such that

$$F(I) = \int T(s,s)\,ds \tag{19}$$

and

$$F(K) = \iint T(t,s)\,K(s,t)\,ds\,dt \tag{20}$$

for every integral operator K with a kernel $K(s,t)$. Then the operator $T = T_F$ is the integral operator determined by the kernel $T(s,t)$:

$$\xi T x = \iint \xi(s)\,T(s,t)\,x(t)\,ds\,dt.$$

To write some integral formulas for the subdeterminants of an integral quasi-nucleus F, it is convenient to introduce a formal expression $\delta(s,t)$ which is a substitute of the Dirac delta distribution. Viz. we define axiomatically $\delta(s,t)$ by the equations

$$\int \delta(s,t)\,x(t)\,dt = x(s), \qquad \int \xi(s)\,\delta(s,t)\,dt = \xi(t), \tag{21}$$

$$\iint \delta(s,t)\,T(t,s)\,ds\,dt = F(I). \tag{22}$$

If every point in Γ has a positive measure, then of course there exists a function $\delta(s,t)$ satisfying this identity. In the general case, the identities (21) should be interpreted quite formally as an integral notation for the identity operator I. By (21), we have

$$\iint \xi(s)\,\delta(s,t)\,x(t)\,ds\,dt = \xi x,$$

$$\iiint \delta(s,t)\,K_1(t,r)\,K_2(r,s)\,dr\,dt\,ds = \iint K_1(t,r)\,K_2(r,t)\,dr\,dt$$

for any kernels $K_1(t,r)$, $K_2(r,s)$, etc.

THEOREM 3. *If F is an integral quasi-nucleus satisfying* (19) *and* (20), *then*

$$A_{n,m}(F) = A_{n,m}\begin{pmatrix}\xi_1,\ldots,\xi_n\\ x_1,\ldots,x_n\end{pmatrix}$$

$$= \iint a_{n,m}\begin{pmatrix}s_1,\ldots,s_n\\ t_1,\ldots,t_n\end{pmatrix}\xi_1(s_1)\ldots\xi_n(s)\,x_1(t_1)\ldots x_n(t_n)\,ds_1\ldots ds_n\,dt_1\ldots dt_n$$

where

$$a_{n,m}\begin{pmatrix}s_1,\ldots,s_n\\ t_1,\ldots,t_n\end{pmatrix}$$

$$= \frac{1}{m!}\int\ldots\int\begin{vmatrix}\delta(s_1,t_1),\ldots,\delta(s_1,t_n), & \delta(s_1,r_1),\ldots,\delta(s_1,r_m)\\ \cdots & \cdots\\ \delta(s_n,t_1),\ldots,\delta(s_n,t_n), & \delta(s_n,r_1),\ldots,\delta(s_n,r_m)\\ T(r_1,t_1),\ldots,T(r_1,t_n), & T(r_1,r_1),\ldots,T(r_1,r_m)\\ \cdots & \cdots\\ T(r_m,t_1),\ldots,T(r_m,t_n), & T(r_m,r_1),\ldots,T(r_m,r_m)\end{vmatrix}dr_1\ldots dr_m.$$

Consequently

$$D_n(F) = D_n\begin{pmatrix}\xi_1,\ldots,\xi_n\\ x_1,\ldots,x_n\end{pmatrix}$$

$$= \int\ldots\int\vartheta_n\begin{pmatrix}s_1,\ldots,s_n\\ t_1,\ldots,t_n\end{pmatrix}\xi_1(s_1)\ldots\xi_n(s_n)\,x_1(t_1)\ldots x_n(t_n)\,ds_1\ldots ds_n\,dt_1\ldots dt_n,$$

where

$$\vartheta_n\begin{pmatrix}s_1,\ldots,s_n\\ t_1,\ldots,t_n\end{pmatrix} = \sum_{m=0}^{\infty} a_{n,m}\begin{pmatrix}s_1,\ldots,s_n\\ t_1,\ldots,t_n\end{pmatrix}.$$

In fact,

$$\theta_k\begin{pmatrix}\xi_1,\dots,\xi_k\\ x_1,\dots,x_k\end{pmatrix}$$

$$=\begin{vmatrix} \iint \xi_1(s_1)\delta(s_1,t_1)x_1(t_1)\,ds_1dt_1, \dots, \iint \xi_1(s_1)\delta(s_1,t_k)x_k(t_k)\,ds_1dt_k \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \iint \xi_k(s_k)\delta(s_k,t_1)x_1(t_1)\,ds_kdt_1,\dots,\iint\xi_k(s_k)\delta(s_k,t_k)x_k(t_k)\,ds_kdt_k\end{vmatrix} =$$

$$\int\dots\int \begin{vmatrix}\delta(s_1,t_1),\dots,\delta(s_1,t_k)\\ \dots\dots\dots\dots\\ \delta(s_k,t_1),\dots,\delta(s_k,t_k)\end{vmatrix}\xi_1(s_1)\dots\xi_k(s_k)x_1(t_1)\dots x_k(t_k)\,ds_1\dots ds_k\,dt_1\dots dt_k.$$

Hence, for $k=m+n$,

$$m!A_{n,m}(F)$$

$$=\int\dots\int\begin{vmatrix}\delta(s_1,t_1),\dots,\delta(s_1,t_{n+m})\\ \dots\dots\dots\dots\dots\\ \delta(s_{n+m},t_1),\dots,\delta(s_{n+m},t_{n+m})\end{vmatrix}\cdot\xi_1(s_1)\dots\xi_n(s_n)\cdot x_1(t_1)\dots x_n(t_n)\times$$

$$\times T(t_{n+1},s_{n+1})\dots T(t_{n+m},s_{n+m})\,ds_1\dots ds_{n+m}\,dt_1\dots dt_{n+m}$$

$$=\int\dots\int \begin{vmatrix}\delta(s_1,t_1),\dots,\delta(s_1,t_{n+m})\\ \dots\dots\dots\dots\dots \\ \delta(s_n,t_1),\dots,\delta(s_n,t_{n+m})\\ T(t_{n+1},t_1),\dots,T(t_{n+1},t_{n+m})\\ \dots\dots\dots\dots\dots\\ T(t_{n+m},t_1),\dots,T(t_{n+m},t_{n+m})\end{vmatrix}\times$$

$$\times\xi_1(s_1)\dots\xi_n(s_n)x_1(t_1)\dots x_n(t_1)\,ds_1\dots ds_n\,dt_1\dots dt_{n+m}.$$

Replacing $t_{n+1},\dots,t_{n+m}$ by $r_1,\dots,r_m$ we get the required formulas for $A_{n,m}(F)$, and consequently for $D_n(F)$.

THEOREM 4. *If F is an integral quasi-nucleus satisfying* (19) *and* (20), *then*

$$A^*_{n,m}(F)=A^*_{n,m}\begin{pmatrix}\xi_1,\dots,\xi_n\\x_1,\dots,x_n\end{pmatrix}$$

$$=\iint a^*_{n,m}\begin{pmatrix}s_1,\dots,s_n\\t_1,\dots,t_n\end{pmatrix}\xi_1(s_1)\dots\xi_n(s_n)x_1(t_1)\dots x_n(t_n)\,ds_1\dots ds_n\,dt_1\dots dt_n$$

where

$$a^*_{n,m}\begin{pmatrix}s_1\dots s_n\\t_1\dots t_n\end{pmatrix}=$$

$$\frac{1}{m!}\int\dots\int\begin{vmatrix}T(s_1,t_1),\dots,T(s_1,t_n),T(s_1,r_1),\dots,T(s_1,r_m)\\ \dots\dots\dots\dots\dots\dots\dots\\ T(s_n,t_1),\dots,T(s_n,t_n),T(s_n,r_1),\dots,T(s_n,r_m)\\ T(r_1,t_1),\dots,T(r_1,t_n),T(t_1,r_1),\dots,T(r_1,r_m)\\ \dots\dots\dots\dots\dots\dots\dots\\ T(r_m,t_1),\dots,T(r_m,t_n),T(r_m,r_1),\dots,T(r_m,r_m)\end{vmatrix}dr_1\dots dr_m.$$

Consequently

$$D^*_n(F)=D^*_n\begin{pmatrix}\xi_1,\dots,\xi_n\\x_1,\dots,x_n\end{pmatrix}$$

$$=\int\dots\int\vartheta^*_n\begin{pmatrix}s_1,\dots,s_n\\t_1,\dots,t_n\end{pmatrix}\xi_1(s_1)\dots\xi_n(s_n)x_1(t_1)\dots x_n(t_n)\,ds_1\dots ds_n\,dt_1\dots dt_n$$

where

$$\vartheta^*_n\begin{pmatrix}s_1,\dots,s_n\\t_1,\dots,t_n\end{pmatrix}=\sum_{m=0}^{\infty}a^*_{n,m}\begin{pmatrix}s_1,\dots,s_n\\t_1,\dots,t_n\end{pmatrix}.$$

Theorem 4 follows immediately from Theorem 3 and (13), (14).

§ 5. Multiplication in $\mathfrak{O}\mathfrak{N}$. As we have observed, $\mathfrak{O}\mathfrak{N}$ is a Banach space under the ordinary linear operations. We can also introduce in $\mathfrak{O}\mathfrak{N}$ operations of multiplication.

First observe that if $F\in\mathfrak{O}\mathfrak{N}$ and $C\in\mathfrak{O}$, then the functional

$$F_0(A)=F(AC)\quad\text{for}\quad A\in\mathfrak{O}$$

is also a quasi-nucleus since $\xi T_{F_0}x=F(x\cdot\xi C)=\xi CT_Fx$, i. e. $T_{F_0}=CT_F\in\mathfrak{O}$. The quasi-nucleus F_0 will be denoted by CF. Observe that

$$(C_1+C_2)F=C_1F+C_2F,\qquad C(F_1+F_2)=CF_1+CF_2,$$

$$C_1(C_2F)=(C_1C_2)F.$$

Moreover

$$T_{CF}=CT_F\quad\text{and}\quad|CF|\leqslant|C|\cdot|F|.$$

Similarly, if $F\in\mathfrak{O}\mathfrak{N}$, and $C\in\mathfrak{O}$, then the functional

$$F_0(A)=F(CA)\quad\text{for}\quad A\in\mathfrak{O}$$

is a quasi-nucleus. We denote it by FC. Observe that

$$F(C_1+C_2) = FC_1+FC_2, \qquad (F_1+F_2)C = F_1C+F_2C,$$

$$F(C_1C_2) = (FC_1)C_2.$$

Moreover,

$$T_{FC} = T_F C \quad \text{and} \quad |FC| \leqslant |F|\cdot|C|.$$

For any quasi-nucleus F_1, F_2, the quasi-nucleus $T_{F_1}F_2$ will be denoted by $F_1 \ogreaterthan F_2$, and the quasi-nucleus $F_1T_{F_2}$ will be denoted by $F_1 \olessthan F_2$. It follows immediately from the definition that both $F_1 \ogreaterthan F_2$ and $F_1 \olessthan F_2$ are multiplications in $\mathfrak{O}\mathfrak{N}$, i. e.

$$F_1 \ogreaterthan (F_2+F_3) = F_1 \ogreaterthan F_2 + F_1 \ogreaterthan F_3, \quad (F_1+F_2)\ogreaterthan F_3 = F_1 \ogreaterthan F_3 + F_2 \ogreaterthan F_3,$$

$$(F_1 \ogreaterthan F_2)\ogreaterthan F_3 = F_1 \ogreaterthan (F_2 \ogreaterthan F_3), \quad |F_1 \ogreaterthan F_2| \leqslant |F_1|\cdot|F_2|,$$

and similarly

$$F_1 \olessthan (F_2+F_3) = F_1 \olessthan F_2 + F_1 \olessthan F_3, \quad (F_1+F_2)\olessthan F_3 = F_1 \olessthan F_3 + F_2 \olessthan F_3,$$

$$(F_1 \olessthan F_2)\olessthan F_3 = F_1 \olessthan (F_2 \olessthan F_3), \quad |F_1 \olessthan F_2| \leqslant |F_1|\cdot|F_2|,$$

Moreover

$$T_{F_1 \ogreaterthan F_2} = T_{F_1 \olessthan F_1} = T_{F_1}T_{F_2}.$$

and for $C \in \mathfrak{O}$

$$C(F_1 \ogreaterthan F_2) = (CF_1)\ogreaterthan F_2, \qquad (F_1 \olessthan F_2)C = F_1 \olessthan (F_2C).$$

The multiplications $\ogreaterthan$ and $\olessthan$ have been introduced by Leżański [4] (see also Sikorski [6]).

If F_1, F_2 are finitely dimensional, then, by an easy calculation,

$$F_1 \ogreaterthan F_2 = F_1 \olessthan F_2 \tag{23}$$

and

$$\operatorname{tr}(F_1 \ogreaterthan F_2) = \operatorname{tr}(F_2 \ogreaterthan F_1), \qquad \operatorname{tr}(F_1 \olessthan F_2) = \operatorname{tr}(F_2 \olessthan F_1). \tag{24}$$

By continuity, this equalities hold also for arbitrary nucleus F_1, F_2. They do not hold, in general, for quasi-nucleus which are not nucleus (e. g. if $X = \boldsymbol{L}$, $\Xi = \boldsymbol{M}$, there exists a quasi-nuclear operator T which is not compact—see Sikorski [7]; given any distinct numbers a, b, there exist then quasi-nucleus F_1, F_2 such that $T_{F_1} = T = T_{F_2}$ and $F_1(T) = a$, $F_2(T) = b$; the quasi-nucleus F_1, F_2 satisfy neither (23) nor (24)).

Since the multiplications $\ogreaterthan$, $\olessthan$ do not satisfy, in general, the equalities (24) characteristic for the trace, it is convenient to introduce another multiplication $F_1 \circ F_2$ in $\mathfrak{O}\mathfrak{N}$ such that

$$\operatorname{tr}(F_1 \circ F_2) = \operatorname{tr}(F_2 \circ F_1) \quad \text{for every} \quad F_1, F_2 \in \mathfrak{O}\mathfrak{N}. \tag{25}$$

Viz. we define

$$F_1 \circ F_2 = \tfrac{1}{2}(F_1 \ogreaterthan F_2 + F_1 \olessthan F_2).$$

It is easy to verify that the distributive laws hold:

$$F_1 \circ (F_2 \circ F_3) = F_1 \circ F_2 + F_1 \circ F_3, \qquad (F_1+F_2)\circ F_3 = F_1 \circ F_3 + F_2 \circ F_3.$$

The associative law

$$(F_1 \circ F_2)\circ F_3 = F_1 \circ (F_2 \circ F_3)$$

also holds but its proof is not elementary. It is based on a theorem of Grothendieck [2] on nuclear and quasi-nuclear operators. More exactly, the associativity of $\circ$ follows from the fact that all the products

$$\begin{cases} F_1 \olessthan (F_2 \olessthan F_3), & (F_1 \olessthan F_2)\olessthan F_3, & F_1 \olessthan (F_2 \ogreaterthan F_3), \\ F_1 \ogreaterthan (F_2 \ogreaterthan F_3), & (F_1 \ogreaterthan F_2)\ogreaterthan F_3, & (F_1 \olessthan F_2)\ogreaterthan F_3, \\ & F_1 \ogreaterthan (F_2 \olessthan F_3), \quad (F_1 \ogreaterthan F_2)\olessthan F_3 & \end{cases} \tag{26}$$

are equal. It is easy to verify that the products written in the same line are equal (this follows immediately from the definition of $\ogreaterthan$ and $\olessthan$). It follows from a theorem proved by Grothendieck [2] that also products in different lines are equal.

Consequently all the products (26) are also equal to $F_1 \circ F_2 \circ F_3$, $F_1 \circ (F_2 \olessthan F_3)$ etc. Thus the multiplications $\ogreaterthan$, $\olessthan$, $\circ$ coincide for products of at least three factors. They do not coincide, in general, for two factors.

Observe else that

$$|F_1 \circ F_2| \leqslant |F_1|\cdot|F_2| \quad \text{and} \quad T_{F_1 \circ F_2} = T_{F_1}T_{F_2}.$$

In the sequel we shall consider $\mathfrak{O}\mathfrak{N}$ as a Banach algebra with the multiplication $\circ$. The map $F \to T_F$ is a ring homomorphism of $\mathfrak{O}\mathfrak{N}$ into $\mathfrak{O}$. Sometimes it is convenient to add an abstract unit E to the Banach algebra $\mathfrak{O}\mathfrak{N}$. The map $\nu E + F \to \nu I + T_F$ is a ring homomorphism of the extended algebra $\mathfrak{O}\mathfrak{N}$ into $\mathfrak{O}$. Instead of $D_0(F)$ we shall now write $D(E+F)$ for any $F \in \mathfrak{O}\mathfrak{N}$.

THEOREM 5. *For any* $F_1, F_2 \in \mathfrak{O}\mathfrak{N}$.

$$D\big((E+F_1)\circ(E+F_2)\big) = D(E+F_1)D(E+F_2).$$

The proof of this theorem is the same as the proof of Theorem 2 in my paper [6]. It is based on the fundamental identity (25). Theorem 5 follows also immediately from (25) and a general theorem proved by Michel and Martin [5].

§ 6. The differential equation of the determinant. We recall that by the first differential $D_0'(F; F_1)$ of $D_0(F)$ ($F, F_1 \epsilon \mathfrak{Q}\mathfrak{N}$) we understand the limit

$$\lim_{\varepsilon\to 0} \frac{D_0(F+\varepsilon F_1)-D(F)}{\varepsilon}.$$

By induction, the n-th differential $D_0^{(n)}(F; F_1, \ldots, F_n)$ is the limit

$$\lim_{\varepsilon\to 0} \frac{D_0^{(n-1)}(F+\varepsilon F_n; F_1, \ldots, F_{n-1})-D_0^{(n-1)}(F; F_1, \ldots, F_n)}{\varepsilon}.$$

Of course, $D_0^{(n)}(F; F_1, \ldots, F_n)$ is analytic in the variable F, and linear and symmetric in variables $F_1, \ldots, F_n$.

The following theorem is a slight modification of a general theorem due to Michel and Martin [5]:

THEOREM 6. *The determinant $D_0(F)$ is the only entire function on $\mathfrak{Q}\mathfrak{N}$ which is a solution of the differential equation*

$$D_0'\big(F; (I+T_F)F_1\big) = D_0(F)\cdot \mathrm{tr}(F_1), \tag{27}$$

satisfying the initial condition

$$D_0(0) = 1. \tag{28}$$

In fact, it is easy to verify that the first differential $A_{0,m}'(F; F_1)$ of $A_{0,m}(F)$ is equal to $F_1(A_{1,m-1})$. Hence it follows that

$$D_0'(F; F_1) = F_1\big(D_1(F)\big)$$

and consequently

$$D_0'\big(F; (I+T_F)F_1\big) = F_1\big(D_1(F)(I+T_F)\big) = F_1\big(D_0(F)I\big) = D_0(F)\mathrm{tr}(F_1).$$

Since $A_{0,0}(F) = 1$, we have $D_0(0) = 1$.

To prove the uniqueness, let us observe that for $|F|<1$ the differential equation (27) can be written in the form

$$D_0'(F; F_1) = D_0(F)\cdot \mathrm{tr}\big((I+T_F)^{-1}F_1\big) \tag{29}$$

since then $|T_F|<1$ and consequently $(I+T_F)^{-1}$ exists. Hence it follows that, in the set of all F with norm <1, the equation (27) have at most one solution satisfying the initial condition (28). By analyticity, we infer that there exists only one solution in the whole space $\mathfrak{Q}\mathfrak{N}$.

Observe also that the equation (27) can be replaced in Theorem 6 by each of the following equations:

$$D_0'\big(F; F_1(I+T_F)\big) = D_0(F)\cdot \mathrm{tr}(F_1),$$

$$D_0'\big(F; (E+F)\circ F_1\big) = D_0(F)\cdot \mathrm{tr}(F_1).$$

The subdeterminants $D_n(F)$ can be easily obtained from $D_0(F)$ by differentiation. Viz. the following formula holds for $n = 1, 2, \ldots$ (see Grothendieck [1]):

$$D_n\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix} = D_0^{(n)}(F; \xi_1\otimes x_1, \ldots, \xi_n\otimes x_n).$$

References

[1] A. Grothendieck, *La théorie de Fredholm*, Bull. Soc. Math. France (1956), p. 319-384.

[2] — *The trace of certain operators*, Studia Math. 20 (1961), p. 141-143.

[3] T. Leżański, *The Fredholm theory of linear equations in Banach spaces*, Studia Math. 13 (1953), p. 244-276.

[4] — *Sur les fonctionnelles multiplicatives*, ibidem 14 (1953), p. 13-23.

[5] A. D. Michel and R. S. Martin, *Some expansions in vector space*, Journ. Math. Pures et Appl. 13 (1934), p. 69-91.

[6] R. Sikorski, *On Leżański's determinants of linear equations in Banach spaces*, Studia Math. 14 (1953), p. 24-48.

[7] — *On determinants of Leżański and Ruston*, ibidem 16 (1957), p. 99-112.

[8] — *Determinant systems*, ibidem 18 (1959), p. 161-186.

Reçu par la Rédaction le 2. 7. 1960