und $\{b_n\}$ gegen Null konvergieren, eine integrierbare Funktion $x(t)$ gibt, für welche die Relationen

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos k_n t\,dt, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin k_n t\,dt \qquad (n=1,2,\ldots)$$

erfüllt sind.

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On the partial sums of Fourier series

by

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1. Let $f(\theta)$ $(0 \leqslant \theta \leqslant 2\pi)$ belong to L^p $(p \geqslant 1)$, and let $s_n(\theta) = s_n(f;\theta)$ and $\sigma_n(\theta) = \sigma_n(f;\theta)$ $(n = 0, 1, 2, \ldots)$ denote respectively the partial sums and the first arithmetical means of

$$(1.1) \qquad \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta),$$

the Fourier series of $f(\theta)$. A number of papers [1]) have appeared recently on the behaviour of integrals

$$(1.2) \qquad \int_0^{2\pi} s_{n_\theta}(\theta)\,d\theta\,, \qquad\qquad (1.3) \qquad \int_0^{2\pi} \sigma_{n_\theta}(\theta)\,d\theta\,,$$

where n_θ depends arbitrarily on θ.

It is evident that the necessary and sufficient condition that the integral (1.2) should be *finite* (or, what in this case is equivalent, *bounded*) is the existence of a function $\Phi(\theta)$ integrable L, such that

$$(1.4) \qquad |s_n(\theta)| \leqslant \Phi(\theta) \qquad (n = 0, 1, 2, \ldots).$$

Similarly the neccessary and sufficient condition that the integral (1.2) may be always greater than $-\infty$ is the existence of a function $\Phi^*(\theta) \subset L$, such that

$$(1.5) \qquad s_n(\theta) \geqslant -\Phi^*(\theta) \qquad (n = 0, 1, 2, \ldots).$$

The existence of a function $\Phi \subset L^r$ $(r > o)$, such that (1.4) is satisfied, is equivalent to the inequality

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[1]) Kolmogoroff and Seliverstoff [3]; Plessner [5]; Hardy and Littlewood [2]; Paley [4].

$$\int_0^{2\pi} |s_{n_\theta}(\theta)|^r d\theta = O(1).$$

2. The object of this paper is to prove the following theorem:

THEOREM. *If $f \subset L^p\,(p>1)$, and if $s_n(f;\,\theta)$ satisfies the condition* (1.5), *where $\Phi^*(\theta) \subset L^p$, then there exists a function $\psi(\theta) \subset L^p$, such that*

$$(2.1)\qquad -\Phi^*(\theta) \leqslant s_n(\theta) \leqslant \psi(\theta), \qquad -\psi(\theta) \leqslant \bar{s}_n(\theta) \leqslant \psi(\theta),$$

$\bar{s}_n(\theta)$ denoting the n-th partial sum of the conjugate series

$$\sum_{n=1}^{\infty} (a_n \sin n\theta - b_n \cos n\theta).$$

If $p=1$, then we can still assert (2.1) *with $\psi(\theta) \subset L^{1-\varepsilon}$ (ε being an arbitrary positive number).*

Let $t(\theta)$ denote a trigonometrical polynomial of order n. Then

$$\frac{t'(\theta)}{n+1} = \frac{1}{\pi(n+1)} \int_0^{2\pi} t(\theta+u)\,[\sin u + 2\sin 2u + \ldots + n \sin nu]\, du.$$

As $t(\theta)$ is of order n, we may add to the expression in square brackets the polynomial

$$(n+1)\sin(n+1)u + n\sin(n+2)u + \ldots + \sin(2n+1)u.$$

Adding together the terms $k \sin ku$ and $k\sin(2n+2-k)u$, we get

$$(2.2)\qquad \frac{t'(\theta)}{n+1} = \frac{2}{\pi}\int_0^{2\pi} t(\theta+u)\sin(n+1)u\,.\,k_n(u)\,du,$$

where $k_n(u)$ denotes Fejér's well known kernel. The formula (2.2) is one due to F. Riesz[2]). Similarly we get the formula for the conjugate polynomial[3]),

$$(2.3)\qquad \frac{\bar{t}'(\theta)}{n+1} = \frac{2}{\pi}\int_0^{2\pi} t(\theta+u)\cos(n+1)u\,.\,k_n(u)\,du.$$

[2]) F. Riesz [6].
[3]) Szegö [8].

We now substitute $s_n(f;\theta)$ for $t(\theta)$ in (2.2), and write

$$s_n(\theta+u) = s_n(\theta+u) + \Phi^*(\theta+u) - \Phi^*(\theta+u),$$

where, without loss of generality we may suppose that $\Phi^*(\theta)$ is non-negative. Then, since $|\sin(n+1)u| \leqslant 1$, and the expressions $s_n+\Phi^*$, Φ^*, k_n are everywhere positive, we have

$$\left|\frac{s_n'(\theta)}{n+1}\right| \leqslant \frac{2}{\pi}\int_0^{2\pi}\{s_n(\theta+u)+\Phi^*(\theta+u)\}\,k_n(u)\,du$$

$$+\frac{2}{\pi}\int_0^{2\pi}\Phi^*(\theta+u)\,k_n(u)\,du$$

$$=\frac{2}{\pi}\int_0^{2\pi}\{f(\theta+u)+\Phi^*(\theta+u)+\Phi^*(\theta+u)\}\,k_n(u)\,du$$

$$=2\,[\sigma_n(f;\theta)+2\,\sigma_n(\Phi^*;\theta)].$$

From (2.3) we may obtain the same result for $|\bar{s}_n'(\theta)/(n+1)|$. From the equations

$$s_n(f;\theta)-\sigma_n(f;\theta) = \frac{\bar{s}_n'(\theta)}{n+1},$$

$$\bar{s}_n(f;\theta)-\bar{\sigma}_n(f;\theta) = -\frac{s_n'(\theta)}{n+1},$$

we thus obtain

$$(2.4)\qquad |s_n(\theta)| \leqslant 3\,|\sigma_n(f;\theta)| + 4\,\sigma_n(\Phi^*;\theta)$$

$$(2.5)\qquad |\bar{s}_n(\theta)| \leqslant 2\,|\sigma_n(f;\theta)| + |\sigma_n(\bar{f};\theta)| + 4\,\sigma_n(\Phi^*;\theta),$$

$\bar{f}$ denoting the function conjugate to f.

To prove the theorem in the case $p>1$, we apply the following result due to Hardy and Littlewood[4]).

If $\xi(\theta) \subset L^p\,(p>1)$, then there exists a function $\psi(\theta) \subset L^p$, such that

$$(2.6)\qquad -\psi(\theta) \leqslant \sigma_n(\xi;\theta) \leqslant \psi(\theta).$$

Observing that, by M. Riesz's well known theorem[5]), $\bar{f}(\theta) \subset L^p$, and that

$$|\sigma_n(f;\theta)| \leqslant \sigma_n(|f|;\theta), \qquad |\sigma_n(\bar{f};\theta)| \leqslant \sigma_n(|\bar{f}|;\theta),$$

[4]) Hardy ad Littlewood [2].
[5]) M. Riesz [7].

we apply Hardy and Littlewood's result (2.6) with $\xi(\theta)=3|f(\theta)|+|\bar{f}(\theta)|+4\,\Phi^*(\theta)$. This proves the theorem in the case $p>1$.

3. The case $p=1$ is not dealt with in Hardy and Littlewood's paper, but it is not difficult to deduce the required results from their considerations. We first need two lemmas. We use the letter B throughout to denote an absolute positive constant (not always the same constant in different contexts).

Lemma 1. *Let $h(\theta)$ be a real function, periodic in 2π, and integrable in the Lebesgue sense in the interval $(0, 2\pi)$. Let $h(\varrho,\theta)$, $\bar{h}(\varrho,\theta)$ denote respectively the integrals*

$$\frac{1}{2\pi}\int_0^{2\pi} h(\theta+u)\,\frac{1-\varrho^2}{1-2\varrho\cos u+\varrho^2}\,du,$$

$$\frac{1}{\pi}\int_0^{2\pi} h(\theta+u)\,\frac{\varrho\sin u}{1-2\varrho\cos u+\varrho^2}\,du.$$

We denote by $H(\theta)$ the upper bound $(0\leqslant\varrho<1)$ of

$$|h(\varrho,\theta)+i\bar{h}(\varrho,\theta)|.$$

Then $H(\theta)$ belongs to the class $L^{1-\varepsilon}$ for all positive ε.

By one of the results[6]) of Hardy and Littlewood's paper, we have, for $r<1$,

$$\int_0^{2\pi}\operatorname*{Max}_{0\leqq\varrho\leqq r}|h(\varrho,\theta)+i\bar{h}(\varrho,\theta)|^{1-\varepsilon}d\theta$$

$$\leqslant B\int_0^{2\pi}|h(r,\theta)+i\bar{h}(r,\theta)|^{1-\varepsilon}d\theta$$

$$(3.1)\qquad \leqslant B\int_0^{2\pi}|h(r,\theta)|^{1-\varepsilon}d\theta+B\int_0^{2\pi}|\bar{h}(r,\theta)|^{1-\varepsilon}d\theta.$$

The first integral on the right hand of (3.1) is clearly bounded, and so is the second by virtue of a known theorem, which has been proved recently by Hardy[7]). From this the result of the lemma follows.

6) Hardy and Littlewood [2], theorem 27.
7) Hardy [1].

Lemma 2. *With the notation of* lemma 1, *let $H_1(\theta)$ denote the upper bound of*

$$\frac{1}{2\varrho}\int_{\theta-\varrho}^{\theta+\varrho}|h(x)|\,dx.$$

Then $H_1(\theta)\subset L^{1-\varepsilon}$, for all positive ε.

Without loss of generality we may suppose that $h(\theta)\geqslant 0$. Then, by a known result[8]), we have,

$$H_1(\theta)\leqslant B\sup_{0\leqq\varrho<1}h(\varrho,\theta)\leqslant BH_1(\theta),$$

from which the result follows[9]).

We write

$$F_1(\theta)=\sup\left\{\frac{1}{2\varrho}\int_{\theta-\varrho}^{\theta+\varrho}|f(x)|\,dx\right\},$$

$$\Phi_1(\theta)=\sup\left\{\frac{1}{2\varrho}\int_{\theta-\varrho}^{\theta+\varrho}|\Phi^*(x)dx|\right\}.$$

Then (as Hardy and Littlewood show in their paper)

$$(3.2)\qquad |\sigma_n(f;\theta)|\leqslant BF_1(\theta),\quad \sigma_n(\Phi^*;\theta)\leqslant B\,\Phi_1(\theta).$$

We denote by $f(\varrho,\theta)$, $\bar{f}(\varrho,\theta)$ respectively the integrals

$$\frac{1}{2\pi}\int_0^{2\pi}f(\theta+u)\frac{1-\varrho^2}{1-2\varrho\cos u+\varrho^2}du,\ \frac{1}{\pi}\int_0^{2\pi}f(\theta+u)\frac{\varrho\sin u}{1-2\varrho\cos u+\varrho^2}du,$$

and by $F(\theta)$ the upper bound $(0\leqslant\varrho<1)$ of

$$|f(\varrho,\theta)+i\bar{f}(\varrho,\theta)|.$$

Since

$$\bar{\sigma}_n(f;\theta)=\frac{1}{\pi}\int_0^{2\pi}f(\theta+u)\left\{\frac{1}{2}\cot\frac{1}{2}u-\frac{\sin(n+1)u}{4(n+1)\,.\,\sin^2\frac{1}{2}u}\right\}du,$$

it is not difficult to see that[10])

8) Paley [4], lemma 6.
9) The argument is sketched in Theorem I, Paley [4].
10) In fact, this difference is equal to

$$\frac{1}{\pi}\int_0^{\pi}[f(\theta+u)-f(\theta-u)]\cdot\left\{\frac{\sin u}{1+4n(n+1)\sin^2\frac{u}{2}}-\frac{\sin(n+1)u}{n+1}\right\}\frac{du}{4\sin^2\frac{u}{2}}$$

$$|\bar{\sigma}_n(f;\theta)-\bar{f}(1-\frac{1}{n+1},\theta)|\leqslant B F_1(\theta),$$

from which it follows that

$$|\sigma_n(\bar{f};\theta)|\leqslant F(\theta)+B F_1(\theta).$$

Combining this with (2.4), (2.5), (3.2) we obtain

$$|s_n(\theta)|\leqslant B F_1(\theta)+B\,\Phi_1(\theta),$$

$$|\bar{s}_n(\theta)|\leqslant F(\theta)+B F_1(\theta)+\Phi_1(\theta).$$

Now it follows from lemma 1 that $F(\theta)\subset L^{1-\varepsilon}$, and from lemma 2 that $F_1(\theta)\subset L^{1-\varepsilon}$, $\Phi_1(\theta)\subset L^{1-\varepsilon}$, and the required result follows at once.

It is not difficult to prove that if $|f|.\log|f|$ and $|\Phi^*|.\log|\Phi^*|$ are integrable then $\psi\subset L.$[11])

The following theorem is evident:

THEOREM. If the partial sums of the Fourier series of a function f, such that $|f|\leqslant 1$, verify an inequality $s_n\geqslant A$ ($A=$ const.; $n=0,1,2,\ldots$), then there exists a constant $B=B(A)$ such that $s_n\leqslant B$ ($n=0,1,2,\ldots$).

If f is continuous and, for any positive ε, $s_n(x)\geqslant f(x)-\varepsilon$ ($0\leqslant x\leqslant 2\pi$, $n>n(\varepsilon)$), then s_n converges uniformly towards $f(x)$

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Let us divide the integral into two, extended respectively over the intervals $(0,1/n)$ and $(1/n,\pi)$. In the first case the expression in the brackets { } is absolutely less than $B n\theta^2$ and so the coresponding integral is less than

$$B n\int_0^{1/n}(|f(\theta+u)|+|f(\theta-u)|)\,du\leqslant B F_1(\theta)$$

The second integral is less than

$$\frac{B}{n}\int_{1/n}(|f(\theta+u)|+|f(\theta-u)|)\frac{du}{u^2}$$

and an integration by parts shows also this expression to be less than $B F_1(\theta)$.

[11]) Cf. Hardy and Littlewood [2].

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